UDC 517.9

Liu Yang, Liping Luo, Zhenguo Luo (Hengyang Normal Univ., China)

INFINITELY MANY FAST HOMOCLINIC SOLUTIONS FOR SOME SECOND-ORDER NONAUTONOMOUS SYSTEMS*

НЕСКІНЧЕННА КІЛЬКІСТЬ ШВИДКИХ ГОМОКЛІНІЧНИХ РОЗВ'ЯЗКІВ НЕАВТОНОМНИХ СИСТЕМ ДРУГОГО ПОРЯДКУ

We investigate the existence of infinitely many fast homoclinic solutions for a class of second-order nonautonomous systems. Our main tools are based on the variant fountain theorem. A criterion guaranteeing that the second-order system have infinitely many fast homoclinic solutions is obtained. Recent results from the literature are generalized and significantly improved.

Досліджено існування нескінченної кількості швидких гомоклінічних розв'язків для класу неавтономних систем другого порядку. Наш основний метод базується на модифікації теореми про фонтан. Отримано критерій, що гарантує наявність нескінченної кількості швидких гомоклінічних розв'язків системи другого порядку. Узагальнено та значно покращено нещодавно опубліковані результати.

1. Introduction. In this article, we are concerned with the existence of infinitely many fast homoclinic solutions for the following second-order nonautonomous systems:

$$\ddot{u}(t) + c\dot{u} - L(t)u(t) + W_u(t, u(t)) = 0 \qquad \forall t \in \mathbb{R}, \tag{FHS}$$

where $u \in \mathbb{R}^n, c \geq 0$ is a constant, $W(t,u) \in C^1(\mathbb{R}, \mathbb{R}^n)$, and $L(t) \in C(\mathbb{R}, \mathbb{R}^{n\times n})$ is a symmetric matrix valued function. A nontrivial solution u of (FHS) is said to be homoclinic to zero if $u \in C^2(\mathbb{R}, \mathbb{R}^n), u(t) \to 0$ and $\dot{u}(t) \to 0$ as $|t| \to \infty$.

When $c = 0$, (FHS) is just the following second-order Hamiltonian system:

$$\ddot{u}(t) - L(t)u(t) + W_u(t, u(t)) = 0. \tag{HS}$$

In the last ten years, the existence and multiplicity of homoclinic solutions of (HS) have been intensively studied by many mathematicians (see [1 – 14] and the references therein). Compared with the case that $W(t,u)$ is superquadratic growth as $|u| \to \infty$, there is less literature for the case that $W(t,u)$ is subquadratic growth as $|u| \to \infty$ (see [12 – 14]). In [13], Zhang and Yuan established the following theorem.

Theorem 1.1 [13]. *Assume that L and W satisfy the following conditions*:

$(H1)$ *$L(t) \in C(\mathbb{R}, \mathbb{R}^{n\times n})$ is a symmetric and positive definite matrix for all $t \in \mathbb{R}$ and there is a continuous function $\alpha\colon \mathbb{R} \to \mathbb{R}$ such that $\alpha(t) > 0$ for all $t \in \mathbb{R}$ and $(L(t)u, u) \geq \alpha(t)|u|^2$ and $\alpha(t) \to \infty$ as $|t| \to +\infty$;*

$(H2)$ *$W(t,u) = a(t)|u|^\gamma$ where $a(t)\colon \mathbb{R} \to \mathbb{R}^+$ is a positive continuous function such that $a(t) \in L^2(\mathbb{R}, \mathbb{R}) \cap L^{\frac{2}{2-\gamma}}(\mathbb{R}, \mathbb{R})$ and $1 < \gamma < 2$ is a constant.*

Then (HS) possesses a nontrivial homoclinic solution.

* This work was supported by the Natural Science Foundation of Hunan Province (12JJ9001), Hunan Provincial Science and Technology Department of Science and Technology Project (2012SK3117) and Construct program of the key discipline in Hunan Province.

There are many mathematicians introduced the concept of fast heteroclinic solutions for the second-order ordinary differential equation $u'' + cu' + f(u) = 0$, (see [15]). When $c \neq 0$ in (FHS), as far as we know, there is few research about the existence of homoclinic solutions for (FHS). In [15], Zhang and Yuan introduced the concept of fast homoclinic solutions for (FHS) and established some criteria to guarantee the existence of fast homoclinic solutions for the first time. In order to state the concept of the fast homoclinic solutions conveniently, we first introduce some properties of the weighted Sobolev space E_c. For $c \geq 0$, we define the weighted Sobolev space E_c as follows:

$$E_c = \left\{ u \in H^1(\mathbb{R}, \mathbb{R}^n) \colon \int\limits_{\mathbb{R}} e^{ct} \left[|u'|^2 + (L(t)u(t), u(t)) \right] dt < +\infty \right\}.$$

