UDC 517.9

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COMMON FIXED POINT THEOREMS FOR NONLINEAR WEAKLY CONTRACTIVE MAPPINGS
СПІЛЬНІ ТЕОРЕМИ ПРО НЕРУХОМУ ТОЧКУ ДЛЯ НЕЛІНІЙНИХ СЛАБКОСТИСКАЛЬНИХ ВІДОБРАЖЕНЬ

Some common fixed point results for mappings satisfying a nonlinear weak contractive condition in the framework of ordered metric spaces are obtained. The accumulated results generalize and extend several comparable results well-known from the literature.

Отримано деякі спільні теореми про нерухому точку для відображень, що задовольняють нелінійну слабкостискальну умову в рамках упорядкованих метричних просторів. Отримані результати узагальнюють та розширюють декілька порівняльних результатів, відомих із літературних джерел.

Introduction and preliminaries. Banach contraction principle is one of the pivotal results of metric fixed point theory. It is a popular tool for solving existence problems in different fields of mathematics. There are several generalizations of Banach contraction principle in the related literature on metric fixed point theory.

Ran and Reurings [15] extended Banach contraction principle in partially ordered metric spaces with some applications to linear and nonlinear matrix equations. While Nieto and López [14] extended the result of Ran and Reurings and applied their main result to obtain a unique solution for a first order ordinary differential equation with periodic boundary conditions. Bhaskar and Lakshmikantham [3] introduced a concept of mixed monotone mappings and obtained some coupled fixed point results. Also, they applied their results to a first order differential equation with periodic boundary conditions.

Alber and Guerre-Delabriere [1] introduced a concept of weakly contractive mappings and proved the existence of fixed point of such mappings in Hilbert spaces. Thereafter, in 2001, Rhoades [17] proved the fixed point theorem which is one of the generalizations of Banach's contraction principle. Weakly contractive mappings are closely related to the mappings of Boyd and Wong [4] and of Reich types [16]. Recently, Doric [9] proved a common fixed point theorem for a generalized (ψ, ϕ)-weakly contractive mappings. Fixed point problems involving weak contractions and mappings satisfying weak contractive type inequalities have been studied by many authors (see [1, 5 – 10, 17] and references cited therein). In this paper, we generalize Chatterjea type contraction mappings to (μ, ψ)-generalized Chatterjea type contraction mappings and derive some common fixed point results for single-valued mappings on ordered metric spaces.

First, we recall some basic definitions and notations.

Let (X, d) be a metric space. A mapping $T\colon X \to X$ is said to be:

(a) *Kannan type* (see [11]) if there exists a $k \in \left(0, \dfrac{1}{2}\right]$ such that $d(Tx, Ty) \leq k[d(x, Tx) + d(y, Ty)]$ for all $x, y \in X$;

(b) *Chatterjea type* [7] if there exists a $k \in \left(0, \dfrac{1}{2}\right]$ such that $d(Tx, Ty) \leq k[d(x, Ty) + d(y, Tx)]$ for all $x, y \in X$.

Khan et al. [12] initiated the use of a control function that alters distance between two points in a metric space. So they called it an altering distance function.

A function $\mu\colon [0,\infty)\to[0,\infty)$ is called an *altering distance function* if the following properties are satisfied:

(i) μ is monotone increasing and continuous;

(ii) $\mu(t)=0$ if and only if $t=0$.

Using the control function, we generalize the Chatterjea type contraction mappings as follows:

Suppose that T and f are self-mappings defined on a metric space X. A pair of mappings (T,f) is said to satisfy (μ,ψ)*-generalized Chatterjea type contractive condition* if for all $x,y\in X$,

$$\mu(d(Tx,fy))\le\mu\left(\frac{1}{2}[d(x,fy)+d(y,Tx)]\right)-\psi(d(x,fy),d(y,Tx)), \tag{1}$$

holds, where $\mu\colon[0,\infty)\to[0,\infty)$ is an altering distance function and $\psi\colon[0,\infty)^2\to[0,\infty)$ is a lower semicontinuous mapping such that $\psi(x,y)=0$ if and only if $x=y=0$.

