UDC 512.5

**Fucai Lin** (Minnan Normal Univ., China),
**Chuan Liu** (Ohio Univ. Zanesville Campus, USA),
**Li-Hong Xie** (Wuyi Univ., China)

# REMAINDERS OF SEMITOPOLOGICAL GROUPS OR PARATOPOLOGICAL GROUPS*

# ЗАЛИШКОВІ ЧЛЕНИ НАПІВТОПОЛОГІЧНИХ ГРУП АБО ПАРАТОПОЛОГІЧНИХ ГРУП

We mainly discuss the remainders of Hausdorff compactifications of paratopological groups or semitopological groups. Thus, we show that if a nonlocally compact semitopological group $G$ has a compactification $bG$ such that the remainder $Y = bG \setminus G$ possesses a locally countable network, then $G$ has a countable $\pi$-character and is also first-countable, that if $G$ is a nonlocally compact semitopological group with locally metrizable remainder, then $G$ and $bG$ are separable and metrizable, that if a nonlocally compact paratopological group has a remainder with sharp base, then $G$ and $bG$ are separable and metrizable, and that if a nonlocally compact $\mathbb{R}_1$-factorizable paratopological group has a remainder which is a $k$-semistratifiable space, then $G$ and $bG$ are separable and metrizable. These results improve some results obtained by C. Liu (Topology and Appl. – 2012. – **159**. – P. 1415 – 1420) and A. V. Arhangel'skiĭ, M. M. Choban (Topology Proc. – 2011. – **37**. – P. 33 – 60). Moreover, some open questions are posed.

У даній статті, в основному, розглядаються залишкові члени хаусдорфових компактифікацій паратопологічних груп або напівтопологічних груп. Так, показано, що у випадку, коли нелокально компактна напівтопологічна група $G$ має компактифікацію $bG$ таку, що залишковий член $Y = bG \setminus G$ має локально зліченну мережу, група $G$ має зліченний $\pi$-характер, а також є першозліченною. Також доведено, що для нелокально компактної напівтопологічної групи з локально метризовним залишковим членом групи $G$ і $bG$ є сепарабельними і метризовними. Крім того, якщо нелокально компактна паратопологічна група має залишковий член з точною базою, то групи $G$ і $bG$ є сепарабельними і метризовними, а якщо нелокально компактна $\mathbb{R}_1$-факторизовна паратопологічна група має залишковий член, який є простором, що допускає $k$-напівспрямлення, то групи $G$ і $bG$ є також сепарабельними і метризовними. Наведені результати покращують деякі результати, отримані C. Liu (Topology and Appl. – 2012. – **159**. – P. 1415 – 1420) і A. V. Arhangel'skiĭ, M. M. Choban (Topology Proc. – 2011. – **37**. – P. 33 – 60). Крім того, сформульовано деякі відкриті питання.

**1. Introduction.** Throughout this paper, all spaces are assumed to be Tychonoff. Denote the set of positive natural numbers by $\mathbb{N}$. We refer the reader to [4, 12] for notations and terminology not explicitly given here.

A *semitopological group* $G$ is a group $G$ with a topology such that the product map of $G \times G$ into $G$ is separately continuous. If $G$ is a semitopological group and the inverse map of $G$ onto itself associating $x^{-1}$ with arbitrary $x \in G$ is continuous, then $G$ is called a *quasitopological group*. A *paratopological group* $G$ is a group $G$ with a topology such that the product maps of $G \times G$ into $G$ is jointly continuous. If $G$ is a paratopological group and the inverse map of $G$ onto itself associating $x^{-1}$ with arbitrary $x \in G$ is continuous, then $G$ is called a *topological group*. However, there exists a paratopological group which is not a topological group; Sorgenfrey line [12] (Example 1.2.2) is such an example. Paratopological groups were discussed and many results have been obtained [4, 5, 7, 17 – 20].

Recall that a space $X$ is of *countable type* if every compact subspace $F$ of $X$ is contained in a compact subspace $K \subset X$ with a countable base of open neighborhoods in $X$.

* Supported by the NSFC (No. 11201414), the Natural Science Foundation of Fujian Province (No. 2012J05013) of China and Training Programme Foundation for Excellent Youth Researching Talents of Fujian's Universities (JA13190).

By a remainder of a space $X$ we understand the subspace $bX \setminus X$ of a Hausdorff compactification $bX$ of $X$. Remainders in compactifications of topological spaces have been studied by some topologists in the last few years. A famous classical result in this study is the following theorem of M. Henriksen and J. Isbell's [15]:

A space $X$ is of countable type if and only if the remainder in any (in some) compactification of $X$ is Lindelöf.

**2. Paratopological groups with locally metrizable remainders.** In this section we shall prove that if a nonlocally compact semitopological group with a remainder which is locally metrizable or has locally a countable network then $G$ and $bG$ are separable and metrizable.

First, we give some technical lemmas.

**Lemma 2.1** [7]. *Suppose that $X$ is a regular space with a countable network*[1] $\mathcal{S}$. *Then $X = Y \cup Z$, where $Y$ is a separable metrizable, and $Z$ has a countable network $\mathcal{P}$ such that every element of $\mathcal{P}$ is nowhere dense in $X$.*

The following lemma maybe was proved somewhere.

