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NONEXISTENCE OF NONZERO DERIVATIONS ON SOME CLASSES OF ZERO-SYMMETRIC 3-PRIME NEAR-RINGS

НЕІСНУВАННЯ НЕНУЛЬОВИХ ПОХІДНИХ НА ДЕЯКИХ КЛАСАХ 3-ПРОСТИХ МАЙЖЕ-КІЛЕЦЬ З НУЛЬОВОЮ СИМЕТРІЄЮ

We give some classes of zero-symmetric 3-prime near-rings such that every member in these classes has no nonzero derivation. Moreover, we extend the concept of "3-prime" to subsets of near-rings and use it to generalize Theorem 1.1 due to Fong, Ke, and Wang concerning the transformation near-rings $M_o(G)$ by using a different technique and a more simple proof.

Наведено деякі класи 3-простих майже-кілець з нульовою симетрією таких, що будь-який елемент цих класів не має ненульової похідної. Крім того, поняття „3-простих“ узагальнено на підмножини майже-кілець і застосовано, щоб узагальнити теорему 1.1 Фонга, Ке і Ванга про трансформацію майже-кілець $M_o(G)$ за допомогою іншої техніки та більш простого доведення.

1. Introduction. Throughout this paper all near-rings are left near-rings. A derivation d on a near-ring R is an additive mapping satisfying $d(xy) = xd(y) + d(x)y$ for all $x, y \in R$. If R is a subnear-ring of a near-ring N and $d\colon R \to N$ is a map satisfies $d(a+b) = d(a)+d(b)$ and $d(ab) = ad(b)+d(a)b$ for all $a, b \in S$, where S is a nonempty subset of R, then we say that d acts as a derivation on S [1]. An element $x \in R$ is called a left (right) zero divisor in R if there exists a nonzero element $y \in R$ such that $xy = 0$ ($yx = 0$). A zero divisor is either a left or a right zero divisor. By an integral near-ring we mean a near-ring without nonzero zero divisors. A near-ring R is called a constant near-ring, if $xy = y$ for all $x, y \in R$ and is called a zero-symmetric near-ring, if $0x = 0$ for all $x \in R$. A trivial zero-symmetric near-ring R is a zero-symmetric near-ring such that $xy = y$ for all $x \in R - \{0\}, y \in R$ [6]. For any group $(G, +)$, $M(G)$ denotes the near-ring of all maps from G to G with the two operations of addition and composition of maps. $M_o(G) = \{f \in M(G)\colon 0f = 0\}$ is the zero-symmetric subnear-ring of $M(G)$ consists of all zero preserving maps from G to itself. We refer the reader to the books of Meldrum [6] and Pilz [7] for basic results of near-ring theory and their applications. We say that a near-ring R is 3-prime if, for all $x, y \in R$ ($xRy = \{0\}$ implies $x = 0$ or $y = 0$). Notice that every trivial zero-symmetric near-ring is 3-prime.

In Section 2 we extend the concept of "3-prime" for subsets of a near-ring and use it to show the nonexistence of nonzero derivation on special kinds of zero-symmetric 3-prime subnear-rings of $M_o(G)$. This result generalizes Theorem 1.1 due to Fong, Ke and Wang in [3].

It is easy to show that each member of the following classes has no nonzero derivations:

1. The class of all trivial zero-symmetric near-rings.

2. The class $\{R\colon R$ is a zero-symmetric 3-prime near-ring such that $(R, +)$ is a cyclic group$\}$.

3. The class $\{R\colon R$ is a direct sum of R_i and $i \in \Lambda$ such that R_i is a zero-symmetric 3-prime near-ring and $(R_i, +)$ is a cyclic group for all $i \in \Lambda\}$.

