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I-n-COHERENT RINGS, I-n-SEMIHEREDITARY RINGS AND I-REGULAR RINGS

I-n-КОГЕРЕНТНІ КІЛЬЦЯ, I-n-НАПІВСПАДКОВІ КІЛЬЦЯ ТА I-РЕГУЛЯРНІ КІЛЬЦЯ

Let R be a ring, I an ideal of R and n a fixed positive integer. We define and study I-n-injective modules, I-n-flat modules. Moreover, we define and study left I-n-coherent rings, left I-n-semihereditary rings and I-regular rings. By using the concepts of I-n-injectivity and I-n-flatness of modules, we also present some characterizations of left I-n-coherent rings, left I-n-semihereditary rings, and I-regular rings.

Нехай R — кільце, I — ідеал R, а n — фіксоване додатне ціле число. Ми визначаємо та вивчаємо I-n-ін'єктивні модулі та I-n-плоскі модулі. Крім того, визначаємо та вивчаємо ліві I-n-когерентні кільця, ліві I-n-напівспадкові кільця та I-регулярні кільця. За допомогою концепцій I-n-ін'єктивності та I-n-пологості модулів також наводимо деякі характеристики лівих I-n-когерентних кілець, лівих I-n-напівспадкових кілець та I-регулярних кілець.

1. Introduction. Throughout this paper, n is a positive integer, R is an associative ring with identity, I is an ideal of R, $J = J(R)$ is the Jacobson radical of R and all modules considered are unitary.

Recall that a ring R is called left coherent if every finitely generated left ideal of R is finitely presented; a ring R is called left semihereditary if every finitely generated left ideal of R is projective; a ring R is called von Neumann regular (or regular for short) if for any $a \in R$, there exists $b \in R$ such that $a = aba$. Left coherent rings, left semihereditary rings, von Neumann regular rings and their generalizations have been studied by many authors. For example, a ring R is said to be *left n-coherent* [1] if every n-generated left ideal of R is finitely presented; a ring R is said to be *left J-coherent* [8] if every finitely generated left ideal in $J(R)$ is finitely presented; a ring R is said to be *left n-semihereditary* [24, 25] if every n-generated left ideal of R is projective; a ring R is said to be *left J-semihereditary* [8] if every finitely generated left ideal of R is projective; a commutative ring R is called an *n-von Neumann regular ring* [14] if every n-presented right R-module is projective.

In this article, we extend the concepts of left n-coherent rings and left J-coherent rings to *left I-n-coherent* rings, extend the concepts of left n-semihereditary rings and left J-semihereditary rings to *left I-n-semihereditary* rings, and extend the concept of regular rings to *I-regular rings*, respectively. We call a ring R left I-n-coherent (resp., left I-n-semihereditary, I-regular) if every finitely generated left ideal in I is finitely presented (resp., projective, a direct summand of ${}_RR$). Left I-1-coherent rings and left I-1-semihereditary rings are also called left I-P-coherent rings and left IPP rings respectively.

To characterize left I-n-coherent rings, left I-n-semihereditary rings and I-regular rings, in Sections 2 and 3, I-n-injective modules and I-n-flat modules are introduced and studied. I-1-injective modules and I-1-flat modules are also called I-P-injective modules and I-P-flat modules respectively. In Sections 4, 5, and 6, I-n-coherent rings, I-n-semihereditary and I-regular rings are investigated respectively. It is shown that there are many similarities between I-n-coherent rings and coherent rings, I-n-semihereditary rings and semihereditary rings, and between I-regular rings and regular rings. For instance, we prove that R is a left I-n-coherent ring $\Leftrightarrow$ any direct product of I-

n-flat right R-modules is I-n-flat $\Leftrightarrow$ any direct limit of I-n-injective left R-modules is I-n-injective $\Leftrightarrow$ every right R-module has an I-n-flat preenvelope; R is a left I-n-semihereditary ring $\Leftrightarrow$ R is left I-n-coherent and submodules of I-n-flat right R-modules are I-n-flat $\Leftrightarrow$ every quotient module of an I-n-injective left R-module is I-n-injective $\Leftrightarrow$ every left R-module has a monic I-n-injective cover $\Leftrightarrow$ every right R-module has an epic I-n-flat envelope; R is an I-regular ring $\Leftrightarrow$ every left R-module is I-P-injective $\Leftrightarrow$ every left R-module is I-P-flat $\Leftrightarrow$ R is left IPP and left I-P-injective.

For any module M, M^* denotes $\mathrm{Hom}_R(M,R)$, and M^+ denotes $\mathrm{Hom}_{\mathbb{Z}}(M,\mathbb{Q}/\mathbb{Z})$, where $\mathbb{Q}$ is the set of rational numbers, and $\mathbb{Z}$ is the set of integers. In general, for a set S, we write S^n for the set of all formal $(1\times n)$-matrices whose entries are elements of S, and S_n for the set of all formal $(n\times 1)$-matrices whose entries are elements of S. Let N be a left R-module, $X\subseteq N_n$ and $A\subseteq R^n$. Then we definite $\mathbf{r}_{N_n}(A)=\{u\in N_n\colon au=0\ \forall a\in A\}$, and $\mathbf{l}_{R^n}(X)=\{a\in R^n\colon ax=0\ \forall x\in X\}$.

2. *I*-*n*-injective modules. Recall that a left R-module M is called *F-injective* [11] if every R-homomorphism from a finitely generated left ideal to M extends to a homomorphism of R to M, a left R-module M is called *n-injective* [16] if every R-homomorphism from an n-generated left ideal to M extends to a homomorphism of R to M, 1-injective modules are also called P-injective [16], a ring R is called left P-injective [16] if ${}_RR$ is P-injective. P-injective ring and its generalizations have been studied by many authors, for example, see [16, 17, 19, 22, 26]. A left R-module M is called *J-injective* [8] if every R-homomorphism from a finitely generated left ideal in $J(R)$ to M extends to a homomorphism of R to M. We extends the concepts of n-injective modules and J-injective modules as follows.