Let M be a nonempty subset of a metric space X, a point $x\in M$ is a *common fixed* (*coincidence*) *point* of f and T if $x=fx=Tx$ $(fx=Tx)$. The set of fixed points (respectively, coincidence points) of f and T is denoted by $F(f,T)$ (respectively, $C(f,T)$).

Definition 1. *Let $(X,\le)$ be a partially ordered set. Two mappings $f,g\colon X\to X$ are said to be weakly increasing if $fx\le gfx$ and $gx\le fgx$ for all $x\in X$.*

The following example shows that there exist discontinuous not nondecreasing mappings which are weakly increasing.

***Example* 1.** Let $X=(0,\infty)$, endowed with usual ordering. Let $f,g: X\to X$ be defined by

$$fx=\begin{cases}3x+2 & \text{if } 0<x<1,\\[2mm] 2x+1 & \text{if } 1\le x<\infty\end{cases}$$

and

$$gx=\begin{cases}4x+1 & \text{if } 0<x<1,\\[2mm] 3x & \text{if } 1\le x<\infty.\end{cases}$$

For $0<x<1,$ $fx=3x+2\le 3(3x+2)=gfx$ and $gx=4x+1\le 4x+3=2(2x+1)+1=fgx$ and for $1\le x<\infty,$ $fx=2x+1\le 3(2x+1)=gfx$ and $gx=3x\le 2(3x)+1=fgx$. Thus f and g are weakly increasing maps but not nondecreasing.

Common fixed point theorem in ordered metric spaces. Suppose that $(X,\preceq)$ is a partially ordered set. A mapping $T\colon X\to X$ is said to be *monotone increasing* if for all $x,y\in X$,

$$x\preceq y \quad \text{if and only if} \quad Tx\preceq Ty. \tag{2}$$

A subset W of a partially ordered set X is said to be *well ordered* if every two elements of W are comparable.

Theorem 1. *Let $(X,\preceq)$ be a partially ordered set such that there exists a complete metric d on X. Suppose that T and f are weakly increasing self mappings on X, and satisfy* (1) *for all comparable elements $x,y\in X$.*

Also suppose that either

(i) *if* $\{x_n\} \subset X$ *is a nondecreasing sequence with* $x_n \to z$ *in* X, *then* $x_n \preceq z$, *for every* $n \in \mathbb{N}$, *or*

(ii) T *or* f *is continuous.*

Then T *and* f *have a common fixed point. Moreover, the set of common fixed points of* f *and* T *is well ordered if and only if* f *and* T *have one and only one common fixed point.*

Proof. Let $x_0 \in X$. We can choose $x_1, x_2 \in X$ such that $x_1 = Tx_0$ and $x_2 = fx_1$. By induction, we construct a sequence $\{x_n\}$ in X such that $x_{2n+1} = Tx_{2n}$ and $x_{2n+2} = fx_{2n+1}$, for every $n \geq 0$. As T and f are weakly increasing mappings, so we obtain

$$x_1 = Tx_0 \preceq fx_1 = x_2 \preceq Tx_2 = x_3.$$

By induction on n, we conclude that

$$x_1 \preceq x_2 \preceq \ldots \preceq x_{2n+1} \preceq x_{2n+2} \preceq \ldots .$$

Since x_{2n+1} and x_{2n+2} are comparable, by inequality (1) we have

$$\mu(d(x_{2n+1}, x_{2n+2})) = \mu(d(Tx_{2n}, fx_{2n+1})) \leq$$

$$\leq \mu\left(\frac{1}{2}[d(x_{2n}, fx_{2n+1}) + d(x_{2n+1}, Tx_{2n})]\right) - \psi(d(x_{2n}, fx_{2n+1}), d(x_{2n+1}, Tx_{2n})) =$$

$$= \mu\left(\frac{1}{2}d(x_{2n}, x_{2n+2})\right) - \psi(d(x_{2n}, x_{2n+2}), 0) \leq$$

$$\leq \mu\left(\frac{1}{2}d(x_{2n}, x_{2n+2})\right).$$

Since μ is a monotone increasing function, for all $n = 1, 2, \ldots$, we get

$$d(x_{2n+1}, x_{2n+2}) \leq \frac{1}{2}d(x_{2n}, x_{2n+2}) \leq \frac{1}{2}\Big[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})\Big].$$