Let $R = I \times I \times \ldots \times I = I^n$, where I is a prime ring and n is an integer greater than two. Define the addition on R by

$$(a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n) = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$$

and define the multiplication on R by

$$(a_1, a_2, \ldots, a_n)(b_1, b_2, \ldots, b_n) = (a_1 b_n + b_1, \ldots, a_{n-1} b_n + b_{n-1}, a_n b_n)$$

if $(a_1, a_2, \ldots, a_n) \neq (0, 0, \ldots, 0) = \underline{0}$ and $\underline{0}(b_1, b_2, \ldots, b_n) = \underline{0}$. By the same way as in Example 2.14 of [5], this gives us a large class of zero-symmetric 3-prime near-rings which are not rings and the zero map is the only derivation on any near-ring of the class.

2. Subsets satisfy the 3-prime condition. In this section we extend the concept of "3-prime" for subsets. This extension will be useful in Theorem 2.1 to prove that each member of a certain class of subnear-rings of $M_o(G)$ has no nonzero derivations.

Definition 2.1. *Let U be a nonempty subset of a near-ring R. We say that U satisfies the 3-prime condition if, for all $x, y \in R$ ($xUy = \{0\}$ implies $x = 0$ or $y = 0$). We say that the element $r \in R$ satisfies the 3-prime condition if $\{r\}$ satisfies the 3-prime condition.*

In the next two examples we give some near-rings contain subsets satisfy the 3-prime condition.

***Example* 2.1.** (i) Any 3-prime near-ring satisfies the 3-prime condition.

(ii) Any nonzero subset of R, where R is an integral near-ring, satisfies the 3-prime condition.

(iii) In any constant near-ring R, every element (even 0) satisfies the 3-prime condition, since $xzy = y$ for all $x, y, z \in R$.

***Example* 2.2.** Let G be any group. Then $M(G)$ and $M_o(G)$ are near-rings have subsets satisfy the 3-prime condition. To show that take R to be one of $M(G)$ and $M_o(G)$. For all $g \in G$ define $\beta_g \colon G \to G$ by $0\beta_g = 0$ and $t\beta_g = g$ for all $t \in G - \{0\}$. Let B be the set $\{\beta_g | g \in G\}$. Now, suppose that $fBh = \{0\}$ for some $f, h \in R$. If $f \neq 0$, then there exists $t \in G$ such that $tf \neq 0$. Therefore, $tf\beta_g = g$ and hence $0 = tf\beta_g h = gh$ for all $g \in G$. Thus, $h = 0$. So B satisfies the 3-prime condition. A similar proof can be done for $B_1 = \{\beta_g | g \in G - \{0\}\}$ as a subset of $M_o(G)$ and for the subset of all constant maps $A = \{\alpha_g | g \in G\}$ as a subset of $M(G)$, where $t\alpha_g = g$ for all $t \in G$.

Lemma 2.1. (i) *Let R be a near-ring with a subset U satisfies the 3-prime condition. Then R is 3-prime. In particular, if R has an element which satisfies the 3-prime condition, then R is 3-prime.*

(ii) *Every subnear-ring of $M_o(G)$ contains the subset B_1 is 3-prime and every subnear-ring of $M(G)$ contains either B or A is 3-prime. In particular, $M_o(G)$ and $M(G)$ are 3-prime near-rings.*

Proof. (i) If $xRy = \{0\}$ for some $x, y \in R$, then $xUy = \{0\}$. Thus, either $x = 0$ or $y = 0$.

(ii) The proof is direct from Example 2.2 and (i).

If R has an element which satisfies the 3-prime condition, then R is 3-prime by Lemma 2.1(i), but the converse need not be true as the following example shows.

***Example* 2.3.** Let $R = M_n(F)$ for a field F. Then it is well-known that R is a prime ring and for every singular matrix A of R there exists a singular nonzero matrix B such that $AB = 0$. Therefore, the elements of R do not satisfy the 3-prime condition.

The following lemma extends known results about derivations on near-rings to subsets of near-rings satisfy the 3-prime condition.