**Lemma 2.2.** *Let $F$ be a compact subset of a space $X$ and have a countable base $\{U_n\}$ with $\overline{U_{n+1}} \subset U_n$ in $X$, and let $H = \cap_n V_n$ ($V_{n+1} \subset V_n$ and each $V_n$ is open in $F$) is a compact $G_\delta$-set of $F$. For $n \in \mathbb{N}$, let $W_n$ be an open set in $X$ such that $V_n = W_n \cap F$, $W_n \subset U_n$, $\overline{W_{n+1}} \subset W_n$, then $\{W_n\}$ is a countable base at $H$ in $X$.*

***Proof.*** $H = \cap_n W_n = \cap_n \overline{W_n}$. Suppose that $\{W_n\}$ is not a countable base at $H$, then there is an open subset $U$ of $X$ such that $H \subset U$ and $W_n \setminus U \neq \varnothing$ for every $n$. By induction, choose $x_n \in W_n \setminus U$ with $x_i \neq x_j$ if $i \neq j$. Since $x_n \in U_n$ for each $n \in \mathbb{N}$, then $\{x_n\}$ has a cluster point $x \in F$. Therefore, we have $x \in \overline{W_n}$ for each $n$, then $x \in H \subset U$, and hence $U$ contains infinitely many $x_n's$, which is a contradiction.

**Lemma 2.3** [22]. *Let $X$ be a Lindelöf space with locally a $G_\delta$-diagonal*[2]. *Then $X$ has a $G_\delta$-diagonal.*

Recall that a family $\mathcal{U}$ of nonempty open sets of a space $X$ is called a *$\pi$-base* if for each nonempty open set $V$ of $X$, there exists an $U \in \mathcal{U}$ such that $U \subset V$. The *$\pi$-character* of $x$ in $X$ is defined by $\pi\chi(x, X) = \min\{|\mathcal{U}| : \mathcal{U}$ is a local $\pi$-base at $x$ in $X\}$. The *$\pi$-character of $X$* is defined by $\pi\chi(X) = \sup\{\pi\chi(x, X) : x \in X\}$.

**Lemma 2.4** [2]. *If $X$ is a Lindelöf $p$-space, then any remainder of $X$ is a Lindelöf $p$-space.*

**Theorem 2.1.** *If a nonlocally compact semitopological group $G$ has a Hausdorff compactification $bG$ such that the remainder $Y = bG \setminus G$ has locally a countable network, then $G$ has a countable $\pi$-character and is also first-countable.*

***Proof.*** Since $Y$ has locally a countable network, there exists an open subset $U$ in $Y$ such that $\overline{U}^Y$ has a countable network. Let $V$ be an open subset of $bG$ such that $V \cap Y = U$. Since $G$ is not locally compact semitopological group, the remainder $Y$ is dense in $bG$. Therefore, $\overline{V}^{bG} = \overline{U}^{bG}$. By Lemma 2.1, we have $U = X_1 \cup X_2$, where $X_1$ is a separable metrizable subspace, and $X_2$ has a countable network $\mathcal{P}$ such that each element of $\mathcal{P}$ is nowhere dense in $U$.

*Case* 1: $X_1$ is dense in $\overline{U}^Y$.

[1] Let $\mathcal{P}$ be a family of subsets of a space $X$. The family is called a *network for* $X$ if, for each $x \in U$ with $U$ open in $X$, there exists a $P \in \mathcal{P}$ such that $x \in P \subset U$.

[2] A space $X$ has a *$G_\delta$-diagonal* if there exists a sequence $\{\mathcal{G}_n\}_n$ of open covers of $X$ such that, for each point $x \in X$, we have $\bigcap_{n \in \mathbb{N}} \mathrm{st}(x, \mathcal{G}_n) = \{x\}$.

Since $U$ is dense $\overline{U}^{bG}$, $X_1$ is dense in $\overline{U}^{bG}$. Then $\overline{U}^{bG}$ has a countable $\pi$-base since $X_1$ has a countable $\pi$-base. Therefore, $V \cap G$ has a countable $\pi$-base, and thus $G$ has a countable $\pi$-character.

*Case* 2: $X_1$ is not dense in $\overline{U}^{Y}$.

Put $W = \overline{U}^{bG} \setminus \overline{X_1}^{bG}$. Then $W$ is a nonempty open subspace of $\overline{U}^{bG}$. For an arbitrary $P \in \mathcal{P}$, let $F_P = \overline{P}^{bG}$. Since each $P$ is nowhere dense in $U$, each $F_P$ is nowhere dense in $\overline{U}^{bG}$, and therefore, each $W_P = W \setminus F_P$ is a dense open subspace of $W$. Obvious, $\overline{U}^{bG}$ is compact, and thus it follows that the subspace $H = \bigcap\{W_P : P \in \mathcal{P}\}$ of $W$ is a Čech-complete dense subspace in $W$. Moreover, it is easy to see that $(V \cap G) \setminus \overline{X_1}^{bG} \neq \varnothing$. It follows by a standard argument that $G$ has a dense Čech-complete subspace, or see the proof of [6] (Theorem 1.2). Then $G$ is a Čech-complete topological group [7] (Corollary 5.4). Since $Y$ has locally a countable network, $G$ is separable and metrizable [22]. Then $G$ is a Lindelöf $p$-space, and thus, by Lemma 2.4 $Y$ is a Lindelöf $p$-space. Since $Y$ is a Lindelöf space with locally a $G_\delta$-diagonal, $Y$ has a $G_\delta$-diagonal by Lemma 2.3, and hence $Y$ is separable and metrizable.

Since $\overline{U}^{Y}$ is Lindelöf, $\overline{V}^{bG} \setminus \overline{U}^{Y}$ is of countable type, and it follows from the homogeneity of $G$ and Lemma 2.2 that $G$ is of countable type. Moreover, since every Tychonoff semitopological group with a countable $\pi$-character has a $G_\delta$-diagonal [4] (Corollary 5.7.5), $G$ has a $G_\delta$-diagonal. Hence $G$ is first-countable.

Next we shall prove that if a nonlocally compact semitopological group $G$ has a Hausdorff compactification $bG$ such that the remainder $bG \setminus G$ is locally metrizable then $G$ and $bG$ are separable and metrizable.