This implies that $d(x_{2n+1}, x_{2n+2}) \leq d(x_{2n}, x_{2n+1})$. Following the similar arguments, we obtain $d(x_{2n+2}, x_{2n+3}) \leq d(x_{2n+1}, x_{2n+2})$. Hence, $d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n)$. Thus $\{d(x_n, x_{n+1})\}$ is a monotone decreasing sequence of nonnegative real numbers. Hence there exists $r \geq 0$ such that $d(x_n, x_{n+1}) \to r$. As

$$d(x_{2n+1}, x_{2n+2}) \leq \frac{1}{2}d(x_{2n}, x_{2n+2}) \leq \frac{1}{2}\big[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})\big].$$

Taking limit as $n \to \infty$, we have $r \leq \lim \frac{1}{2}d(x_{2n}, x_{2n+2}) \leq \frac{1}{2}r + \frac{1}{2}r$. Therefore $\lim_{n\to\infty} d(x_{2n}, x_{2n+2}) = 2r$. Using the continuity of μ and lower semicontinuity of ψ, we have $\mu(r) \leq \mu(r) - \psi(2r, 0)$. This implies that $\psi(2r, 0) = 0$ and hence $r = 0$. Thus $d(x_{n+1}, x_n) \to 0$.

Now, we prove that $\{x_n\}$ is a Cauchy sequence. It is sufficient to show that $\{x_{2n}\}$ is a Cauchy sequence. On contrary, suppose that $\{x_{2n}\}$ is not a Cauchy sequence. Then there exists $\epsilon > 0$ for which we can find subsequences $\{x_{2m(k)}\}$ and $\{x_{2n(k)}\}$ of $\{x_{2n}\}$ such that $n(k)$ is the smallest index for which $n(k) > m(k) > k$, $d(x_{2m(k)}, x_{2n(k)}) \geq \epsilon$. This means that $d(x_{2m(k)}, x_{2n(k)-2}) < \epsilon$. So, we have

$$\epsilon \le d(x_{2m(k)}, x_{2n(k)}) \le$$

$$\le d(x_{2m(k)}, x_{2n(k)-2}) + d(x_{2n(k)-2}, x_{2n(k)-1}) + d(x_{2n(k)-1}, x_{2n(k)}) <$$

$$< \epsilon + d(x_{2n(k)-2}, x_{2n(k)-1}) + d(x_{2n(k)-1}, x_{2n(k)}).$$

Taking limit as $k \to \infty$, we get

$$\lim_{n\to\infty} d(x_{2m(k)}, x_{2n(k)}) = \epsilon. \tag{3}$$

Also,

$$\epsilon \le d(x_{2m(k)}, x_{2n(k)}) \le d(x_{2m(k)}, x_{2m(k)-1}) + d(x_{2m(k)-1}, x_{2n(k)}) \le$$

$$\le 2d(x_{2m(k)}, x_{2m(k)-1}) + d(x_{2m(k)}, x_{2n(k)}).$$

On letting $k \to \infty$, we obtain

$$\lim_{n\to\infty} d(x_{2m(k)-1}, x_{2n(k)}) = \epsilon. \tag{4}$$

On the other hand, we have

$$d(x_{2m(k)}, x_{2n(k)}) \le d(x_{2m(k)}, x_{2n(k)+1}) + d(x_{2n(k)+1}, x_{2n(k)}) \le$$

$$\le d(x_{2m(k)}, x_{2n(k)}) + 2d(x_{2n(k)+1}, x_{2n(k)}).$$

On taking limit as $k \to \infty$, we get

$$\lim_{n\to\infty} d(x_{2m(k)}, x_{2n(k)+1}) = \epsilon.$$

Also,

$$d(x_{2m(k)-1}, x_{2n(k)}) \le d(x_{2m(k)-1}, x_{2n(k)+1}) + d(x_{2n(k)+1}, x_{2n(k)}) \le$$

$$\le d(x_{2m(k)-1}, x_{2n(k)}) + 2d(x_{2n(k)+1}, x_{2n(k)}).$$

On taking limit as $k \to \infty$, we obtain

$$\lim_{n\to\infty} d(x_{2m(k)-1}, x_{2n(k)+1}) = \epsilon.$$

Consider

$$\mu(\epsilon) \le \mu(d(x_{2m(k)}, x_{2n(k)})) = \mu(d(Tx_{2m(k)-1}, fx_{2n(k)-1})) \le$$