Definition 2.1. *A left R-module M is called I-n-injective, if every R-homomorphism from an n-generated left ideal in I to M extends to a homomorphism of R to M. A left R-module M is called I-P-injective if it is I-1-injective.*

It is easy to see that direct sums and direct summands of I-n-injective modules are I-n-injective. A left R-module M is n-injective if and only if M is R-n-injective, a left R-module M is J-injective if and only if M is J-n-injective for every positive integer n. Follow [2], a ring R is said to be left Soc-injective if every R-homomorphism from a semisimple submodule of ${}_RR$ to R extends to R. Clearly, if $\mathrm{Soc}({}_RR)$ is finitely generated, then R is left Soc-injective if and only if ${}_RR$ is $\mathrm{Soc}({}_RR)$-n-injective for every positive integer n. We remark that J-P-injective modules are called JP-injective in [22].

Theorem 2.1. *Let M be a left R-module. Then the following statements are equivalent*:

(1) *M is I-n-injective.*

(2) $\mathrm{Ext}^1(R/T,M)=0$ *for every n-generated left ideal T in I.*

(3) $\mathbf{r}_{M_n}\mathbf{l}_{R^n}(\alpha)=\alpha M$ *for all* $\alpha\in I_n$.

(4) *If* $x=(m_1,m_2,\ldots,m_n)'\in M_n$ *and* $\alpha\in I_n$ *satisfy* $\mathbf{l}_{R^n}(\alpha)\subseteq\mathbf{l}_{R^n}(x)$, *then* $x=\alpha y$ *for some* $y\in M$.

(5) $\mathbf{r}_{M_n}(R^nB\cap\mathbf{l}_{R^n}(\alpha))=\mathbf{r}_{M_n}(B)+\alpha M$ *for all* $\alpha\in I_n$ *and* $B\in R^{n\times n}$.

(6) *M is I-P-injective and* $r_M(K\cap L)=r_M(K)+r_M(L)$, *where K and L are left ideals in I such that $K+L$ is n-generated.*

(7) *M is I-P-injective and* $r_M(K\cap L)=r_M(K)+r_M(L)$, *where K and L are left ideals in I such that K is cyclic and L is $(n-1)$-generated.*

(8) *For each n-generated left ideal T in I and any* $f\in\mathrm{Hom}(T,M)$, *if* (α,g) *is the pushout of* (f,i) *in the following diagram*:

$$\begin{array}{ccc} T & \xrightarrow{\ i\ } & R \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ M & \xrightarrow{\ \alpha\ } & P \end{array}$$

where i is the inclusion map, then there exists a homomorphism $h\colon P \to M$ such that $h\alpha = 1_M$.

Proof. (1) $\Leftrightarrow$ (2) and (8) $\Rightarrow$ (1) are clear.

(1) $\Rightarrow$ (3). Always $\alpha M \subseteq \mathbf{r}_{M_n}\mathbf{l}_{R^n}(\alpha)$. If $x \in \mathbf{r}_{M_n}\mathbf{l}_{R^n}(\alpha)$, then the mapping $f\colon R^n\alpha \to M$; $\beta\alpha \mapsto \beta x$ is a well-defined left R-homomorphism. Since M is I-n-injective and $R^n\alpha$ is an n-generated left ideal in I, f can be extended to a homomorphism g of R to M. Let $g(1) = y$, then $x = \alpha y \in \alpha M$. So $\mathbf{r}_{M_n}\mathbf{l}_{R^n}(\alpha) \subseteq \alpha M$. And thus $\mathbf{r}_{M_n}\mathbf{l}_{R^n}(\alpha) = \alpha M$.

(3) $\Rightarrow$ (1). Let $T = \sum_{i=1}^{n} Ra_i$ be an n-generated left ideal in I and f be a homomorphism from T to M. Write $u_i = f(a_i)$, $i = 1, 2, \ldots, n, u = (u_1, u_2, \ldots, u_n)'$, $\alpha = (a_1, a_2, \ldots, a_n)'$, then $u \in \mathbf{r}_{M_n}\mathbf{l}_{R^n}(\alpha)$. By (3), there exists some $x \in M$ such that $u = \alpha x$. Now we define $g\colon R \to M$; $r \mapsto rx$, then g is a left R-homomorphism which extends f.

(3) $\Rightarrow$ (4). If $\mathbf{l}_{R^n}(\alpha) \subseteq \mathbf{l}_{R^n}(x)$, where $\alpha \in I_n$, $x \in M_n$, then $x \in \mathbf{r}_{M_n}\mathbf{l}_{R^n}(x) \subseteq \mathbf{r}_{M_n}\mathbf{l}_{R^n}(\alpha) = = \alpha M$ by (3). Thus (4) is proved.

(4) $\Rightarrow$ (5). Let $x \in \mathbf{r}_{M_n}(R^nB \cap \mathbf{l}_{R^n}(\alpha))$, then $\mathbf{l}_{R^n}(B\alpha) \subseteq \mathbf{l}_{R^n}(Bx)$. By (4), $Bx = B\alpha y$ for some $y \in M$. Hence $x - \alpha y \in \mathbf{r}_{M_n}(B)$, proving that $\mathbf{r}_{M_n}(R^nB \bigcap \mathbf{l}_{R^n}(\alpha)) \subseteq \mathbf{r}_{M_n}(B) + \alpha M$. The other inclusion always holds.

(5) $\Rightarrow$ (3). By taking $B = E$ in (5).

(1) $\Rightarrow$ (6). Clearly, M is I-P-injective and

$$r_M(K) + r_M(L) \subseteq r_M(K \cap L).$$

Conversely, let $x \in r_M(K \cap L)$. Then $f\colon K + L \to M$ is well defined by $f(k + l) = kx$ for all $k \in K$ and $l \in L$. Since M is I-n-injective, $f = \cdot y$ for some $y \in M$. Hence, for all $k \in K$ and $l \in L$, we have $ky = f(k) = kx$ and $ly = f(l) = 0$. Thus $x - y \in r_M(K)$ and $y \in r_M(L)$, so $x = (x - y) + y \in r_M(K) + r_M(L)$.