**Lemma 2.5** [17]. *Let $G$ be a nonlocally compact semitopological group. If the remainder $Y = bG \setminus G$ is metrizable, then $G$ and $bG$ are separable and metrizable.*

**Lemma 2.6.** *Let $X$ be a space with a $\sigma$-locally countable base. Then $X$ is of countable type.*

***Proof.*** Let $K$ be an arbitrary compact subset of $X$. For each $x \in K$, there exists open neighborhoods $V_x$ and $W_x$ of $x$ in $X$ such that $\overline{W_x} \subset V_x$ and the subspace $V_x$ has a $\sigma$-locally countable base. Then the family of the open subsets $\{W_x : x \in K\}$ is an open covering for $K$, and it follows from the compactness of $K$ that there exist finite set $\{x_i : 1 \leq i \leq n_0\} \subset K$ such that $K \subset \bigcup\{W_{x_i} : 1 \leq i \leq n_0\}$. For each $1 \leq i \leq n_0$, let $K_i = F \cap \overline{W_x}$. Then each $K_i$ is compact and $K = \bigcup_{1 \leq i \leq n_0} K_i$. For each $1 \leq i \leq n_0$, the subspace $V_{x_i}$ has a $\sigma$-locally countable base $\mathcal{B}_i = \bigcup_{n \in \mathbb{N}} \mathcal{B}_{in}$, where each $\mathcal{B}_{in}$ is locally countable in $V_{x_i}$, and then, for each $n \in \mathbb{N}$ the family $\mathcal{D}_{in} = \{B \cap K_i \neq \varnothing : B \in \mathcal{B}_{in}\}$ is countable by the compactness of $K_i$. Let $\mathcal{B} = \bigcup_{1 \leq i \leq n_0, n \in \mathbb{N}} \mathcal{D}_{in}$. Obviously, $\mathcal{B}$ is countable and each element of $\mathcal{B}$ is also open in $X$ since each $V_{x_i}$ is open in $X$. Let

$$\mathcal{K} = \left\{\bigcup \mathcal{C} : K \subset \bigcup \mathcal{C} \text{ and } \mathcal{C} \text{ is a finite subfamily of } \mathcal{B}\right\}.$$

Then $\mathcal{K}$ is countable. Next we shall show that $\mathcal{K}$ is a countable base for $K$.

Fix arbitrary $K \subset U$ with $U$ open in $X$. For each $x \in K$, then there exists $1 \leq i \leq n_0$ such that $x \in V_{x_i}$, and thus there exists an open set $B_x$ such that $x \in B_x \subset U$ and $B_x \in \mathcal{D}_{in}$ for some $n$. Then $\{B_x : x \in K\}$ is an open covering for $K$, and thus there is a finite subfamily $\mathcal{K}_1 \subset \{B_x : x \in K\}$ such that $K \subset \bigcup \mathcal{K}_1$. Obviously, $\bigcup \mathcal{K}_1 \in \mathcal{K}$. Therefore, $\mathcal{K}$ is a countable base for $K$.

**Theorem 2.2.** *If a nonlocally compact semitopological group $G$ has a compactification $bG$ such that the remainder $Y = bG \setminus G$ is locally a $\Sigma$-space with a $\sigma$-locally countable base, then $G$ and $bG$ are separable and metrizable.*

***Proof.*** We firstly claim that $Y$ is nowhere locally countably compact. Indeed, suppose that there exists $a \in Y$ such that $a$ has a neighborhood $U(a)$ in $Y$ with $\overline{U(a)}^Y$ countably compact. Since $Y$ is locally $\Sigma$-space with a $\sigma$-locally countable base, we may assume that $\overline{U(a)}^Y$ is a $\Sigma$-subspace with a $\sigma$-locally countable base. $\overline{U(a)}^Y$ is compact metrizable [11]. Then $\overline{U(a)}^{bG} = \overline{U(a)}^Y \subset Y$. Let $U$ be an open subset of $bG$ such that $U(a) = U \cap Y$. We have $G$, $Y$ are dense in $bG$ since $G$ is not locally compact, and therefore, $U \cap G \neq \varnothing$ and $\overline{U}^{bG} = \overline{U \cap Y}^{bG} = \overline{U(a)}^{bG} = \overline{U(a)}^Y \subset Y$. This is a contradiction. Therefore, $Y$ is nowhere locally countably compact. Then it follows by a standard argument that $G$ has a countable $\pi$-character, and hence $G$ has a $G_\delta$-diagonal by [4] (Corollary 5.7.5). By [11] (Corollary 7.11), $Y$ is locally developable, hence $Y$ is local a $\sigma$-space.

*Claim:* There is a point $y \in Y$ such that $U_y \subset Y$ is separable for some open neighborhood $U_y$ at $y$.

Suppose that $Y$ is nowhere locally separable. Since $Y$ is locally a $\Sigma$-space with a $\sigma$-locally countable base, there exists an open subset $U$ of $Y$ such that $\overline{U}^Y$ is a $\Sigma$-space with $\sigma$-locally countable base. Let $\mathcal{P} = \cup_{n\in\mathbb{N}}\mathcal{P}_n$ be a $\sigma$-discrete network of $U$, and let $F_n$ be the set of all accumulation points of $\mathcal{P}_n$ in $\overline{U}^{bG}$ for each $n \in \mathbb{N}$. Then each $F_n \subset G$ is compact and $\cup_{n\in\mathbb{N}}F_n$ is dense in $\overline{U}^{bG}$. Since $G$ has a $G_\delta$-diagonal, $F_n$ is compact metrizable for each $n \in \mathbb{N}$. Then $G$ is locally separable and $c\big(\overline{U}^{bG} \cap G\big) \leq \omega$. Then it follows that $c(\overline{U}^{bG}) \leq \omega$, and hence $c(\overline{U}^Y) \leq \omega$. By [9] (Lemma 8.1(iii)), every locally countable open collection in $\overline{U}^Y$ is countable, and hence $U$ has a countable base. Thus $U$ is separable and metrizable, which is a contradiction.