If L satisfies $(H1)$, E_c is a Hilbert space with the inner product

$$(x, y) = \int\limits_{\mathbb{R}} e^{ct} \left[(x'(t), y'(t)) + (L(t)x(t), y(t)) \right] dt$$

and the corresponding norm $\|x\|_{E_c}^2 = (x, x)$. Here, we denote by $L^p(e^{ct})$, $2 \leq p < +\infty$, the Banach space of functions on $\mathbb{R}$ with values in $\mathbb{R}^n$ under the norm

$$\|u\|_p := \left(\int\limits_{\mathbb{R}} e^{ct} |u(t)|^p dt \right)^{1/p}.$$

Here, we still use the notation $\| \cdot \|_p$ to denote the norm of $L^p(e^{ct})$. Hence, there exists a constant $\beta = \min\{\alpha(t), t \in \mathbb{R}\} > 0$ such that

$$\beta \|u\|_2^2 \leq \|u\|_{E_c}^2 \qquad \forall u \in E_c. \tag{1.1}$$

Definition 1.1. *For $c > 0$, a homoclinic solution u of (FHS) is called one fast homoclinic solution if $u \in E_c$.*

Theorem 1.2 [15]. *Assume that L and W satisfy $(H1)$ and the following condition*:

$(H2)$ $W(t, u) = a(t)|u|^\gamma$ *where $a(t) \colon \mathbb{R} \to \mathbb{R}$ is a continuous function such that $a(t_1) > 0$ for some $t_1 \in \mathbb{R}$ and $a(t) \in L^{\frac{2}{2-\gamma}}(e^{ct})$ and $1 < \gamma < 2$ is a constant.*

Then (FHS) has at least one nontrivial fast homoclinic solution.

Motivated by the above facts, in this paper, we will use the following conditions to generalize and improve Theorem 1.2. To the best of our knowledge, there is no paper studying the existence of infinitely many fast homoclinic solutions for (FHS).

$(H2')$ $a(t)|u|^\gamma \leq W_u(t, u)u$, $|W_u(t, u)| \leq b(t)|u|^{\gamma-1} + c(t)|u|^{\delta-1}$ where $a(t)$, $b(t)$, $c(t) \colon \mathbb{R} \to \to \mathbb{R}^+$ are positive continuous functions such that $a(t)$, $b(t) \in L^{\frac{2}{2-\gamma}}(e^{ct})$, $c(t) \in L^{\frac{2}{2-\delta}}(e^{ct})$ and $1 < \gamma < 2$, $1 < \delta < 2$ are constants, $W(t, 0) = 0$, $W(t, u) = W(t, -u)$.

We can see that if $b(t) = a(t)$, $c(t) = 0$, then $W(t, u) = \dfrac{a(t)}{\gamma}|u|^\gamma$. Therefore, the condition of $(H2)$ is a special case of the condition of $(H2')$. Here is our main result.

Theorem 1.3. *Suppose that the conditions of $(H1)$ and $(H2')$ hold. Then (FHS) possesses infinitely many fast homoclinic solutions.*

The organization of this paper is as follows. In Section 2, we shall give some lemmas and some preliminary results. In Section 3, main result are verified.

2. Preliminaries. In this section, we will present some lemmas that will be used in the proof of our main result.

Lemma 2.1 [15]. *Suppose that L satisfies $(H1)$. Then the embedding of E_c in $L^2(e^{ct})$ is compact.*

Lemma 2.2. *Suppose that $(H1), (H2^{'})$ hold. If $u_k \rightharpoonup u$ in E_c, then $W_u(t, u_k) \to W_u(t, u)$ in $L^2(e^{ct})$.*

Proof. Assume that $u_k \rightharpoonup u$ in E_c. By $(H2^{'})$ we have

$$|W_u(t, u_k) - W_u(t, u)| \le b(t)\left[|u_k|^{\gamma-1} + |u|^{\gamma-1}\right] + c(t)\left[|u_k|^{\delta-1} + |u|^{\delta-1}\right], \tag{2.1}$$

which yields that

$$|W_u(t, u_k) - W_u(t, u)|^2 \le 4b^2(t)\left[|u_k|^{2\gamma-2} + |u|^{2\gamma-2}\right] + 4c^2(t)\left[|u_k|^{2\delta-2} + |u|^{2\delta-2}\right]. \tag{2.2}$$

Multiplying e^{ct} and integrating on $\mathbb{R}$, by (1.1) and Hölder inequality, we get

$$\int\limits_{\mathbb{R}} e^{ct}|W_u(t, u_k) - W_u(t, u)|^2 dt \le$$

$$\le 4\int\limits_{\mathbb{R}} e^{ct}b^2(t)[|u_k(t)|^{2\gamma-2} + |u(t)|^{2\gamma-2}]dt + 4\int\limits_{\mathbb{R}} e^{ct}c^2(t)[|u_k(t)|^{2\delta-2} + |u(t)|^{2\delta-2}]dt \le$$

$$\le 4\|b\|^2_{\frac{2}{2-\gamma}}(\|u_k\|_2^{2\gamma-2} + \|u\|_2^{2\gamma-2}) + 4\|c\|^2_{\frac{2}{2-\delta}}(\|u_k\|_2^{2\delta-2} + \|u\|_2^{2\delta-2}) \le$$

$$\le 4\beta^{1-\gamma}\|b\|^2_{\frac{2}{2-\gamma}}(\|u_k\|_{E_c}^{2\gamma-2} + \|u\|_{E_c}^{2\gamma-2}) + 4\beta^{1-\delta}\|c\|^2_{\frac{2}{2-\delta}}(\|u_k\|_{E_c}^{2\delta-2} + \|u\|_{E_c}^{2\delta-2}). \tag{2.3}$$