(6) $\Rightarrow$ (7) is trivial.

(7) $\Rightarrow$ (1). We proceed by induction on n. If $n = 1$, then (1) is clearly holds by hypothesis. Suppose $n > 1$. Let $T = Ra_1 + Ra_2 + \ldots + Ra_n$ be an n-generated left ideal in I, $T_1 = Ra_1$ and $T_2 = Ra_2 + \ldots + Ra_n$. Suppose $f\colon T \to M$ is a left R-homomorphism. Then $f|_{T_1} = \cdot y_1$ by hypothesis and $f|_{T_2} = \cdot y_2$ by induction hypothesis for some $y_1, y_2 \in R$. Thus $y_1 - y_2 \in \in r_M(T_1 \cap T_2) = r_M(T_1) + r_M(T_2)$. So $y_1 - y_2 = z_1 + z_2$ for some $z_1 \in r_M(T_1)$ and $z_2 \in r_M(T_2)$. Let $y = y_1 - z_1 = y_2 + z_2$. Then for any $a \in T$, let $a = b_1 + b_2$, $b_1 \in T_1$, $b_2 \in T_2$, we have $b_1z_1 = 0$, $b_2z_2 = 0$. Hence $f(a) = f(b_1) + f(b_2) = b_1y_1 + b_2y_2 = b_1(y_1 - z_1) + b_2(y_2 + z_2) = b_1y + b_2y = ay$. So (1) follows.

(1) $\Rightarrow$ (8). Without loss of generality, we may assume that $P = (M \oplus R)/W$, where $W = = \{f(a), -i(a) \mid a \in T\}$, $g(r) = (0, r) + W$, $\alpha(x) = (x, 0) + W$ for $x \in M$ and $r \in R$. Since M is I-n-injective, there is $\varphi \in \mathrm{Hom}_R(R, M)$ such that $\varphi i = f$. Define $h[(x, r) + W] = x + \varphi(r)$ for all $(x, r) + W \in P$. It is easy to check that h is well-defined and $h\alpha = 1_M$.

Theorem 2.1 is proved.

***Corollary* 2.1.** *Let M be a left R-module. Then the following statements are equivalent*:

(1) *M is n-injective.*

(2) $\mathrm{Ext}^1(R/T, M) = 0$ *for every n-generated left ideal T.*

(3) $\mathbf{r}_{M_n}\mathbf{l}_{R^n}(\alpha) = \alpha M$ *for all* $\alpha \in R_n$.

(4) *If* $x = (m_1, m_2, \ldots, m_n)' \in M_n$ *and* $\alpha \in R_n$ *satisfy* $\mathbf{l}_{R^n}(\alpha) \subseteq \mathbf{l}_{R^n}(x)$, *then* $x = \alpha y$ *for some* $y \in M$.

(5) $\mathbf{r}_{M_n}(R^n B \cap \mathbf{l}_{R^n}(\alpha)) = \mathbf{r}_{M_n}(B) + \alpha M$ *for all* $\alpha \in R_n$ *and* $B \in R^{n\times n}$.

(6) M *is* P*-injective and* $r_M(K \cap L) = r_M(K) + r_M(L)$, *where* K *and* L *are left ideals such that* $K + L$ *is* n*-generated.*

(7) M *is* P*-injective and* $r_M(K \cap L) = r_M(K) + r_M(L)$, *where* K *and* L *are left ideals such that* K *is cyclic and* L *is* $(n-1)$*-generated.*

(8) *For each* n*-generated left ideal* T *and any* $f \in \mathrm{Hom}(T, M)$, *if* (α, g) *is the pushout of* (f, i) *in the following diagram*:

$$\begin{array}{ccc} T & \xrightarrow{\ i\ } & R \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ M & \xrightarrow{\ \alpha\ } & P \end{array}$$

where i *is the inclusion map, there exists a homomorphism* $h\colon P \to M$ *such that* $h\alpha = 1_M$.

We note that the equivalence of (1), (3), (6), (7) in Corollary 2.1 appears in [6] (Corollaries 2.5 and 2.10).

***Corollary* 2.2.** *Let* $\{M_\alpha\}_{\alpha\in A}$ *be a family of right* R*-modules. Then* $\prod_{\alpha\in A} M_\alpha$ *is* I*-*n*-injective if and only if each* M_α *is* I*-*n*-injective.*

Proof. It follows from the isomorphism $\mathrm{Ext}^1\left(N, \prod_{\alpha\in A} M_\alpha\right) \cong \prod_{\alpha\in A} \mathrm{Ext}^1(N, M_\alpha)$.

Recall that an element $a \in R$ is called *left I-semiregular* [18] if there exists $e^2 = e \in Ra$ such that $a - ae \in I$, and R is called left I-semiregular if every element is I-semiregular. A ring R is called *semiregular* if $R/J(R)$ is regular and idempotents lift modulo $J(R)$. It is well known that a ring R is semiregular if and only if it is left (equivalently right) J-semiregular [19]. Next, we consider a case when I-n-injective modules are n-injective.

Theorem 2.2. *Let* R *be a left* I*-semiregular ring. Then a left* R*-module* M *is* n*-injective if and only if* M *is* I*-*n*-injective.*

Proof. *Necessity* is clear. To prove *sufficiency*, let T be an n-generated left ideal and $f\colon T \to M$ be a left R-homomorphism. Since R is left I-semiregular, by [18] (Theorem 1.2(2)), $R = H \oplus L$, where $H \leq T$ and $T \cap L \subseteq I$. Hence $R = T + L$, $T = H \oplus (T \cap L)$, and so $T \cap L$ is n-generated. Since M is I-n-injective, there exists a homomorphism $g\colon R \to M$ such that $g(x) = f(x)$ for all $x \in T \cap L$. Now let $h\colon R \to M$; $r \mapsto f(t) + g(l)$, where $r = t + l$, $t \in T$, $l \in L$. Then h is a well-defined left R-homomorphism and h extends f.

Theorem 2.2 is proved.