Since $Y$ is locally a $\Sigma$-space with a $\sigma$-locally countable base, we may assume that the $U_y$ in claim is a $\Sigma$-subspace with a $\sigma$-locally countable base. Let $U$ be an open subset such that $U_y = U \cap Y$. Since $G$ is not locally compact, $Y$ is dense in $bG$. Then it is easy to see that $\overline{U}^{bG} = \overline{U_y}^{bG}$. Thus $\overline{U_y}^{bG} \cap Y$ is separable in $Y$, and hence it is separable and metrizable [11] (Theorem 7.2). Since $\overline{U_y}^{bG} \cap G$ is a remainder of $\overline{U_y}^{bG} \cap Y$, $\overline{U_y}^{bG} \cap G$ is a Lindelöf $p$-space by Lemma 2.4, and hence $\overline{U_y}^{bG} \cap G$ is separable and metrizable since $G$ has a $G_\delta$-diagonal [11] (Corollary 3.20). Then $G$ is locally separable and metrizable since $U \cap G \subset \overline{U_y}^{bG} \cap G$ and $G$ is homogeneous. Since $Y$ has locally a $\sigma$-locally countable base, then $Y$ is of countable type by Lemma 2.6.

Therefore, $G$ is Lindelöf, and thus $G$ is separable and metrizable. Then $Y$ is a Lindelöf $p$-space by Lemma 2.4, and hence $Y$ is locally separable metrizable since a Lindelöf developable space are separable and metrizable [11] (Theorem 1.2). Then $Y$ is separable and metrizable since $Y$ is a Lindelöf locally separable metrizable space. By Lemma 2.5, $G$ and $bG$ are separable and metrizable.

***Corollary* 2.1.** *Let $G$ be a nonlocally compact paratopological group. If the remainder $Y = bG \setminus G$ is locally metrizable, then $G$ and $bG$ are separable and metrizable.*

## 3. Paratopological groups with weakly developable remainders.

**Lemma 3.1** [5]. *Let $G$ be a paratopological group. Then the following conditions are equivalent*:

(1) *some remainder $Y = bG \setminus G$ is Ohio-complete*[3];

(2) *every remainder $Y = bG \setminus G$ is Ohio-complete*;

(3) *$G$ is $\sigma$-compact or $G$ is a space of countable type.*

[3]A space $X$ is *Ohio complete* [2] if in each compactification $bX$ of $X$ there is a $G_\delta$-subset $Z$ such that $X \subset Z$ and each point $y \in Z \setminus X$ is separated from $X$ by a $G_\delta$-subset of $Z$.

**Lemma 3.2** [5]**.** *Let $G$ be a paratopological group. If there exists a nonempty compact subset of $G$ of countable character in $G$, then $G$ is of countable type.*

**Theorem 3.1.** *Let $G$ be a nonlocally compact paratopological group. If the remainder $Y = bG \setminus G$ satisfies the following conditions, then $G$ and $bG$ are separable and metrizable.*

(1) *$Y$ is Ohio-complete.*

(2) *$Y$ is a locally $p$-space with a point countable base.*

***Proof.*** Since $Y$ is Ohio-complete, it follows from Lemma 3.1 that $G$ is $\sigma$-compact or $G$ is a space of countable type.

*Case* 1: $G$ is a space of countable type.

By Henriksen and Isbell's theorem, $Y$ is Lindelöf. Since $Y$ is a locally $p$-space with a point countable base, $Y$ is locally metrizable since a paracompact $p$-space with a point-countable base is metrizable [11] (Corollaries 3.20 and 7.11), and then $G$ and $bG$ are separable and metrizable by Corollary 2.1.

*Case* 2: $G$ is $\sigma$-compact.

Since $G$ is a $\sigma$-compact paratopological group, the Souslin number $c(G)$ of $G$ is countable [4] (Corollary 5.7.12). Therefore, $c(bG) \leq \omega$. $Y$ is dense in $bG$, since $G$ is nonlocally compact. It follows that $c(Y) \leq \omega$ as well. Since $Y$ is Čech-complete, there exists a dense subspace $Z \subset Y$ such that $Z$ is a paracompact and Čech-complete subspace of $Y$ by [24]. Since $Z$ is a locally paracompact Čech-complete subspace with a point-countable base, $Z$ is locally metrizable [11] (Corollaries 3.20 and 7.11). Since $c(Y) \leq \omega$ and $Z$ is dense for $Y$, $c(Z) \leq \omega$ as well. It follows that $Z$ is locally separable, and hence $Y$ is locally separable since $Z$ is dense in $Y$. Then $Y$ is locally separable space with a point-countable base, and hence $Y$ has locally a countable base, which implies that $Y$ is locally metrizable. Then $G$ and $bG$ are separable and metrizable by Corollary 2.1.

***Corollary* 3.1.** *Let $G$ be a nonlocally compact paratopological group. If the remainder $Y = bG \setminus G$ satisfies one of the following conditions, then $G$ and $bG$ are separable and metrizable.*

(1) *$Y$ is a $p$-space with a point-countable base.*

(2) *$Y$ has a sharp base*[4].

***Proof.*** (1) Since a $p$-space is Ohio-complete [2], it follows from Theorem 3.1 that if $Y$ is a $p$-space with a point-countable base then $G$ and $bG$ are separable and metrizable.

(2) Since $Y$ has a sharp base, it follows from [8] (Theorem 3.4) that $Y$ is a weakly developable[5] space. Therefore, $Y$ is a $p$-space by [8] (Theorem 2.4), and hence $Y$ is Ohio-complete [2]. Since $Y$ has a sharp base, $Y$ has a point-countable base [1]. Then $Y$ is a $p$-space with a point-countable base, and hence $G$ and $bG$ are separable and metrizable by (1).