Moreover, since $u_k \rightharpoonup u$ in E_c, there exists a constant $M > 0$ such that, by Banach – Steinhaus theorem,

$$\|u_k\|_{E_c} \le M, \qquad \|u\|_{E_c} \le M.$$

Therefore, we can obtain

$$\int\limits_{\mathbb{R}} e^{ct}|W_u(t, u_k) - W_u(t, u)|^2 dt \le 8\beta^{1-\gamma}\|b\|^2_{\frac{2}{2-\gamma}}M^{2\gamma-2} + 8\beta^{1-\delta}\|c\|^2_{\frac{2}{2-\delta}}M^{2\delta-2}.$$

Since, by Lemma 2.1, $u_k \to u$ in $L^2(e^{ct})$, which yields that $e^{ct}u_k(t) \to e^{ct}u(t)$ for almost every $t \in \mathbb{R}$, i.e., $u_k(t) \to u(t)$ for almost every $t \in \mathbb{R}$ since $e^{ct} > 0$ for every $t \in \mathbb{R}$. Then, by the using the Lebesgue convergence theorem.

Lemma 2.2 is proved.

Define the functional

$$I(u) = \frac{1}{2}\int\limits_{\mathbb{R}} e^{ct}[|\dot{u}|^2 + (L(t)u(t), u(t))]dt - \int\limits_{\mathbb{R}} e^{ct}W(t, u(t))dt = \frac{1}{2}\|u\|_{E_c}^2 - B(u), \tag{2.4}$$

where $B(u) = \int_{\mathbb{R}} e^{ct}W(t, u(t))dt$.

Lemma 2.3. *Under the conditions of Theorem* 1.3, *we get*

$$I'(u)v = \int\limits_{\mathbb{R}} e^{ct}[(\dot{u},\dot{v}) + (L(t)u(t),v(t))]dt - \int\limits_{\mathbb{R}} e^{ct}(W_u(t,u(t)),v(t))dt =$$

$$= \int\limits_{\mathbb{R}} e^{ct}[(\dot{u},\dot{v}) + (L(t)u(t),v(t))]dt - B'(u)v \tag{2.5}$$

for any $u, v \in E_c$, *which yields that*

$$I'(u)u = \|u\|^2_{E_c} - \int\limits_{\mathbb{R}} e^{ct}(W_u(t,u(t)),u(t))dt. \tag{2.6}$$

Moreover, $I \in C^1(E_c,\mathbb{R}), B' \colon E_c \to E_c^*$ *is compact, and any critical point of* I *on* E_c *is a classical solution of* (FHS) *satisfying* $u \in C^2(\mathbb{R},\mathbb{R}^n), u(t) \to 0$ *and* $\dot{u}(t) \to 0$ *as* $|t| \to \infty$.

Proof. We firstly show that $I \colon E_c \to \mathbb{R}$. Since $W(t,0) = 0$, by $(H2')$, we have

$$0 \le \int\limits_{\mathbb{R}} e^{ct}\left[\int\limits_0^1 a(t)|u|^\gamma h^{\gamma-1}dh\right]dt \le \int\limits_{\mathbb{R}} e^{ct}W(t,u(t))dt =$$

$$= \int\limits_{\mathbb{R}} e^{ct}\left[\int\limits_0^1 W_u(t,hu)udh\right]dt \le \int\limits_{\mathbb{R}} e^{ct}\left[\int\limits_0^1 |W_u(t,hu)||u|dh\right]dt \le$$

$$\le \int\limits_{\mathbb{R}} e^{ct}\frac{b(t)}{\gamma}|u(t)|^\gamma dt + \int\limits_{\mathbb{R}} e^{ct}\frac{c(t)}{\delta}|u(t)|^\delta dt \le \frac{1}{\gamma}\|b\|_{\frac{2}{2-\gamma}}\|u\|_2^\gamma + \frac{1}{\delta}\|c\|_{\frac{2}{2-\delta}}\|u\|_2^\delta \le$$

$$\le \frac{1}{\gamma}\|b\|_{\frac{2}{2-\gamma}}\beta^{-\gamma}\|u\|^\gamma_{E_c} + \frac{1}{\delta}\|c\|_{\frac{2}{2-\delta}}\beta^{-\delta}\|u\|^\delta_{E_c}. \tag{2.7}$$

Next we prove that $I \in C^1(E_c,\mathbb{R})$. Rewrite I as follows:

$$I = A(u) - B(u), \tag{2.8}$$

where

$$A(u) = \frac{1}{2}\int\limits_{\mathbb{R}} e^{ct}\left[|\dot{u}|^2 + (L(t)u(t),u(t))\right]dt.$$

It is easy to check that $A \in C^1(E_c,\mathbb{R})$ and $A'(u)v = \int_{\mathbb{R}} e^{ct}\left[(\dot{u},\dot{v}) + (L(t)u(t),v(t))\right]dt$. Therefore, it is sufficient to show that this is the case for B. In the process we will see that

$$B'(u)v = \int\limits_{\mathbb{R}} e^{ct}(W_u(t,u(t)),v(t))dt. \tag{2.9}$$

For any given $u \in E_c$, let us define $J(u) \colon E_c \to \mathbb{R}$ as follows:

$$J(u)v = \int\limits_{\mathbb{R}} e^{ct}(W_u(t, u(t)), v(t))dt, \qquad v \in E_c. \tag{2.10}$$