***Corollary* 2.3.** *Let* R *be a left semiregular ring. Then*:

(1) *A left* R*-module* M *is* P*-injective if and only if* M *is* JP*-injective.*

(2) *A left* R*-module* M *is* F*-injective if and only if* M *is* J*-injective.*

Theorem 2.3. *Every pure submodule of an* I*-*n*-injective module is* I*-*n*-injective. In particular, every pure submodule of an* n*-injective module is* n*-injective.*

Proof. Let N be a pure submodule of an I-n-injective left R-module M. For any n-generated left ideal T in I, we have the exact sequence

$$\mathrm{Hom}(R/T, M) \to \mathrm{Hom}(R/T, M/N) \to \mathrm{Ext}^1(R/T, N) \to \mathrm{Ext}^1(R/T, M) = 0.$$

Since R/T is finitely presented and N is pure in M, the sequence $\mathrm{Hom}(R/T, M) \to \mathrm{Hom}(R/T, M/N) \to 0$ is exact. Hence $\mathrm{Ext}^1(R/T, N) = 0$, and so N is I-n-injective.

Theorem 2.3 is proved.

3. *I-n*-flat modules. Recall that a right R-module V is said to be *n-flat* [1, 9], if for every n-generated left ideal T, the canonical map $V \otimes T \to V \otimes R$ is monic. 1-flat modules are called P-flat by some authors such as Couchot [7]. A right R-module V is said to be *J-flat* [8], if for every finitely generated left ideal T in $J(R)$, the canonical map $V \otimes T \to V \otimes R$ is monic. We extend the concepts of n-flat modules and J-flat modules as follows.

Definition 3.1. *A right R-module V is said to be I-n-flat, if for every n-generated left ideal T in I, the canonical map $V \otimes T \to V \otimes R$ is monic. V_R is said to be I-P-flat if it is I-1-flat. V_R is said to be I-flat if it is I-n-flat for every positive integer n.*

It is easy to see that direct sums and direct summands and of I-n-flat modules are I-n-flat.

Theorem 3.1. *For a right R-module V, the following statements are equivalent*:

(1) *V is I-n-flat.*

(2) $\mathrm{Tor}_1(V, R/T) = 0$ *for every n-generated left ideal T in I.*

(3) *V^+ is I-n-injective .*

(4) *For every n-generated left ideal T in I, the map $\mu_T \colon V \otimes T \to VT$; $\sum v_i \otimes a_i \mapsto \sum v_i a_i$ is a monomorphism.*

(5) *For all $x \in V^n$, $a \in I_n$, if $xa = 0$, then exist positive integer m and $y \in V^m$, $C \in R^{m\times n}$, such that $Ca = 0$ and $x = yC$.*

Proof. (1) $\Leftrightarrow$ (2) follows from the exact sequence $0 \to \mathrm{Tor}_1(V, R/T) \to V \otimes T \to V \otimes R$.

(2) $\Leftrightarrow$ (3) follows from the isomorphism $\mathrm{Tor}_1(M, R/T)^+ \cong \mathrm{Ext}^1(R/T, M^+)$.

(1) $\Leftrightarrow$ (4) follows from the commutative diagram

$$\begin{array}{ccc} V \otimes T & \xrightarrow{1_V \otimes i_T} & V \otimes R \\ \mu_T \big\downarrow & & \big\downarrow \sigma \\ VT & \xrightarrow{i_{VT}} & V \end{array}$$

where σ is an isomorphism.

(4) $\Rightarrow$ (5). Let $x = (v_1, v_2, \ldots, v_n)$, $a = (a_1, a_2, \ldots, a_n)'$, $T = \sum_{j=1}^n Ra_j$. Write e_j be the element in R^n with 1 in the jth position and 0's in all other positions, $j = 1, 2, \ldots, n$. Consider the short exact sequence

$$0 \to K \overset{i_K}{\to} R^n \overset{f}{\to} T \to 0$$

where $f(e_j) = a_j$ for each $j = 1, 2, \ldots, n$. Since $xa = 0$, by (4), $\sum_{j=1}^n (v_j \otimes f(e_j)) = \sum_{j=1}^n (v_j \otimes \otimes a_j) = 0$ as an element in $V \otimes_R T$. So in the exact sequence

$$V \otimes K \overset{1_V \otimes i_K}{\to} V \otimes R^n \overset{1_V \otimes f}{\to} V \otimes T \to 0$$

we have $\sum_{j=1}^n (v_j \otimes e_j) \in \mathrm{Ker}(1_V \otimes f) = \mathrm{Im}(1_V \otimes i_K)$. Thus there exist $u_i \in V$, $k_i \in K$, $i = 1, 2, \ldots, m$, such that $\sum_{j=1}^n (v_j \otimes e_j) = \sum_{i=1}^m (u_i \otimes k_i)$. Let $k_i = \sum_{i=1}^n c_{ij} e_j$, $j = 1, 2, \ldots, m$. Then $\sum_{j=1}^n c_{ij} a_j = \sum_{j=1}^n c_{ij} f(e_j) = f(k_i) = 0$, $i = 1, 2, \ldots, m$. Write $C = (c_{ij})_{mn}$, then $Ca = 0$.

Moreover, this also gives $\sum_{j=1}^{n}(v_j\otimes e_j)=\sum_{i=1}^{m}(u_i\otimes k_i)=\sum_{i=1}^{m}\left(u_i\otimes\left(\sum_{j=1}^{n}c_{ij}e_j\right)\right)=$ $=\sum_{j=1}^{n}\left(\left(\sum_{i=1}^{m}u_ic_{ij}\right)\otimes e_j\right)$. So $v_j=\sum_{i=1}^{m}u_ic_{ij}$, $j=1,2,\ldots,n$. Let $y=(u_1,u_2,\ldots,u_m)$, then $y\in V^m$ and $x=yC$.