$$\le \mu\left(\frac{1}{2}\left[d(x_{2m(k)-1}, fx_{2n(k)-1}) + d(x_{2n(k)-1}, Tx_{2m(k)-1})\right]\right) -$$

$$-\psi(d(x_{2m(k)-1}, fx_{2n(k)-1}), d(x_{2n(k)-1}, Tx_{2m(k)-1})) =$$

$$= \mu\left(\frac{1}{2}[d(x_{2m(k)-1}, x_{2n(k)}) + d(x_{2n(k)-1}, x_{2m(k)})]\right) -$$

$$-\psi(d(x_{2m(k)-1}, x_{2n(k)}), d(x_{2n(k)-1}, x_{2m(k)})).$$

Taking limit as $k \to \infty$, and using the continuity of μ and lower semicontinuity of ψ, we have $\mu(\epsilon) \leq \mu\left(\frac{1}{2}[\epsilon+\epsilon]\right) - \psi(\epsilon,\epsilon)$ and consequently $\psi(\epsilon,\epsilon) \leq 0$, a contradiction as $\epsilon > 0$. Thus $\{x_{2n}\}$ is a Cauchy sequence and hence $\{x_n\}$ is a Cauchy sequence. As X is a complete metric space, there exists $t \in X$ such that $\lim_{n\to\infty} x_n = t$. Since $\{x_n\}$ is a nondecreasing sequence, by (i), we have $x_n \preceq t$. Consider

$$\mu(d(x_{2n+1}, ft)) = \mu(d(Tx_{2n}, ft)) \leq$$

$$\leq \mu\left(\frac{1}{2}[d(x_{2n}, ft) + d(t, Tx_{2n})]\right) - \psi(d(x_{2n}, ft), d(t, Tx_{2n})) =$$

$$= \mu\left(\frac{1}{2}[d(x_{2n}, ft) + d(t, x_{2n+1})]\right) - \psi(d(x_{2n}, ft), d(t, x_{2n+1})).$$

Taking limit as $n \to \infty$, we have $\mu(d(t, ft)) \leq \mu\left(\frac{1}{2}d(t, ft)\right) - \psi(d(t, ft), 0)) \leq \mu\left(\frac{1}{2}d(t, ft)\right)$. This implies that $d(t, ft) = 0$ and hence $t = ft$.

Again, consider

$$\mu(d(Tt, t)) = \mu(d(Tt, ft)) \leq \mu\left(\frac{1}{2}[d(t, ft) + d(t, Tt)]\right) - \psi(d(t, ft), d(t, Tt)) =$$

$$= \mu\left(\frac{1}{2}d(t, Tt)\right) - \psi(0, d(t, Tt)) \leq \mu\left(\frac{1}{2}d(t, Tt)\right).$$

This implies that $d(Tt, t) = 0$, $Tt = t$. Therefore, $t = Tt = ft$, i.e., t is a common fixed point of T and f.

If condition (ii) holds: Assume that T is continuous. Then $t = \lim_{n\to\infty} Tx_n = x_{2n+1} = Tt$. Now

$$\mu(d(t, ft)) = \mu(d(Tt, ft)) \leq \mu\left(\frac{1}{2}\left[d(t, ft) + d(t, Tt)\right]\right) - \psi(d(t, ft), d(t, Tt)) =$$

$$= \mu\left(\frac{1}{2}d(t, ft)\right) - \psi(d(t, ft), 0) \leq \mu\left(\frac{1}{2}d(t, ft)\right)$$

implies that $d(t, ft) = 0$, $ft = t$. Therefore, $t = Tt = ft$, i.e., t is a common fixed point of T and f.

If f is continuous, then following arguments similar to those given above, the result follows.