It is obvious that $J(u)$ is linear. Now we show that $J(u)$ is bounded. Indeed, for any given $u \in E_c$, we have

$$|J(u)v| = \int\limits_{\mathbb{R}} e^{ct}(W_u(t, u(t)), v(t))dt \le$$

$$\le \int\limits_{\mathbb{R}} e^{ct} b(t)|u(t)|^{\gamma-1}||v(t)|dt + \int\limits_{\mathbb{R}} e^{ct} c(t)|u(t)|^{\delta-1}||v(t)|dt \le$$

$$\le \left(\int\limits_{\mathbb{R}} e^{ct} b^2(t)|u(t)|^{2\gamma-2}dt\right)^{1/2} \left(\int\limits_{\mathbb{R}} e^{ct}|v(t)|^2 dt\right)^{1/2} +$$

$$+ \left(\int\limits_{\mathbb{R}} e^{ct} c^2(t)|u(t)|^{2\delta-2}dt\right)^{1/2} \left(\int\limits_{\mathbb{R}} e^{ct}|v(t)|^2 dt\right)^{1/2} \le$$

$$\le \left(\int\limits_{\mathbb{R}} e^{ct} b^2(t)|u(t)|^{2\gamma-2}\right)^{1/2} \|v\|_2 + \left(\int\limits_{\mathbb{R}} e^{ct} c^2(t)|u(t)|^{2\delta-2}\right)^{1/2} \|v\|_2 \le$$

$$\le \|b\|_{\frac{2}{2-\gamma}} \|u\|_2^{\gamma-1} \|v\|_2 + \|c\|_{\frac{2}{2-\delta}} \|u\|_2^{\delta-1} \|v\|_2 \le$$

$$\le \beta^{-\gamma} \|b\|_{\frac{2}{2-\gamma}} \|u\|_{E_c}^{\gamma-1} \|v\|_{E_c} + \beta^{-\delta} \|c\|_{\frac{2}{2-\delta}} \|u\|_{E_c}^{\delta-1} \|v\|_{E_c}.$$

Moreover, for $u, v \in E_c$, by the mean value theorem, we obtain

$$\int\limits_{\mathbb{R}} e^{ct} W(t, u(t) + v(t))dt - \int\limits_{\mathbb{R}} e^{ct} W(t, u(t))dt = \int\limits_{\mathbb{R}} e^{ct}(W_u(t, u(t) + h(t)v(t)), v(t))dt,$$

where $h(t) \in (0, 1)$. Therefore, by Lemma 2.2, we get

$$\int\limits_{\mathbb{R}} e^{ct}(W_u(t, u(t) + h(t)v(t)), v(t))dt - \int\limits_{\mathbb{R}} e^{ct}(W_u(t, u(t)), v(t))dt =$$

$$= \int\limits_{\mathbb{R}} e^{ct} \left(W_u(t, u(t) + h(t)v(t)) - W_u(t, u(t)), v(t)\right) dt \to 0$$

as $v \to 0$. Suppose that $u \to u_0$ in E_c and note that

$$B'(u)v - B'(u_0)v = \int\limits_{\mathbb{R}} e^{ct}(W_u(t, u(t)) - W_u(t, u_0(t)), v(t))dt. \tag{2.11}$$

By Lemma 2.2 and the Hölder inequality, we obtain that

$$B'(u)v - B'(u_0)v \to 0 \text{ as } \quad u \to u_0, \tag{2.12}$$

which implies the continuity of B' and we show that $I \in C^1(E_c, \mathbb{R})$. Let $u_k \rightharpoonup u$ in E_c, we have

$$\|B'(u_k) - B'(u)\|_{E_c^*} = \sup_{\|v\|=1} \|(B'(u_k) - B'(u))v\| =$$

$$= \sup_{\|v\|=1} \left| \int_{\mathbb{R}} e^{ct} \langle W_u(t, u_k) - W_u(t, u), v(t)\rangle dt \right| \leq$$

$$\leq \sup_{\|v\|=1} \left(\int_{\mathbb{R}} e^{ct} |W_u(t, u_k) - W_u(t, u)|^2 dt \right)^{1/2} \|v\|_2 \leq$$

$$\leq C_2 \left(\int_{\mathbb{R}} e^{ct} |W_u(t, u_k) - W_u(t, u)|^2 dt \right)^{1/2} \to 0$$

as $k \to \infty$. Consequently, B' is weakly continuous. Therefore, B' is compact by the weakly continuity of B' since E is a Hilbert space. Proofs of the other conclusions can be found in Lemma 3.1 of [15], so we omit them here.

In order to prove our main results, we recall the variant fountain theorem. Let E be a Banach space with the norm $\|\cdot\|$ and $E = \overline{\bigoplus_{j=0}^{k} X_j}$ with $\dim X_j < \infty$ for any $j \in \mathbb{N}$. Set $Y_k = \bigoplus_{j=0}^{k} X_j$, $Z_k = \overline{\bigoplus_{j=k}^{\infty} X_j}$. Consider the following C^1-functional $I_\lambda \colon E \to \mathbb{R}$ defined by

$$I_\lambda(u) = A(u) - \lambda B(u), \qquad \lambda \in [1, 2]. \tag{2.13}$$

Theorem 2.1 [16]. *Suppose that the functional $I_\lambda(u)$ defined above satisfies:*