(5) ⇒ (4). Let $T=\sum_{j=1}^{n}Rb_j$ be an n-generated left ideal in I and suppose $a_i=\sum_{j=1}^{n}r_{ij}b_j\in$ $\in T$, $v_i\in V$ with $\sum_{i=1}^{k}v_ia_i=0$. Then $\sum_{j=1}^{n}\left(\sum_{i=1}^{k}v_ir_{ij}\right)b_j=0$. By (5), there exists elements $u_1,\ldots,u_m\in V$ and elements $c_{ij}\in R$, $i=1,\ldots,m$, $j=1,\ldots,n$, such that $\sum_{j=1}^{n}c_{ij}b_j=0$, $i=1,\ldots,m$, and $\sum_{i=1}^{m}u_ic_{ij}=\sum_{i=1}^{k}v_ir_{ij}$, $j=1,\ldots,n$. Thus, $\sum_{i=1}^{k}v_i\otimes a_i=\sum_{i=1}^{k}v_i\otimes$ $\otimes\left(\sum_{j=1}^{n}r_{ij}b_j\right)=\sum_{j=1}^{n}\left(\sum_{i=1}^{k}v_ir_{ij}\right)\otimes b_j=\sum_{j=1}^{n}\left(\sum_{i=1}^{m}u_ic_{ij}\right)\otimes b_j=\sum_{i=1}^{m}\Big(u_i\otimes$ $\otimes\sum_{j=1}^{n}c_{ij}b_j\Big)=0$. Thus (4) is proved.

Theorem 3.1 is proved.

***Corollary* 3.1.** *For a right R-module V, the following statements are equivalent*:

(1) *V is n-flat.*

(2) $\mathrm{Tor}_1(V,R/T)=0$ *for every n-generated left ideal T.*

(3) *V^+ is n-injective.*

(4) *For every n-generated left ideal T of R, the map $\mu_T\colon V\otimes T\to VT$; $\sum v_i\otimes x_i\mapsto\sum v_ix_i$ is a monomorphism.*

(5) *For all $x\in V^n$, $a\in R_n$, if $xa=0$, then exist positive integer m and $y\in V^m$, $C\in R^{m\times n}$, such that $Ca=0$ and $x=yC$.*

***Corollary* 3.2.** *Let R be a left I-semiregular ring. Then*:

(1) *A right R-module M is n-flat if and only if M is I-n-flat.*

(2) *A right R-module M is flat if and only if M is I-flat.*

Proof. (1) follows from Corollary 3.1, Theorems 2.3 and 3.1.

(2) follows from (1).

***Corollary* 3.3.** *Let R be a left semiregular ring. Then*:

(1) *A right R-module M is n-flat if and only if M is J-n-flat.*

(2) *A right R-module M is flat if and only if M is J-flat.*

We note that Corollary 3.3(2) improves the result of [8] (Proposition 2.17).

***Corollary* 3.4.** *Let $\{M_\alpha\}_{\alpha\in A}$ be a family of right R-modules and n be a positive integer. Then*

(1) $\bigoplus\limits_{\alpha\in A}M_\alpha$ *is I-n-flat if and only if each M_α is I-n-flat.*

(2) $\prod_{\alpha\in A}M_\alpha$ *is I-n-injective if and only if each M_α is I-n-injective.*

Proof. (1) follows from the isomorphism $\mathrm{Tor}_1\left(\bigoplus\limits_{\alpha\in A}M_\alpha,N\right)\cong\bigoplus\limits_{\alpha\in A}\mathrm{Tor}_1(M_\alpha,N)$.

(2) follows from the isomorphism $\mathrm{Ext}^1\left(N,\prod_{\alpha\in A}M_\alpha\right)\cong\prod_{\alpha\in A}\mathrm{Ext}^1(N,M_\alpha)$.

***Remark* 3.1.** From Theorem 3.1, the I-n-flatness of V_R can be characterized by the I-n-injectivity of V^+. On the other hand, by [5] (Lemma 2.7(1)), the sequence $\mathrm{Tor}_1(V^+,M)\to\mathrm{Ext}^1(M,V)^+\to 0$ is exact for all finitely presented left R-module M, so if V^+ is I-n-flat, then V is I-n-injective.

Theorem 3.2. *Every pure submodule of an I-n-flat module is I-n-flat. In particular, pure submodules of n-flat modules are n-flat.*

Proof. Let A be a pure submodule of an I-n-flat right R-module B. Then the pure exact sequence $0 \to A \to B \to B/A \to 0$ induces a split exact sequence $0 \to (B/A)^+ \to B^+ \to A^+ \to 0$. Since B is I-n-flat, by Theorem 3.1, B^+ is I-n-injective, and so A^+ is I-n-injective. Thus A is I-n-flat by Theorem 3.1 again.

Definition 3.2. *Given a right R-module U with submodule U'. If $a = (a_1, a_2, \ldots, a_n)' \in R_n$ and $T = \sum_{i=1}^n Ra_i$, then U' is called a-pure in U if the canonical map $U' \otimes_R R/T \to U \otimes_R R/T$ is a monomorphism*; *U' is called I-n-pure in U if U' is a-pure in U for every $a \in I_n$. U' is called I-P-pure in U if U' is I-1-pure in U.*

Clearly, if U' is I-n-pure in U then U' is I-m-pure in U for every positive integer $m \leq n$.

Theorem 3.3. *Let $U'_R \leq U_R$ and $a = (a_1, a_2, \ldots, a_n)' \in R_n$, $T = \sum_{i=1}^n Ra_i$. Then the following statements are equivalent*:

(1) *U' is a-pure in U.*

(2) *The canonical map $\mathrm{Tor}_1(U, R/T) \to \mathrm{Tor}_1(U/U', R/T)$ is surjective.*

(3) *$U' \cap U^n a = (U')^n a$.*

(4) *$U' \cap UT = U'T$.*

(5) *The canonical map $\mathrm{Hom}_R(R_n/aR, U) \to \mathrm{Hom}_R(R_n/aR, U/U')$ is surjective.*

(6) *Every commutative diagram*

$$\begin{array}{ccc} aR & \xrightarrow{i_{aR}} & R_n \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ U' & \xrightarrow{i_{U'}} & U \end{array}$$

there exists $h\colon R_n \to U'$ with $f = hi_{aR}$.