Now suppose that the set of common fixed points of T and f is well ordered. Now, we claim the uniqueness of common fixed points of T and f. Assume on contrary that $Tu = fu = u$ and $Tv = fv = v$ but $u \neq v$. Consider

$$\mu(d(u, v)) = \mu(d(Tu, fv)) \leq$$

$$\leq \mu\left(\frac{1}{2}[d(u, fv) + d(v, Tu)]\right) - \psi(d(u, fv), d(v, Tu)) =$$

$$= \mu\left(\frac{1}{2}[d(u,v) + d(v,u)]\right) - \psi(d(u,v), d(v,u)) =$$

$$= \mu(d(u,v)) - \psi(d(u,v), d(u,v)).$$

This implies that $d(u,v) = 0$, by the property of ψ. Hence $u = v$. Conversely, if T and f have only one common fixed point then the set of common fixed point of f and T being singleton is well ordered.

Theorem 1 is proved.

If $T = f$, we have the following result.

***Corollary* 1.** *Let $(X, \preceq)$ be a partially ordered set such that there exists a complete metric d on X. Suppose that T is a monotone nondecreasing self mapping on X such that*

$$\mu(d(Tx,Ty)) \leq \mu\left(\frac{1}{2}[d(x,Ty) + d(y,Tx)]\right) - \psi(d(x,Ty), d(y,Tx)),$$

is satisfied for all $x, y \in X$ with x and y comparable.

Also suppose that either

(i) *if $\{x_n\} \subset X$ is a nondecreasing sequence with $x_n \to z$ in X, then $x_n \preceq z$, for every $n \in \mathbb{N}$, or*

(ii) *T is continuous.*

Then T has a fixed point.

If $\mu(t) = t$, we have the following result.

***Corollary* 2** (see [5, 10])**.** *Let $(X, \preceq)$ be a partially ordered set such that there exists a complete metric d on X. Suppose that T is a monotone nondecreasing self mapping on X such that*

$$\mu(d(Tx,Ty)) \leq \mu\left(\frac{1}{2}[d(x,Ty) + d(y,Tx)]\right) - \psi(d(x,Ty), d(y,Tx)),$$

is satisfied for all comparable elements $x, y \in X$.

Also suppose that either

(i) *if $\{x_n\} \subset X$ is a nondecreasing sequence with $x_n \to z$ in X, then $x_n \preceq z$, for every $n \in \mathbb{N}$, or*

(ii) *T is continuous.*

Then T has a fixed point.

***Example* 2.** Let $M = [0,1]$ be endowed with partial order $x \preceq y$ if and only if $x \geq y$. Let d be defined by $d(x,y) = |x - y|$. We set $Tx = 0$ and $fx = \dfrac{x^2}{8}$ for all $x \in M$. It is easy to see that f and g are weakly increasing maps. We define $\mu\colon [0,\infty) \to [0,\infty)$ and $\psi\colon [0,\infty) \times [0,\infty) \to [0,\infty)$ by

$$\mu(t) = \frac{t}{2} \qquad \text{and} \qquad \psi(t,s) = \frac{t+s}{16}.$$

Then for $x, y \in M$, we have

$$\mu(d(Tx,fy)) = \mu\left(d\left(0, \frac{y^2}{8}\right)\right) = \mu\left(\frac{y^2}{8}\right) = \frac{y^2}{16},$$

and

$$\mu\left(\frac{1}{2}[d(x,fy)+d(y,Tx)]\right)-\psi(d(x,fy),d(y,Tx))=$$

$$=\mu\left(\frac{1}{2}\left[d\left(x,\frac{y^2}{8}\right)+d(y,0)\right]\right)-\psi\left(d\left(x,\frac{y^2}{8}\right),\,d(y,0)\right)=$$

$$=\mu\left(\frac{1}{2}\left[\left|x-\frac{y^2}{8}\right|+y\right]\right)-\psi\left(\left|x-\frac{y^2}{8}\right|,\,y\right)=$$

$$=\frac{1}{4}\left[\left|x-\frac{y^2}{8}\right|+y\right]-\frac{\left|x-\frac{y^2}{8}\right|+y}{16}=\frac{3}{16}\left[\left|x-\frac{y^2}{8}\right|+y\right]\geq\frac{3y}{16}\geq\frac{y^2}{16}.$$

Hence

$$\mu(d(Tx,fy))\leq\mu\left(\frac{1}{2}\left[d(x,fy)+d(y,Tx)\right]\right)-\psi(d(x,fy),d(y,Tx)).$$

Thus all conditions of Theorem 1 are satisfied. Moreover, T and f have a unique common fixed point 0.

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Received 11.06.12,
after revision — 04.04.13