$(C1)$ *I_λ maps bounded sets to bounded sets uniformly for $\lambda \in [1, 2]$. Furthermore, $I_\lambda(-u) = I_\lambda(u)$ for all $(\lambda, u) \in [1, 2] \times E$.*

$(C2)$ *$B(u) \geq 0$; $B(u) \to \infty$ as $\|u\| \to \infty$ on any finite dimensional subspace of E.*

$(C3)$ *There exist $\rho_k > r_k > 0$ such that*

$$a_k(\lambda) := \inf_{u \in Z_k, \|u\|=\rho_k} I_\lambda(u) \geq 0 > b_k(\lambda) := \max_{u \in Y_k, \|u\|=r_k} I_\lambda(u)$$

for all $\lambda \in [1, 2]$ and $d_k(\lambda) := \inf_{u \in Z_k \|u\| \leq \rho_k} I_\lambda(u) \to 0$ as $k \to \infty$ uniformly for $\lambda \in [1, 2]$. Then there exist $\lambda_n \to 1, u_{\lambda_n} \in Y_n$ such that $I'_{\lambda_n}|Y_n(u(\lambda_n)) = 0, I_{\lambda_n}(u(\lambda_n)) \to c_k \in [d_k(2), b_k(1)]$ as $n \to \infty$. In particular, if $\{u(\lambda_n)\}\}$ has a convergent subsequence for every k, then I_1 has infinitely many nontrivial critical points $\{u_n\} \subset E \setminus \{0\}$ satisfying $I_1(u_k) \to 0^-$ as $k \to \infty$.

3. Main results. ***Proof of Theorem* 1.3.** In order to apply Theorem 2.1 to prove Theorem 1.3, we define the functionals A, B and I_λ on our working space E_c by

$$A(u) = \frac{1}{2}\|u\|_{E_c}^2, \qquad B(u) = \int_{\mathbb{R}} e^{ct} W(t, u) dt, \tag{3.1}$$

$$I_\lambda(u) = A(u) - \lambda B(u) \tag{3.2}$$

for all $u \in E_c$ and $\lambda \in [1,2]$. From Lemma 2.3, we know that $I_\lambda \in C^1(E_c, \mathbb{R})$ for all $\lambda \in [1,2]$. We choose a completely orthonormal basis $\{e_j\}$ of E_c and define $X_j := \mathbb{R}e_j$. Then Z_k, Y_k can be defined as that in Section 2.

***Step* 1.** In the condition of Theorem 1.3, we have $B(u) \geq 0$. Moreover, $B(u) \to \infty$ as $\|u\| \to \infty$ on any finite dimensional subspace of E_c.

Obviously, $B(u) \geq 0$ follows by the definition of the functional B and $(H2')$. For any finite dimensional subspace $F \subset E_c$, there exists $\varepsilon_1 > 0$ such that

$$\text{meas}\left\{t \in \mathbb{R} : e^{ct}a(t)|u(t)|^\gamma \geq \varepsilon_1 \|u\|_{E_c}^\gamma\right\} \geq \varepsilon_1 \qquad \forall u \in F \setminus \{0\}, \tag{3.3}$$

where meas denotes that Lebesgue measure in $\mathbb{R}^n$. Otherwise, for any positive integer n, there exists $u_n \in F \setminus \{0\}$ such that

$$\text{meas}\left\{t \in \mathbb{R}\colon e^{ct}a(t)|u(t)|^\gamma \geq \frac{1}{n}\|u\|_{E_c}^\gamma\right\} < \frac{1}{n}. \tag{3.4}$$

Set $v_n(t) := \dfrac{u_n(t)}{\|u_n\|_{E_c}} \in F \setminus \{0\}$, then $\|v_n\|_{E_c} = 1$ for all $n \in \mathbb{N}$ and

$$\text{meas}\left\{t \in \mathbb{R}\colon e^{ct}a(t)|v_n(t)|^\gamma \geq \frac{1}{n}\right\} < \frac{1}{n}.$$

Since $\dim F < \infty$, it follows from the compactness of the unit sphere of F that there exists a subsequence, say $\{v_n\}$, such that v_n converges to some v_0 in F. Hence, we have $\|v_0\|_{E_c} = 1$. By the equivalence of the norms on the finite dimensional space F, we have $v_n \to v_0$ in $L^2(e^{ct})$. By the Hölder inequality, one has

$$\int_{\mathbb{R}} e^{ct}a(t)|v_n - v_0|^\gamma dt \leq \|a\|_{\frac{2}{2-\gamma}} \left(\int_{\mathbb{R}} e^{ct}|v_n - v_0|^2 dt\right)^{\gamma/2} \to 0 \quad \text{as} \quad n \to \infty. \tag{3.5}$$

Thus there exist $\xi_1, \xi_2 > 0$ such that

$$\text{meas}\left\{t \in \mathbb{R}\colon e^{ct}a(t)|v_0(t)|^\gamma \geq \xi_1\right\} \geq \xi_2. \tag{3.6}$$