***Corollary* 3.2.** *Let $G$ be a nonlocally compact paratopological group. If the remainder $Y = bG \setminus G$ has a uniform base* (*that is, a metacompact developable space*), *then $G$ and $bG$ are separable and metrizable.*

**Theorem 3.2.** *Let $G$ be a nonlocally compact paratopological group. If the remainder $Y = bG \setminus G$ is weakly developable and irresolvable, then $G$ and $bG$ are separable and metrizable.*

[4] A *sharp base* $\mathcal{B}$ of a space $X$ is a base of $X$ such that, for every sequence $\{B_n : n \in \mathbb{N}\}$ of distinct members of $\mathcal{B}$ and every $x \in \bigcap_{n\in\mathbb{N}} B_n$, the sequence $\{\bigcap_{i\leq n} B_i : n \in \mathbb{N}\}$ is a base at $x$.

[5] A space $X$ is called *weakly developable* if there exists a sequence $\{\mathcal{G}_n : n \in \mathbb{N}\}$ of open covers on $X$ such that for every sequence $\{B_n \in \mathcal{G}_n : n \in \mathbb{N}\}$ and every $x \in \bigcap_{n\in\mathbb{N}} B_n$, the sequence $\{\bigcap_{i\leq n} B_i : n \in \mathbb{N}\}$ is a base at $x$.

***Proof.*** By the proof of Theorem 3.1, it is suffice to consider the case of $G$ is $\sigma$-compact. Moreover, it follows from the proof of Theorem 3.1 that the Souslin number $c(Y)$ of $Y$ is countable and there exists a dense subspace $Z \subset Y$ such that $Z$ is separable and metrizable subspace of $Y$. Put $X_1 = bG \setminus Z$ and $X_2 = Y \setminus Z$.

Obvious, $\overline{Z}^{bG} = bG$, and therefore $X_1$ is the remainder of $Z$. Since $Z$ is separable and metrizable, $Z$ is a Lindelöf $p$-space, and hence $X_1$ is a Lindelöf $p$-space by Lemma 2.4. Since $Y$ is irresolvable, we have $\overline{X_2}^{bG} \neq bG$, and thus $\overline{X_2}^{bG} \cap G \neq G$. Therefore, $X_1 \setminus \overline{X_2}^{bG} \subset G$ is a nonempty open subset in $X_1$. Since $X_1$ is a $p$-space, $X_1$ is a space of point-countable type. Take a point $x_0 \in X_1 \setminus \overline{X_2}^{bG}$. Then there exists a compact subset $F \subset X_1$ such that $x_0 \in F$ and $F$ has a countable neighborhoods base at $F$. By Lemma 2.2, there exists a compact subset $L \subset X_1 \setminus \overline{X_2}^{bG}$ such that $x_0 \in L \subset F$ and $L$ has a countable neighborhoods base at $L$. It follows from Lemma 3.2 that $G$ is of countable type. By Henriksen and Isbell's theorem, $Y$ is Lindelöf. Since $Y$ is weakly developable, $Y$ is metrizable by [8] (Proposition 2.6), and then $G$ and $bG$ are separable and metrizable by Lemma 2.5.

**Theorem 3.3.** *Let $G$ be a nonlocally compact paratopological group which is a generalized ordered space* (*that is, GO-space*). *If the remainder $Y = bG \setminus G$ is locally weakly developable, then $G$ and $bG$ are separable and metrizable.*

***Proof.*** In view of proof Theorem 2.2, we have $Y$ is nowhere locally countably compact, and hence $Y$ is not countably compact, Then it follows by a standard argument that $G$ has a countable $\pi$-character, and hence $G$ has a $G_\delta$-diagonal by [4] (Corollary 5.7.5). Since a GO-space with a $G_\delta$-diagonal is first-countable [10] (Lemma 5.1 and Proposition 5.5). Therefore, $G$ is countable type by Lemma 3.2. By Henriksen and Isbell's theorem, $Y$ is Lindelöf. Since $Y$ is locally weakly developable, $Y$ is locally metrizable by [8] (Proposition 2.6), and then $G$ and $bG$ are separable and metrizable by Corollary 2.1.

***Corollary* 3.3.** *Let $G$ be a nonlocally compact paratopological group which is GO-space. If the remainder $Y = bG \setminus G$ is locally developable, then $G$ and $bG$ are separable and metrizable.*

However, the following question is still open.

***Question* 3.1.** Let $G$ be a nonlocally compact paratopological group. If the remainder $Y = bG \setminus G$ is developable, are $G$ and $bG$ separable and metrizable?

**Lemma 3.3** [5]**.** *Let $G$ be a $k$-gentle paratopological group, and $Y$ be a remainder of $G$. Then $Y$ is Lindelöf or pseudocompact.*

**Lemma 3.4** [5]**.** *Let $G$ be a $k$-gentle paratopological group such that some remainder of $G$ is Lindelöf. Then $G$ is a topological group.*

**Theorem 3.4.** *Suppose that $G$ is a nonlocally compact, $k$-gentle paratopological group, and $Y = bG \setminus G$ is a remainder of $G$. If $Y$ has a weakly uniform base*[6]*, then $G$, $bG$ and $Y$ are separable and metrizable spaces.*

***Proof.*** Since $Y$ has a weakly uniform base, $Y$ has a $G_\delta$-diagonal [16], and therefore, $Y$ is Ohio-complete. By Lemma 3.1, $G$ is a space of countable type or $G$ is $\sigma$-compact.

*Case* 1: $G$ is a space of countable type.

By Henriksen and Isbell's theorem, $Y$ is Lindelöf. By Lemma 3.4, $G$ is a topological group. Since $Y$ has a $G_\delta$-diagonal, $G, bG$ and $Y$ are separable and metrizable spaces [3].

*Case* 2: $G$ is $\sigma$-compact.