(7) *The canonical map $\mathrm{Ext}^1(R_n/aR, U') \to \mathrm{Ext}^1(R_n/aR, U)$ is a monomorphism.*

(8) *$\mathbf{l}_{U^n}^{U'}(a) = (U')^n + \mathbf{l}_{U^n}(a)$, where $\mathbf{l}_{U^n}^{U'}(a) = \{x \in U^n | xa \in U'\}$.*

Proof. (1) $\Leftrightarrow$ (2). This follows from the exact sequence

$$\mathrm{Tor}_1(U, R/T) \to \mathrm{Tor}_1(U/U', R/T) \to U' \otimes R/T \to U \otimes R/T.$$

(1) $\Rightarrow$ (3). Suppose that $x \in U' \cap U^n a$. Then there exists $y = (y_1, y_2, \ldots, y_n) \in U^n$ such that $x = ya$, and so we have $x \otimes \left(1 + \sum_{i=1}^n Ra_i\right) = \left(\sum_{i=1}^n y_i a_i\right) \otimes \left(1 + \sum_{i=1}^n Ra_i\right) = \sum_{i=1}^n (y_i \otimes 0) = 0$ in $U \otimes \left(R/\sum_{i=1}^n Ra_i\right)$. Since U' is a-pure in U, $x \otimes \left(1 + \sum_{i=1}^n Ra_i\right) = 0$ in $U' \otimes \left(R/\sum_{i=1}^n Ra_i\right)$. Let $\iota\colon \sum_{i=1}^n Ra_i \to R$ be the inclusion map and $\pi\colon R \to R/\sum_{i=1}^n Ra_i$ be the natural epimorphism. Then we have $x \otimes 1 \in \mathrm{Ker}(1_{U'} \otimes \pi) = \mathrm{im}\,(1_{U'} \otimes \iota)$, it follows that there exists $x'_i \in U'$, $i = 1, 2, \ldots, n$, such that $x \otimes 1 = \sum_{i=1}^n x'_i \otimes a_i = \left(\sum_{i=1}^n x'_i a_i\right) \otimes 1$ in $U' \otimes R$, and so $x = \sum_{i=1}^n x'_i a_i \in (U')^n a$. But $(U')^n a \subseteq U' \cap U^n a$, so $U' \cap U^n a = (U')^n a$.

(3) $\Leftrightarrow$ (4) is obvious.

(3) $\Rightarrow$ (5). Consider the following diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & aR & \xrightarrow{i_{aR}} & R_n & \xrightarrow{\pi_2} & R_n/aR & \longrightarrow & 0 \\ & & & & & & \downarrow{\scriptstyle f} & & \\ 0 & \longrightarrow & U' & \xrightarrow{i_{U'}} & U & \xrightarrow{\pi_1} & U/U' & \longrightarrow & 0 \end{array}$$

where $f \in \mathrm{Hom}_R(R_n/aR, U/U')$. Since R_n is projective, there exist $g \in \mathrm{Hom}_R(R_n, U)$ and $h \in$ $\in \mathrm{Hom}_R(aR, U')$ such that the diagram commutes. Now let $u = g(a)$, Then $u = g(a) = h(a) \in U'$. Note that $u = (g(e_1), g(e_2), \ldots, g(e_n))a \in U^n a$, where $e_i \in R_n$, with 1 in the ith position and 0's in all other positions. By (3), $u \in (U')^n a$. Therefore, $u = \sum_{i=1}^{n} u_i' a_i$ for some $u_i' \in U'$, $i = 1, 2, \ldots, n$. Define $\sigma \in \mathrm{Hom}_R(R_n, U')$ such that $\sigma(e_i) = u_i'$, $i = 1, 2, \ldots, n$, then $\sigma i_{aR} = h$. Finally, we define $\tau \colon R_n/aR \to U$ by $\tau(x + aR) = g(x) - \sigma(x)$, then τ is a well-defined right R-homomorphism and $\pi_1 \tau = f$. Whence $\mathrm{Hom}_R(R_n/aR, U) \to \mathrm{Hom}_R(R_n/aR, U/U')$ is surjective.

(5) $\Rightarrow$ (3). Suppose that $x \in U' \cap U^n a$. Then $x = ya$ for some $y = (y_1, y_2, \ldots, y_n) \in U^n$. Thus we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & aR & \xrightarrow{i_{aR}} & R_n & \xrightarrow{\pi_2} & R_n/aR & \longrightarrow & 0 \\
 & & \big\downarrow f_1 & & \big\downarrow f_2 & & & & \\
0 & \longrightarrow & U' & \xrightarrow{i_{U'}} & U & \xrightarrow{\pi_1} & U/U' & \longrightarrow & 0
\end{array}$$

where f_2 is defined by $f_2(e_i) = y_i$, $i = 1, 2, \ldots, n$ and $f_1 = f_2|_{aR}$. Define $f_3 \colon R_n/aR \to U/U'$ by $f_3(z + aR) = \pi_1 f_2(z)$. It is easy to see that f_3 is well defined and $f_3 \pi_2 = \pi_1 f_2$. By hypothesis, $f_3 = \pi_1 \tau$ for some $\tau \in \mathrm{Hom}_R(R_n/aR, U)$. Now we define $\sigma \colon R_n \to U'$ by $\sigma(z) = f_2(z) - \tau\pi_2(z)$. Then $\sigma \in \mathrm{Hom}_R(R_n, U')$ and $\sigma(a) = f_2(a)$ since $\pi_2(a) = 0$. Hence $x = f_2(a) = \sigma(a) =$ $= (\sigma(e_1), \sigma(e_2), \ldots, \sigma(e_n))a \in (U')^n a$. Therefore $U' \cap U^n a = (U')^n a$.