Lemma 2.2. *Let R be a subnear-ring of a near-ring N with a nonzero subsemigroup U of $(R, \cdot)$ and d an additive map from R to N which acts as a derivation on U. Then*

(i) *For all $u, v, w \in U$, we have $(ud(v) + d(u)v)w = ud(v)w + d(u)vw$.*

(ii) *If U satisfies the 3-prime condition on N and $d(U)w = \{0\}$ for some $w \in U$, then either $d(U) = \{0\}$ or $w = 0$. Moreover, if R is zero-symmetric and $xd(U) = \{0\}$ for some $x \in R$, then either $d(U) = \{0\}$ or $x = 0$.*

(iii) *Suppose d is a derivation on R and U satisfies the 3-prime condition on N. If $d(U)x = \{0\}$ for some $x \in R$, then either $d(U) = \{0\}$ or $x = 0$.*

Proof. (i) By the same way of the proof of Lemma 1 in [2].

(ii) Suppose $d(U)w = \{0\}$. Using (i), we have $0 = d(uv)w = ud(v)w + d(u)vw = d(u)vw$ for all $u, v \in U$. Since U satisfying the 3-prime condition, we get $d(U) = \{0\}$ or $w = 0$. The proof of the second case is similar using that $0r = 0$ for all $r \in R$.

(iii) The proof is similar to the proof of (ii) using that $U = R$ in (i).

***Remark* 2.1.** Let G be any group. For all $g \in G$, take $\beta_g \colon G \to G$ as defined in Example 2.2. For all $g, h \in G$, observe that $\beta_g + \beta_h = \beta_{g+h}$ and for all $0 \neq g \in G, h \in G$, we have $\beta_g\beta_h = \beta_h$, $\beta_0\beta_g = 0$ in $M_o(G)$ and $\beta_h f = \beta_{hf}$ for all $f \in M_o(G)$. Let B_1 as defined in Example 2.2 with $G \neq \{0\}$. It is easy to see that $B_1 \cup \{0\}$ is even a subnear-ring of the near-ring $M_o(G)$ which is isomorphic to the trivial zero-symmetric near-ring on G.

In Theorem 1.1 of [3], Fong, Ke and Wang had proved that any subnear-ring of $M_o(G)$ containing all the transformations (maps) with finite range has no nonzero derivations using the maps $\delta_{x,y} \colon G \to G$ defined by $(z)\delta_{x,y} = x$ if $z = y$ and 0 otherwise for all $x \in G$ and $y \in G^*$, where $G^* = G - \{0\}$. The following theorem generalizes Theorem 1.1 of [3] with another technique and simple proof different from the proof of it.

Theorem 2.1. *Let G be any group and R a subnear-ring of $M_o(G)$ containing B_1. Suppose S is a subset of R containing B_1. If d is a map from R to $M_o(G)$ which acts as a derivation on S and $d(0) = 0$, then $d(S) = \{0\}$.*

Proof. If $G = \{0\}$, then $d = 0$ and B_1 is the empty set. So suppose that $G \neq \{0\}$. Assume that for some $0 \neq g \in G$, $d(\beta_g) = f$. If $gf = h \in G - \{0\}$, then

$$f = d(\beta_g) = d(\beta_g\beta_g) = \beta_g d(\beta_g) + d(\beta_g)\beta_g =$$

$$= \beta_g f + f\beta_g = \beta_{gf} + f\beta_g = \beta_h + f\beta_g$$

and hence $f = \beta_h + f\beta_g$. Thus, $h = gf = g(\beta_h + f\beta_g) = g\beta_h + gf\beta_g = h + g$ which implies $g = 0$, a contradiction. Using that $0d(\beta_0) = 0d(0) = 0$, we have