[6]A base $\mathcal{B}$ for a space $X$ is said to be *weakly uniform* if for each countably infinite family $\mathcal{U} \subset \mathcal{B}$ and for each $x \in X$, if $x \in U$ for each $U \in \mathcal{U}$, then $\bigcap \mathcal{U} = \{x\}$.

By the proof of Theorem 3.1, we have $c(Y) \leq \omega$. It follows from Lemma 3.3 that $Y$ is Lindelöf or pseudocompact. By the case 1, it is suffice to consider the case of pseudocompactness of $Y$. Let $Y$ be pseudocompact. Since a pseudocompact ccc space with a weakly uniform base is metrizable [23], $Y$ is metrizable. Then $G$ and $bG$ are separable and metrizable by Lemma 2.5.

However, the following question is still open.

***Question* 3.2** [20]**.** Suppose that $G$ is a nonlocally compact, $k$-gentle paratopological group, and $Y = bG \setminus G$ is a remainder of $G$. If $Y$ has a $G_\delta$-diagonal, are $G, bG$ and $Y$ separable and metrizable spaces?

The following theorem is also a partial answer to Questions 3.1 and 3.2.

**Theorem 3.5.** *Let $G$ be a nonlocally compact paratopological group. If the remainder $Y = bG \setminus G$ satisfies one of the following conditions, then $G$ and $bG$ are separable and metrizable.*

(1) *$Y$ is a meta-Lindelöf*[7] *developable space*;

(2) *$G$ is $k$-gentle and $Y$ is a meta-Lindelöf space with a $G_\delta$-diagonal.*

***Proof.*** Since $Y$ has a $G_\delta$-diagonal, $Y$ is Ohio-complete. By Lemma 3.1, $G$ is a space of countable type or $G$ is $\sigma$-compact.

*Case* 1: $G$ is a space of countable type.

By Henriksen and Isbell's theorem, $Y$ is Lindelöf.

(1) If $Y$ is developable, then $Y$ is metrizable [11], and hence $G$ and $bG$ are separable and metrizable by Corollary 2.1.

(2) If $G$ is $k$-gentle and $Y$ is a meta-Lindelöf space with a $G_\delta$-diagonal, then it follows from Lemma 3.4 that $G$ is a topological group. Since $Y$ has a $G_\delta$-diagonal, $G, bG$ and $Y$ are separable and metrizable spaces [3].

*Case* 2: $G$ is $\sigma$-compact.

By the proof of Theorem 3.1, we have $c(Y) \leq \omega$, $Y$ is Čech-complete, and there exists a dense subspace $Z \subset Y$ such that $Z$ is a paracompact Čech-complete subspace of $Y$. Obvious, we have $c(Z) \leq \omega$. Since a paracompact Čech-complete space with a $G_\delta$-diagonal is metrizable [11] (Corollaries 3.8 and 3.20), $Z$ is metrizable. Then $Z$ is separable since $c(Z) \leq \omega$, and hence $Y$ is separable. Since $Y$ is meta-Lindelöf, then $Y$ is Lindelöf. By case 1, one obtain that $G$ and $bG$ are separable and metrizable.

Finally, we pose some open questions.

***Question* 3.3.** Let $G$ be a nonlocally compact paratopological group. If the remainder $Y = bG \setminus G$ has locally sharp base, are $G$ and $bG$ separable and metrizable?

***Question* 3.4.** Let $G$ be a nonlocally compact semitopological group. If the remainder $Y = bG \setminus G$ has a sharp base, are $G$ and $bG$ separable and metrizable?

***Question* 3.5.** Let $G$ be a nonlocally compact paratopological group which is GO-space. If the remainder $Y = bG \setminus G$ has a point-countable base, are $G$ and $bG$ separable and metrizable?

***Question* 3.6.** Let $G$ be a nonlocally compact paratopological group. If the remainder $Y = bG \setminus G$ has a weakly uniform base, are $G$ and $bG$ separable and metrizable?

[7] A space $X$ is said to be *meta-Lindelöf* if each open cover of $X$ has a locally countable open refined covering.

**4. The remainders of $\mathbb{R}_1$-factorizable paratopological groups.** A paratopological group $H$ is called $\mathbb{R}_1$-*factorizable* [25] if $H$ is a $T_1$-space and for every continuous real-valued function $f$ on $H$, one can find a continuous homomorphism $p\colon H \to K$ onto a paratopological group $K$ of countable weight satisfying the $T_1$ separation axiom and a continuous real-valued function $g$ on $K$ such that $f = g \circ p$.

***Remark* 4.1.** In this paper, we assume that all $H$ in the above definition are Tychonoff.

A space $(X, \tau)$ is called a $k$-*semistratifiable space* if there exists a function $\mathcal{S}\colon \mathbb{N} \times \tau \to \tau^c$ such that:

(a) for each $U \in \tau$, $U = \bigcup\{\mathcal{S}(n, U)\colon n \in \mathbb{N}\}$;

(b) if $U, V \in \tau$ and $U \subset V$, then $\mathcal{S}(n, U) \subset \mathcal{S}(n, V)$ for each $n \in \mathbb{N}$;

(c) for each compact $K$ of $X$ and open neighborhood $U$ of $K$, there exists an $n \in \mathbb{N}$ such that $K \subset \mathcal{S}(n, U)$.

**Lemma 4.1** [25]. *Let $G$ be $\mathbb{R}_1$-factorizable paratopological group. Then $\omega(G) = \chi(G)$.*

By Theorem 2.1 and Lemma 4.1, it is easy to see the following theorem holds.