(3) $\Rightarrow$ (1). Suppose that $\sum_{k=1}^{s} u_k' \otimes \left(b_k + \sum_{i=1}^{n} Ra_i\right) = 0$ in $U \otimes \left(R / \sum_{i=1}^{n} Ra_i\right)$, where $u_k' \in U'$, $b_k \in R$, then $\left(\sum_{k=1}^{s} u_k' b_k\right) \otimes \left(1 + \sum_{i=1}^{n} Ra_i\right) = 0$ in $U \otimes \left(R / \sum_{i=1}^{n} Ra_i\right)$. By the exactness of the sequence $U \otimes \left(\sum_{i=1}^{n} Ra_i\right) \to U \otimes R \to U \otimes \left(R / \sum_{i=1}^{n} Ra_i\right) \to 0$, we have that $\sum_{k=1}^{s} u_k' b_k = xa$ for some $x \in U^n$. By (3), there exists some $y \in (U')^n$ such that $\sum_{k=1}^{s} u_k' b_k = ya$. Thus, $\sum_{k=1}^{s} u_k' \otimes \left(b_k + \sum_{i=1}^{n} Ra_i\right) = ya \otimes \left(1 + \sum_{i=1}^{n} Ra_i\right) = 0$ in $U' \otimes$ $\otimes \left(R / \sum_{i=1}^{n} Ra_i\right)$.

(5) $\Leftrightarrow$ (6). By diagram lemma (see [21, p. 53]).

(5) $\Leftrightarrow$ (7). It follows from the exact sequence

$$\mathrm{Hom}_R(R_n/aR, U) \to \mathrm{Hom}_R(R_n/aR, U/U') \to \mathrm{Ext}^1(R_n/aR, U') \to \mathrm{Ext}^1(R_n/aR, U).$$

(5) $\Rightarrow$ (8). It is sufficient to show that $\mathbf{l}_{U^n}^{U'}(a) \subseteq (U')^n + \mathbf{l}_{U^n}(a)$. Let $x = (x_1, x_2, \ldots, x_n) \in$ $\in \mathbf{l}_{U^n}^{U'}(a)$. Define $f \colon R_n/aR \to U/U'$ via $\alpha + aR \mapsto x\alpha + U'$, then $f \in \mathrm{Hom}_R(R_n/aR, U/U')$. By (5), $f = \pi g$ for some $g \in \mathrm{Hom}_R(R_n/aR, U)$, where $\pi \colon U \to U/U'$ is the natural epimorphism. Let $g(e_i + aR) = y_i$, $i = 1, 2, \ldots, n$, $y = (y_1, y_2, \ldots, y_n)$. Then $y \in \mathbf{l}_{U^n}(a)$, $x_i + U' = f(e_i + aR) =$ $= \pi g(e_i + aR) = y_i + U'$, and so $x_i - y_i \in U'$, $i = 1, 2, \ldots, n$, this implies that $x - y \in (U')^n$. Therefore, $x = (x - y) + y \in (U')^n + \mathbf{l}_{U^n}(a)$.

(8) $\Rightarrow$ (6). Let $x = (g(e_1), g(e_2), \ldots, g(e_n))$. Then $xa = g(a) = f(a) \in U'$, so $x \in \mathbf{l}_{U^n}^{U'}(a)$. By (8), $x = y + z$ for some $y \in (U')^n$ and $z \in \mathbf{l}_{U^n}(a)$. Now we define $h \colon R_n \to U'$; $b \mapsto yb$, then $h(a) = ya = xa = f(a)$. And thus $f = h i_{aR}$.

Theorem 3.3 is proved.

Let M be a right R-module, K be a submodule of M and X a subset of M, then we write $X/K = \{x + K | x \in X\}$.

***Corollary* 3.5.** *Suppose that E, F and G are right R-modules such that $E \subseteq F \subseteq G$, and $a \in R_n$. Then*:

(1) *If E is a-pure in F and F is a-pure in G, then E is a-pure in G.*

(2) *If E is a-pure in G, then E is a-pure in F.*

(3) *If F is a-pure in G, then F/E is a-pure in G/E.*

(4) *If E is a-pure in G and F/E is a-pure in G/E, then F is a-pure in G.*

Proof. (1). Since E is a-pure in F and F is a-pure in G, we have $F \cap G^n a = F^n a$ and $E \cap F^n a = E^n a$. Thus, $E \cap G^n a = E \cap (F \cap G^n a) = E \cap F^n a = E^n a$, and therefore E is a-pure in G.

(2) Since E is a-pure in G, $E \cap G^n a = E^n a$. Note that $E \cap G^n a \supseteq E \cap F^n a \supseteq E^n a$, we get that $E \cap F^n a = E^n a$, and (2) follows.

(3) Since F is a-pure in G, $F \cap G^n a = F^n a$, and so $(F/E) \cap (G/E)^n a = (F \cap G^n a)/E = = (F^n a)/E = (F/E)^n a$. This follows that F/E is a-pure in G/E.

(4) By hypothesis, we have $(F/E) \cap (G/E)^n a = (F/E)^n a$, i.e., $(F \cap G^n a)/E = (F^n a)/E$, and $E \cap G^n a = E^n a$. For any $f \in F \cap G^n a$, write $f = ga$, where $g \in G^n$. Then there exists $f_1 \in F^n$ such that $(g - f_1)a = ga - f_1 a = f - f_1 a \in E \cap G^n a = E^n a$, so $f - f_1 a = ea$ for some $e \in E^n$. This implies that $f = f_1 a + ea = (f_1 + e)a \in F^n a$, and hence F is a-pure in G.

***Corollary* 3.6.** *Let $U'_R \leq U_R$ and $a \in R$. Then the following statements are equivalent*:

(1) *U' is a-pure in U.*

(2) *The canonical map $\operatorname{Tor}_1(U, R/Ra) \to \operatorname{Tor}_1(U/U', R/Ra)$ is surjective.*

(3) *$U' \cap Ua = U'a$.*

(4) *The canonical map $\operatorname{Hom}_R(R/aR, U) \to \operatorname{Hom}_R(R/aR, U/U')$ is surjective.*

(5) *Every commutative diagram*

$$\begin{array}{ccc} aR & \xrightarrow{i_{aR}} & R \\ f\big\downarrow & & \big\downarrow g \\ U' & \xrightarrow{i_{U'}} & U \end{array}$$

there exists $h\colon R \to U'$ with $f = hi_{aR}$.