In fact, if not, we have

$$\text{meas}\left\{t \in \mathbb{R}\colon e^{ct}a(t)|v_0(t)|^\gamma \geq \frac{1}{n}\right\} \geq 0 \tag{3.7}$$

for all positive integer n, which implies that

$$0 \leq \int_{\mathbb{R}} e^{2ct}a(t)|v_0(t)|^{\gamma+2}dt < \frac{1}{n}\|v_0\|_2^2 \leq \frac{1}{n\beta^2}\|v_0\|_{E_c}^2 = \frac{1}{n\beta^2} \to 0 \tag{3.8}$$

as $n \to \infty$. Hence $v_0 = 0$ which contradicts that $\|v_0\|_{E_c} = 1$. Therefore, (3.6) holds. Now let

$$\Omega_0 = \left\{t \in \mathbb{R}\colon e^{ct}a(t)|v_0(t)|^\gamma \geq \xi_1\right\}, \qquad \Omega_n = \left\{t \in \mathbb{R} : e^{ct}a(t)|v_n(t)|^\gamma < \frac{1}{n}\right\}$$

and $\Omega_n^c = \mathbb{R} \setminus \Omega_n$. Then we get

$$\operatorname{meas}(\Omega_n \cap \Omega_0) \geq \operatorname{meas}(\Omega_0) - \operatorname{meas}(\Omega_n^c \cap \Omega_0) \geq \xi_2 - \frac{1}{n} \tag{3.9}$$

for all positive integer n. Let n be large enough such that $\xi_2 - \dfrac{1}{n} \geq \dfrac{1}{2}\xi_2$ and $\dfrac{1}{2^{\gamma-1}}\xi_1 - \dfrac{1}{n} \geq \dfrac{1}{2^\gamma}\xi_1$. Then we have

$$\int\limits_{\mathbb{R}} e^{ct}a(t)|v_n - v_0|^\gamma dt \geq \int\limits_{\Omega_n\cap\Omega_0} e^{ct}a(t)|v_n - v_0|^\gamma dt \geq$$

$$\geq \frac{1}{2^{\gamma-1}} \int\limits_{\Omega_n\cap\Omega_0} e^{ct}a(t)|v_0|^\gamma dt - \int\limits_{\Omega_n\cap\Omega_0} e^{ct}a(t)|v_n|^\gamma dt \geq$$

$$\geq \left(\frac{1}{2^{\gamma-1}}\xi_1 - \frac{1}{n}\right) \operatorname{meas}(\Omega_n \cap \Omega_0) \geq \frac{\xi_1\xi_2}{2^{\gamma+1}} > 0$$

for all large n, which is a contradiction to (3.5). Therefore, (3.3) holds. For the ε_1 given in (3.1), let

$$\Omega_u = \left\{t \in \mathbb{R}\colon e^{ct}a(t)|u(t)|^\gamma \geq \varepsilon_1\|u\|_{E_c}^\gamma\right\} \qquad \forall u \in F \setminus \{0\}. \tag{3.10}$$

Then by (3.1),

$$\operatorname{meas}(\Omega_u) \geq \varepsilon_1 \qquad \forall u \in F \setminus \{0\}.$$

Combining $(H2')$ and (3.10), for any $u \in F \setminus \{0\}$, we obtain

$$B(u) = \int\limits_{\mathbb{R}} e^{ct}[W(t,u) - W(t,0)]dt = \int\limits_{\mathbb{R}} e^{ct}\left[\int\limits_0^1 W_u(t,hu)udh\right] dt \geq$$

$$\geq \int\limits_{\mathbb{R}} e^{ct}\left[\int\limits_0^1 a(t)|u|^\gamma h^{\gamma-1}dh\right] dt \geq \frac{1}{\gamma}\int\limits_{\mathbb{R}} e^{ct}a(t)|u(t)|^\gamma dt \geq$$

$$\geq \frac{1}{\gamma}\int\limits_{\Omega_u} e^{ct}a(t)|u(t)|^\gamma dt \geq \frac{1}{\gamma}\varepsilon_1\|u\|_{E_c}^\gamma \operatorname{meas}(\Omega_u) \geq$$

$$\geq \frac{1}{\gamma}\varepsilon_1^2\|u\|_{E_c}^\gamma.$$

This implies $B(u) \to \infty$ as $\|u\|_{E_c} \to \infty$ on any finite dimensional subspace of E.

***Step* 2.** Under the assumptions of Theorem 1.3, then there exists a sequence $\rho_k \to 0^+$ as $k \to \infty$ such that

$$a_k(\lambda) := \inf_{u\in Z_k, \|u\|_{E_c}=\rho_k} I_\lambda(u) \geq 0,$$

and

$$d_k(\lambda) := \inf_{u\in Z_k, \|u\|_{E_c}\le\rho_k} I_\lambda(u) \to 0 \text{ as } k\to\infty \quad \text{uniformly for} \quad \lambda\in[1,2].$$

Set $\beta_k := \sup_{u\in Z_k, \|u\|_{E_c}=1} \|u\|_2$. Then $\beta_k \to 0$ as $k\to\infty$ since E_c is compactly embedded into $L^2(e^{ct})$. By $(H2')$, we have

$$I_\lambda(u) = \frac{1}{2}\|u\|_{E_c}^2 - \lambda\int_{\mathbb{R}} e^{ct}W(t,u)dt \ge$$

$$\ge \frac{1}{2}\|u\|_{E_c}^2 - 2\int_{\mathbb{R}} e^{ct}W(t,u)dt \ge \frac{1}{2}\|u\|_{E_c}^2 - \frac{2}{\gamma}\|b\|_{\frac{2}{2-\gamma}}\|u\|_2^\gamma - \frac{2}{\delta}\|c\|_{\frac{2}{2-\delta}}\|u\|_2^\delta \ge$$

$$\ge \frac{1}{2}\|u\|_{E_c}^2 - \frac{2}{\gamma}\beta_k^\gamma\|b\|_{\frac{2}{2-\gamma}}\|u\|_{E_c}^\gamma - \frac{2}{\delta}\beta_k^\delta\|c\|_{\frac{2}{2-\delta}}\|u\|_{E_c}^\delta.$$