**Theorem 4.1.** *Let $G$ be a nonlocally compact $\mathbb{R}_1$-factorizable paratopological group. If the remainder $Y = bG \setminus G$ has locally a countable network, then $G$ and $bG$ are separable and metrizable.*

**Theorem 4.2.** *Let $G$ be a nonlocally compact $\mathbb{R}_1$-factorizable paratopological group. If the remainder $Y = bG \setminus G$ is a $k$-semistratifiable space, then $G$ and $bG$ are separable and metrizable.*

***Proof.*** Since $Y$ is a $k$-semistratifiable space, $Y$ is a $\sigma$-space [11], and hence $Y$ has a $G_\delta$-diagonal, and hence $Y$ is Ohio-complete [2]. By Lemma 3.1, $G$ is $\sigma$-compact or $G$ is a space of countable type.

*Case* 1: $G$ is a space of countable type.

By Henriksen and Isbell's theorem, $Y$ is Lindelöf. Then $Y$ is a Lindelöf $\sigma$-space, and hence $Y$ has a countable network by [11] (Theorem 4.4). Therefore, $G$ is first-countable by Theorem 2.1, and thus it follows from Lemma 4.1 that $G$ is separable and metrizable. Then $G$ is a Lindelöf $p$-space, and hence $Y$ is a Lindelöf $p$-space by Lemma 2.4. Thus $Y$ is metrizable by [11] (Corollaries 3.8 and 3.20). Then $G$ and $bG$ are separable and metrizable by Lemma 2.5.

*Case* 2: $G$ is $\sigma$-compact.

Since $G$ is a $\sigma$-compact paratopological group, $Y$ is Čech-complete, and hence $Y$ is first-countable [12]. Then $Y$ is a stratifiable space since a Fréchet $k$-semistratifiable space is stratifiable [14], and hence $Y$ is paracompact. By the proof of Theorem 3.1, we have $c(Y) \leq \omega$, and thus $Y$ is Lindelöf. By case 1, $G$ and $bG$ are separable and metrizable.

***Corollary* 4.1.** *Let $G$ be a nonlocally compact $\mathbb{R}_1$-factorizable paratopological group. If the remainder $Y = bG \setminus G$ is an $\aleph$-space, then $G$ and $bG$ are separable and metrizable.*

By [25] (Corollaries 3.10 and 3.14), we know that if paratopological groups have a countable network or are $\sigma$-compact then they are $\mathbb{R}_1$-factorizable, and hence we have the following corollary.

***Corollary* 4.2.** *Let $G$ be a nonlocally compact paratopological group, and the remainder $Y = bG \setminus G$ be a $k$-semistratifiable space. If $G$ satisfies one of the following conditions, then $G$ and $bG$ are separable and metrizable.*

(1) *$G$ has a countable network.*

(2) *$G$ is $\sigma$-compact.*

However, the following question is still open.

***Question* 4.1.** Let $G$ be a nonlocally compact $\mathbb{R}_1$-factorizable paratopological group. If the remainder $Y = bG \setminus G$ is a $\sigma$-space, then are $G$ and $bG$ separable and metrizable?

The following theorem is also a partial answer to Question 4.1.

**Theorem 4.3.** *Let $G$ be a nonlocally compact $\mathbb{R}_1$-factorizable paratopological group. If the remainder $Y = bG \setminus G$ is a meta-Lindelöf $\sigma$-space, then $G$ and $bG$ are separable and metrizable.*

***Proof.*** By the proof of Theorem 4.2, it is suffice to prove that $Y$ is Lindelöf if $G$ is $\sigma$-compact. Let $G$ be $\sigma$-compact. By the proof of Theorem 3.1, we have $c(Y) \leq \omega$, $Y$ is Čech-complete, and there exists a dense subspace $Z \subset Y$ such that $Z$ is a paracompact Čech-complete subspace of $Y$. Obvious, we have $c(Z) \leq \omega$. Since a paracompact Čech-complete space with a $G_\delta$-diagonal is metrizable [11] (Corollaries 3.8 and 3.20), $Z$ is metrizable. Then $Z$ is separable since $c(Z) \leq \omega$, and hence $Y$ is separable. Since $Y$ is meta-Lindelöf, then $Y$ is Lindelöf.

***Question*** **4.2.** Let $G$ be a nonlocally compact $\mathbb{R}_1$-factorizable paratopological group. If the remainder $Y = bG \setminus G$ is a locally $\aleph$-space, then are $G$ and $bG$ separable and metrizable?

**Theorem 4.4** [21]**.** *Let $G$ be a nonlocally compact paratopological group. Then either every remainder of $G$ has the Baire*[8] *property, or every remainder of $G$ is meager*[9] *and Lindelöf.*

**Theorem 4.5.** *Let $G$ be a $\mathbb{R}_1$-factorizable paratopological group with a $G_\delta$-diagonal. If $G$ is nonmetrizable or nonseparable, then the remainder $Y = bG \setminus G$ is Baire.*

***Proof.*** By Theorem 4.5, $Y$ is meager and Lindelöf or $Y$ is Baire. Suppose that $Y$ is meager and Lindelöf. Then $G$ is of countable type, and thus $G$ is first-countable since $G$ has a $G_\delta$-diagonal. It follows from Lemma 4.1 that $G$ is separable and metrizable, which is a contradiction.

**Theorem 4.6.** *Let $G$ be a $\mathbb{R}_1$-factorizable nonmetrizable or nonseparable paratopological group. If for each point $y \in Y = bG \setminus G$ there is an open neighborhood $V(y)$ of $y$ such that every countably compact subset of $V(y)$ is metrizable and the remainder $Y$ is of countable $\pi$-character, then $Y$ is Baire.*

***Proof.*** If $G$ is locally compact, then the remainder is compact by [12] (Theorem 3.5.8), hence it is Baire. If $G$ is nonlocally compact, then we may use the proof of Theorem 4.4 to prove that the remainder is Baire.