$$gd(\beta_g) = 0 \text{ for all } g \in G. \tag{2.1}$$

Clearly from (2.1) that $\beta_g d(\beta_g) = 0$ for all $g \in G$. Thus, $d(\beta_g) = d(\beta_g\beta_g) = \beta_g d(\beta_g) + d(\beta_g)\beta_g = d(\beta_g)\beta_g$ for all $g \in G$. It follows that $Gd(\beta_g) = Gd(\beta_g)\beta_g$ and hence

$$Gd(\beta_g) \subseteq \{0, g\} \text{ for all } g \in G. \tag{2.2}$$

If $d(\beta_g) = 0$ for some $g \in G - \{0\}$, then we claim first that $d(B_1) = \{0\}$ in $M_o(G)$. Indeed, for all $h \in G - \{0\}$, we get

$$0 = d(\beta_g) = d(\beta_h\beta_g) = \beta_h d(\beta_g) + d(\beta_h)\beta_g = d(\beta_h)\beta_g.$$

Thus, $d(B_1)\beta_g = \{0\}$. But B_1 is a subsemigroup of $M_o(G)$ satisfying the 3-prime condition and β_g is a nonzero element. Therefore, $d(B_1) = \{0\}$ by using Lemma 2.2(ii). After that, we claim that

$d(S) = \{0\}$. Indeed, for all $s \in S, g \in G - \{0\}$, we obtain $d(\beta_{gs}) = 0$ (even for $gs = 0$). It follows that

$$0 = d(\beta_{gs}) = d(\beta_g s) = \beta_g d(s) + d(\beta_g)s = \beta_g d(s) = \beta_h \beta_g d(s)$$

for some $h \in G - \{0\}$. Since B_1 satisfies the 3-prime condition, we have $d(s) = 0$ for all $s \in S$ and $d(S) = \{0\}$.

To complete the proof we will show that $d(\beta_g) \neq 0$ for all $g \in G - \{0\}$ is impossible. If $G = \{0, g\}$, then $d(\beta_g) = 0$ since $gd(\beta_g) = 0$ by (2.1).

Now, suppose G contains more than two elements and $d(\beta_g) \neq 0$ for all $g \in G - \{0\}$. Thus, from (2.1) and (2.2), we obtain

$$\text{for all } g \in G - \{0\} \text{ there exists } h \in G - \{0, g\} \text{ such that } hd(\beta_g) = g. \tag{2.3}$$

Observe that

$$d(\beta_g) = d(\beta_h \beta_g) = \beta_h d(\beta_g) + d(\beta_h)\beta_g. \tag{2.4}$$

Using $gd(\beta_g) = 0$ and (2.4), we have for all $g \in G - \{0\}$

$$0 = g(\beta_h d(\beta_g) + d(\beta_h)\beta_g) = hd(\beta_g) + gd(\beta_h)\beta_g = g + gd(\beta_h)\beta_g. \tag{2.5}$$

Using (2.2), we have $Gd(\beta_h) \subseteq \{0, h\}$. If $gd(\beta_h) = 0$, then $g = 0$ from (2.5), a contradiction. It follows that $gd(\beta_h) = h$. Hence, (2.5) gives us that $g + g = 0$ for all $g \in G$ and so G is a 2-torsion group. From (2.4), we have

$$(g + h)d(\beta_g) = (g + h)\beta_h d(\beta_g) + (g + h)d(\beta_h)\beta_g. \tag{2.6}$$

If $g+h = 0$, then $g = -h = h$ which is a contradiction with (2.3). Thus, we have $(g+h)\beta_h = h$. From (2.3), equation (2.6) will be

$$(g + h)d(\beta_g) = hd(\beta_g) + (g + h)d(\beta_h)\beta_g = g + (g + h)d(\beta_h)\beta_g. \tag{2.7}$$