Let

$$\rho_k = \left(\frac{16\beta_k^\gamma}{\gamma}\|b\|_{\frac{2}{2-\gamma}}\right)^{\frac{1}{2-\gamma}} + \left(\frac{16\beta_k^\delta}{\delta}\|c\|_{\frac{2}{2-\delta}}\right)^{\frac{1}{2-\delta}}.$$

Obviously, $\rho_k\to 0$ as $k\to\infty$. Combining this with the above inequality, straightforward computation shows that

$$a_k(\lambda) \ge \frac{1}{4}\rho_k^2 > 0. \tag{3.11}$$

Furthermore, for any $u\in Z_k$ with $\|u\|_{E_c}\le\rho_k$, we get

$$I_\lambda(u) \ge -\frac{2}{\gamma}\beta_k^\gamma\|b\|_{\frac{2}{2-\gamma}}\|u\|_{E_c}^\gamma - \frac{2}{\delta}\beta_k^\delta\|c\|_{\frac{2}{2-\delta}}\|u\|_{E_c}^\delta.$$

Therefore,

$$0 \ge d_k(\lambda) \ge -\frac{2}{\gamma}\beta_k^\gamma\|b\|_{\frac{2}{2-\gamma}}\|u\|_{E_c}^\gamma - \frac{2}{\delta}\beta_k^\delta\|c\|_{\frac{2}{2-\delta}}\|u\|_{E_c}^\delta. \tag{3.12}$$

Since $\beta_k, \rho_k \to 0$ as $k\to\infty$, we obtain

$$d_k(\lambda) := \inf_{u\in Z_k, \|u\|_{E_c}\le\rho_k} I_\lambda(u) \to 0 \text{ as } k\to\infty \quad \text{uniformly for} \quad \lambda\in[1,2].$$

***Step* 3.** Under the assumptions of Theorem 1.3, for the sequence $\{\rho_k\}_{k\in\mathbb{N}}$ obtained in Step 2, there exist $0<r_k<\rho_k$ for all $k\in\mathbb{N}$ such that

$$b_k(\lambda) := \max_{u\in Y_k, \|u\|_{E_c}=r_k} I_\lambda(u) < 0 \quad \text{for all} \quad \lambda\in[1,2].$$

For any $u\in Y_k$ (a finite dimensional subspace of E_c) and $\lambda\in[1,2]$, we have

$$I_\lambda(u) = \frac{1}{2}\|u\|_{E_c}^2 - \lambda\int_{\mathbb{R}} e^{ct}W(t,u)dt \le$$

$$\leq \frac{1}{2}\|u\|_{E_c}^2 - \int\limits_{\mathbb{R}} e^{ct} W(t,u)dt \leq \frac{1}{2}\|u\|_{E_c}^2 - \frac{1}{\gamma}\int\limits_{\Omega_u} e^{ct} a(t)|u(t)|^{\gamma} dt \leq$$

$$\leq \frac{1}{2}\|u\|_{E_c}^2 - \frac{1}{\gamma}\varepsilon_1 \|u\|_{E_c}^{\gamma} \operatorname{meas}(\Omega_u) \leq \frac{1}{2}\|u\|_{E_c}^2 - \frac{1}{\gamma}\varepsilon_1^2 \|u\|_{E_c}^{\gamma},$$

where Ω_u is defined in (3.10). Choosing $0 < r_k < \min\left\{\rho_k, \left(\frac{\varepsilon_1^2}{\gamma}\right)^{\frac{1}{2-\gamma}}\right\}$. Direct computation shows that

$$b_k(\lambda) \leq -\frac{r_k^2}{2} < 0 \quad \forall k \in \mathbb{N}.$$

***Step* 4.** Evidently, the condition $(C1)$ in Theorem 2.1 holds. By Step 1, 2, 3, Conditions $(C2)$, $(C3)$ in Theorem 2.1 are also satisfied. Therefore, by Theorem 2.1, there exist $\lambda_n \to 1, u(\lambda_n) \in Y_n$ such that

$$I'_{\lambda_n}|_{Y_n}(u(\lambda_n)) = 0, I_{\lambda_n}(u(\lambda_n)) \to c_k \in [d_k(2), b_k(1)]$$

as $n \to \infty$. For the sake of notational simplicity, in what follows we always set $u_n = u_{\lambda_n}$ for all $n \in \mathbb{N}$. Now we show that $\{u_n\}$ is bounded in E_c. Indeed, we have

$$\frac{1}{2}\|u_n\|_{E_c}^2 \leq I_{\lambda_n}(u_n) + \lambda_n \int\limits_{\mathbb{R}} \left[\frac{1}{\gamma} b(t)|u_n(t)|^{\gamma} + \frac{1}{\delta} c(t)|u_n(t)|^{\delta}\right] dt \leq$$