**Theorem 4.7.** *Let $G$ be a $\mathbb{R}_1$-factorizable paratopological group, and the remainder $Y = bG \setminus G$ be a $k$-space with a locally point-countable $k$-network. If $Y$ is not Baire and is of countable $\pi$-character, then $G$ and $bG$ are separable and metrizable.*

***Proof.*** Since a countably compact $k$-space with a point-countable $k$-network is metrizable [13], it follows from Theorem 4.6 that $G$ is metrizable, and hence $G$ is separable and metrizable by Lemma 4.1. Then $G$ is a Lindelöf $p$-space, and thus $Y$ is a Lindelöf $p$-space by Lemma 2.4. Then $Y$ is a Lindelöf $p$-space with a point-countable $k$-network, and thus $Y$ is metrizable by [13]. Then $G$ and $bG$ are separable and metrizable by Lemma 2.5.

***Corollary*** **4.3.** *Let $G$ be a $\mathbb{R}_1$-factorizable paratopological group. If the remainder $Y = bG \setminus G$ is not Baire space with a locally point-countable base, then $G$ and $bG$ are separable and metrizable.*

***Question*** **4.3.** Let $G$ be a $\mathbb{R}_1$-factorizable paratopological group. If the remainder $Y = bG \setminus G$ is a space with a locally point-countable base, then are $G$ and $bG$ separable and metrizable?

***Question*** **4.4.** Let $G$ be a $\mathbb{R}_1$-factorizable paratopological group. Is the remainder $Y = bG \setminus G$ Lindelöf or pseudocompact?

1. *Arhangel'shiĭ A. V., Just W., Rezniczenko E. A., Szeptycki P. J.* Sharp bases and weakly uniform bases verus point-countable bases // Topology and Appl. – 2000. – **100**. – P. 39–46.
2. *Arhangel'skiĭ A. V.* Remainders in compactification and generalized metrizability properties // Topology and Appl. – 2005. – **150**. – P. 79–90.

[8] Recall that a space is *Baire* if the intersection of a sequence of open and dense subsets is dense.
[9] Recall that a space is called *meager* if it can be represented as the union of a sequence of nowhere dense subsets.

3. *Arhangel'skiĭ A. V.* More on remainders close to metrizable spaces // Topology and Appl. – 2007. – **154**. – P. 1084–1088.
4. *Arhangel'skiĭ A. V., Tkachenko M.* Topological groups and related structures. – Atlantis Press and World Sci., 2008.
5. *Arhangel'skiĭ A. V., Choban M. M.* Remainders of rectifiable spaces // Topology and Appl. – 2010. – **157(4)**. – P. 789–799.
6. *Arhangel'skiĭ A. V.* The Baire property in remainders of topological groups and other results // Comment. math. Univ. carol. – 2009. – **50**, № 2. – P. 273–279.
7. *Arhangel'skiĭ A. V., Choban M. M.* Completeness type properties of semitopological groups, and the theorems of Montgomery and Ellis // Topology Proc. – 2011. – **37**. – P. 33–60.
8. *Alleche B., Arhangel'shiĭ A. V., Calbrix J.* Weak developments and metrization // Topology and Appl. – 2000. – **100**. – P. 23–38.
9. *Burke D.* Covering properties // Handbook of Set-Theoretic Topology / Eds K. Kunen, J. E. Vaughan. – Amsterdam: Elsevier Sci. Publ. B. V., 1984. – P. 347–422.
10. *Bennett H. R., Lutzer D. J.* Diagonal conditions in ordered spaces // Fund. math. – 1997. – **153**. – P. 99–123.
11. *Gruenhage G.* Generalized metric spaces // Handbook of Set-Theoretic Topology / Eds K. Kunen, J. E. Vaughan. – Amsterdam: Elsevier Sci. Publ., 1984. – P. 423–501.
12. *Engelking R.* General Topology (revised and completed edition). – Berlin: Heldermann Verlag, 1989.
13. *Gruenhage G., Michael E., Tanaka Y.* Spaces determined by point-countable covvers // Pacif. J. Math. – 1984. – **113**. – P. 303–332.
14. *Gao Z. M.* Some results on $k$-semistratifiable spaces // J. Xibei Univ. – 1985. – **3**. – P. 12–16.
15. *Henriksen M., Isbell J.* Some properties of compactifications // Duke Math. J. – 1958. – **25**. – P. 83–106.
16. *Heath R. W., Lindgren W. E.* Uniform bases // Houston J. Math. – 1976. – **2**. – P. 85–90.
17. *Liu C.* Metrizability of paratopological (semitopological) groups // Topology and Appl. – 2012. – **159**. – P. 1415–1420.
18. *Liu C.* A note on paratopological groups // Comment. math. Univ. carol. – 2006. – **47**. – P. 633–640.
19. *Liu C., Lin S.* Generlized metric spaces with algebraic structures // Topology and Appl. – 2010. – **157**. – P. 1966–1974.
20. *Lin F., Shen R.* On rectfiable spaces and paratopological groups // Topology and Appl. – 2011. – **158**. – P. 597–610.
21. *Lin F., Lin S.* About remainders in compactifications of paratopological groups // arXiv:1106.3836v1.
22. *Lin F.* Local properties on the remainders of the topological groups // Kodai Math. J. – 2011. – **34**. – P. 505–518.
23. *Peregudov S. A.* On pseudocompactness and other covering properties // Questions Answers Gen. Topology. – 1999. – **17**. – P. 153–155.
24. *Šapirovskiĭ B.* On separability and metirzability of spaces with Souslin's condition // Sov. Math. Doke. – 1972. – **13**. – P. 1633–1638.
25. *Sanchis M., Tkachenko M. G.* $\mathbb{R}$-factorizable paratopological groups // Topology and Appl. – 2010. – **157**, № 4. – P. 800–808.
26. *Tkachenko M. G.* On the Souslin property in free topological groups over compact Hausdorff spaces // Mat. Notes. – 1983. – **34**. – P. 790–793.


Received 18.06.12,
after revision — 27.01.13