Using (2.2) and (2.7), if $(g + h)d(\beta_g) = 0$, then $g + (g + h)d(\beta_h)\beta_g = 0$ which means $(g + h)d(\beta_h)\beta_g = -g = g$. Thus, $(g + h)d(\beta_h) = h$. In the other case, if $(g + h)d(\beta_g) = g$, then $(g + h)d(\beta_h)\beta_g = 0$ and hence $(g + h)d(\beta_h) = 0$. Therefore, (2.7) implies that $(g + h)d(\beta_g) + (g + h)d(\beta_h)$ equal either g or h. On the other hand, from (2.1), we have

$$0 = (g + h)d(\beta_{g+h}) = (g + h)d(\beta_g + \beta_h) = (g + h)[d(\beta_g) + d(\beta_h)] =$$

$$= (g + h)d(\beta_g) + (g + h)d(\beta_h).$$

Thus, $g = 0$ or $h = 0$ which is a contradiction with $g \neq 0$ and $h \neq 0$. Therefore, $d(\beta_g) \neq 0$ for all $g \in G - \{0\}$ is impossible.

Theorem 2.1 is proved.

Observe that B_1 is a proper subset of the set of all transformations with finite range of $M_o(G)$. In particular, if G is finite, then $\sum_{x \in G^*} \delta_{g,x} = \beta_g$. Therefore, Theorem 2.1 generalizes Theorem 1.1 of [3] (in the sense that the class of zero-symmetric 3-prime subnear-rings of $M_o(G)$ in Theorem 2.1 is larger than the class of subnear-rings of $M_o(G)$ in Theorem 1.1 of [3]).

***Corollary* 2.1.** *Let G be any group. Any subnear-ring of $M_o(G)$ containing B_1 has no nonzero derivation. In particular, $M_o(G)$ has no nonzero derivation.*

The following example shows that the condition "the subnear-ring of $M_o(G)$ containing the subset B_1" in Theorem 2.1 and Corollary 2.1 is not redundant.

***Example* 2.4.** Take the near-ring $R = \{f \in M_o(\mathbb{Z}_4)\colon \{2,3\}f = \{0\}\} = \mathrm{Ann}_{M_o(\mathbb{Z}_4)}(\{2,3\})$ as a special case of Example 2.7 in [5]. Then R is a subnear-ring of $M_o(\mathbb{Z}_4)$ which is not a ring. Define $D\colon R \to M_o(\mathbb{Z}_4)$ by $D(f_y) = f_{2y}$ for all $y \in \mathbb{Z}_4$. By the same way as in Example 2.7 of [5], we obtain that D acts as a nonzero derivation on R. Notice that $B_1 \nsubseteq R$.

***Remark* 2.2.** Since for any group G, we have any subnear-ring R of $M_o(G)$ containing the subset B_1 is a 3-prime near-ring by Lemma 2.1(ii) and has no nonzero derivation by Corollary 2.1. Therefore, we have a very large class of zero-symmetric 3-prime near-rings which are not rings such that every near-ring of the class has no nonzero derivation.

1. *Bell H. E., Argac N.* Derivations, products of derivations and commutativity in near-rings // Algebra Colloq. – 2001. – **8**. – P. 399 – 407.
2. *Bell H. E., Mason G.* On derivations in near-rings // Proc. Near-rings and Near-fields / Eds G. Betsch et al. North-Holland Math. Stud. – 1987. – P. 31 – 35.
3. *Fong Y., Ke W.-F., Wang C.-S.* Nonexistence of derivations on transformation near-rings // Communs Algebra. – 2000. – **28**. – P. 1423 – 1428.
4. *Hungerford T. W.* Algebra. – New York: Holt, Rinehart and Winston, Inc., 1974.
5. *Kamal Ahmed A. M., Al-Shaalan Khalid H.* Existence of derivations on near-rings // Math. Slovaca. – 2013. – **63**, № 3. – P. 431 – 448.
6. *Meldrum J. D. P.* Near-rings and their links with groups. – Boston, MA: Pitman, 1985.
7. *Pilz G.* Near-rings. – Amsterdam: North-Holland, 1983.

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