$$\leq M + \frac{2}{\gamma}\|b\|_{\frac{2}{2-\gamma}} \|u_n\|_2^{\gamma} + \frac{2}{\delta}\|c\|_{\frac{2}{2-\delta}} \|u_n\|_2^{\delta} \leq$$

$$\leq M + \frac{2}{\gamma}\beta^{-\gamma}\|b\|_{\frac{2}{2-\gamma}} \|u_n\|_{E_c}^{\gamma} + \frac{2}{\delta}\beta^{-\delta}\|c\|_{\frac{2}{2-\delta}} \|u_n\|_{E_c}^{\delta} \qquad \forall n \in \mathbb{N}$$

for some $M > 0$. Since $1 < \gamma < 2, 1 < \delta < 2$, it yields $\{u_n\}$ is bounded in E_c. Finally, we show that $\{u_n\}$ possesses a strong convergent subsequence in E_c. In fact, in view of the boundness of $\{u_n\}$, without loss of generality, we may assume

$$u_n \rightharpoonup u_0 \tag{3.13}$$

as $n \to \infty$ for some $u_0 \in E_c$. By virtue of the Riesz representation theorem, $I'_{\lambda_n}|_{Y_n}\colon Y_n \to Y_n^*$ and $I'\colon E_c \to E_c^*$ can be viewed as $I'_{\lambda_n}|_{Y_n}\colon Y_n \to Y_n$ and $I'\colon E_c \to E_c$ respectively, where Y_n^* is the dual space of Y_n. Note that

$$0 = I'_{\lambda_n}|_{Y_n}(u_n) = u_n - \lambda_n P_n B'(u_n) \qquad \forall n \in \mathbb{N}, \tag{3.14}$$

where P_n is the orthogonal projection for all $n \in \mathbb{N}$. That is,

$$u_n = \lambda_n P_n B'(u_n) \qquad \forall n \in \mathbb{N}. \tag{3.15}$$

Due to the compactness of B' and (3.13), the right-hand side of (3.15) converges strongly in E_c and hence $u_n \to u_0$ in E_c. Now, we know that $I = I_1$ has infinitely many nontrivial critical points. Therefore, (FHS) possesses infinitely many nontrivial fast homoclinic solutions.

1. *Rabinowitz P. H.* Homoclinic orbits for a class of Hamiltonian systems // Proc. Roy. Soc. Edinburgh A. – 1990. – **114**, № 1-2. – P. 33 – 38.
2. *Coti Zelati V., Rabinowitz P. H.* Homoclinic orbits for second order Hamiltonian systems possessing superquadratic potenials // J. Amer. Math. Soc. – 1991. – **4**. – P. 693 – 727.
3. *Ding Y., Girardi M.* Periodic and homoclinic solutions to a class of Hamiltonian systems with the potentials changing sign // Dynam. Systems Appl. – 1993. – **2**. – P. 131 – 145.
4. *Fei G.* The existence of homoclinic orbits for Hamiltonian systems with the potential changing sign // Chinese Ann. Math. Ser. B. – 1996. – **17**. – P. 403 – 410.
5. *Izydorek M., Janczewska J.* Homoclinic solutions for a class of second order Hamiltonian systems // J. Different. Equat. – 2005. – **219**, № 2. – P. 375 – 389.
6. *Ding Y.* Existence and multiplicity results for homoclinic solutions to a class of Hamiltonian systems // Nonlinear Anal. – 1995. – **25**. – P. 1095 – 1113.
7. *Korman P., Lazer A. C.* Homoclinic orbits for a class of symmetric Hamiltonian systems // Electron. J. Different. Equat. – 1994. – **1994**. – P. 1 – 10.
8. *Ou Z., Tang C.* Existence of homoclinic solutions for the second order Hamiltonian systems // J. Math. Anal. and Appl. – 2004. – **291**. – P. 203 – 213.
9. *Zou W.* Infinitely many homoclinic orbits for the second order Hamiltonian systems // Appl. Math. Lett. – 2003. – **16**. – P. 1283 – 1287.
10. *Omana W., Willem M.* Homoclinic orbits for a class of Hamiltonian systems // Different. Integral Equat. – 1992. – **5**, № 5. – P. 1115 – 1120.
11. *Yuan R., Zhang Z.* Infinitely many homoclinic orbits for the second order Hamiltonian systems with super-quadratic potentials // Nonlinear Anal.: Real World Appl. – 2009. – **10**. – P. 1417 – 1423.
12. *Yang L., Chen H., Sun J.* Infinitely many homoclinic solutions for some second order Hamiltonian systems // Nonlinear Anal.: Real World Appl. – 2011. – **74**. – P. 6459 – 6468.
13. *Zhang Z., Yuan R.* Homoclinic solutions for a class of non-autonomous subquadratic second order Hamiltonian systems // Nonlinear Anal. – 2009. – **71**. – P. 4125 – 4130.
14. *Zhang Z., Yuan R.* Homoclinic solutions of some second order non-autonomous systems // Nonlinear Anal. – 2009. – **71**. – P. 5790 – 5798.
15. *Zhang Z., Yuan R.* Fast homoclinic solutions for some second order non-autonomous systems // J. Math. Anal. and Appl. – 2011. – **376**. – P. 51 – 63.
16. *Zou W.* Variant fountain theorems and their applications // Manuscr. math. – 2001. – **104**. – P. 343 – 358.

Received 04.04.12,
after revision — 18.10.13