(6) *The canonical map $\operatorname{Ext}^1(R/aR, U') \to \operatorname{Ext}^1(R/aR, U)$ is a monomorphism.*

(7) *$\mathbf{l}_U^{U'}(a) = U' + \mathbf{l}_U(a)$, where $\mathbf{l}_U^{U'}(a) = \{x \in U \mid xa \in U'\}$.*

***Corollary* 3.7.** *Let U be an n-generated right R-module with submodule U'. If U' is I-n-pure in U, then U' is I-m-pure in U for each positive integer m. In particular, if a right ideal T of R is I-P-pure in R, then it is I-m-pure in R for each positive integer m.*

Proof. For any $a \in I_m$, if $x \in U' \cap U^m a$, then $x = (x_1, x_2, \ldots, x_m)a$, where each $x_i \in U$. Suppose that $u_1, u_2, \ldots, u_n$ is a generating set of U. Then $(x_1, x_2, \ldots, x_m) = (u_1, u_2, \ldots, u_n)C$ for some $C \in R^{n \times m}$, and so $x = (u_1, u_2, \ldots, u_n)(Ca) \in U' \cap U^n(Ca)$. Since U' is I-n-pure in U, by Theorem 3.3, $x \in (U')^n(Ca) = ((U')^n C)a \subseteq (U')^m a$. Thus $U' \cap U^m a = (U')^m a$ and therefore U' is I-m-pure in U.

Proposition 3.1. *Let $U'_R \leq U_R$.*

(1) *If U/U' is I-n-flat, then U' is I-n-pure in U.*

(2) *If U' is I-n-pure in U and U is I-n-flat, then U/U' is I-n-flat.*

Proof. It follows from the exact sequence

$$\mathrm{Tor}_1(U, R/T) \to \mathrm{Tor}_1(U/U', R/T) \to U' \otimes R/T \to U \otimes R/T$$

and Theorem 3.1(2).

Theorem 3.4. *n-Generated I-n-flat module is I-flat.*

Proof. Suppose V is an n-generated I-n-flat module, there exists an exact sequence $0 \to K \to \to F \to V \to 0$ with F free and $\mathrm{rank}(F) = n$. Then K is I-n-pure in F by Proposition 3.1(1) and hence I-m-pure for every positive integer m by Corollary 3.7. So, by Proposition 3.1(2), V is I-m-flat for every positive integer m. Hence, V is I-flat.

Theorem 3.4 is proved.

***Corollary* 3.8.** (1) *n-Generated n-flat module is flat.*

(2) *I-P-flat cyclic module is I-flat.*

4. *I-n*-coherent rings.

Definition 4.1. *A ring R is called left I-n-coherent if every n-generated left ideal in I is finitely presented.*

Clearly, a ring R is left n-coherent if and only if R is left R-n-coherent.

Lemma 4.1. *Let $a \in R_n$. Then $\mathbf{l}_{R^n}(a) \cong P^*$, where $P = R_n/aR$.*

Proof. This is a corollary of [23] (Lemma 5.3).

Theorem 4.1. *The following statements are equivalent for a ring R:*

(1) *R is left I-n-coherent.*

(2) *If $0 \to K \xrightarrow{f} R^n \xrightarrow{g} I$ is an exact sequence of left R-modules, then K is finitely generated.*

(3) *$\mathbf{l}_{R^n}(a)$ is a finitely generated submodule of R^n for any $a \in I_n$.*

(4) *For any $a \in I_n$, $(R_n/aR)^*$ is finitely generated.*

Proof. (1) $\Rightarrow$ (2). Since R is left I-n-coherent and $\mathrm{Im}(g)$ is an n-generated left ideal in I, $\mathrm{Im}(g)$ is finitely presented. Noting that the sequence $0 \to \mathrm{Ker}(g) \to R^n \to \mathrm{Im}(g) \to 0$ is exact, so $\mathrm{Ker}(g)$ is finitely generated. Thus $K \cong \mathrm{Im}(f) = \mathrm{Ker}(g)$ is finitely generated.

(2) $\Rightarrow$ (3). Let $a = (a_1, \ldots, a_n)'$. Then we have an exact sequence of left R-modules $0 \to \to \mathbf{l}_{R^n}(a) \to R^n \xrightarrow{g} I$, where $g(r_1, \ldots, r_n) = \sum_{i=1}^{n} r_i a_i$. By (2), $\mathbf{l}_{R^n}(a)$ is a finitely generated left R-module.

(3) $\Rightarrow$ (1) is obvious. (3) $\Leftrightarrow$ (4) follows from Lemma 4.1.

Theorem 4.1 is proved.

Let $\mathcal{F}$ be a class of right R-modules and M a right R-module. Following [10], we say that a homomorphism $\varphi\colon M \to F$ where $F \in \mathcal{F}$ is an $\mathcal{F}$-preenvelope of M if for any morphism $f\colon M \to F'$ with $F' \in \mathcal{F}$, there is a $g\colon F \to F'$ such that $g\varphi = f$. An $\mathcal{F}$-preenvelope $\varphi\colon M \to F$ is said to be an $\mathcal{F}$-envelope if every endomorphism $g\colon F \to F$ such that $g\varphi = \varphi$ is an isomorphism. Dually, we have the definitions of an $\mathcal{F}$-precover and an $\mathcal{F}$-cover. $\mathcal{F}$-envelopes ($\mathcal{F}$-covers) may not exist in general, but if they exist, they are unique up to isomorphism.

Theorem 4.2. *The following statements are equivalent for a ring R:*

(1) *R is left I-n-coherent.*

(2) $\varinjlim \mathrm{Ext}^1(R/T, M_\alpha) \cong \mathrm{Ext}^1(R/T, \varinjlim M_\alpha)$ *for every n-generated left ideal T in I and direct system $(M_\alpha)_{\alpha \in A}$ of left R-modules.*

(3) $\mathrm{Tor}_1\left(\prod N_\alpha, R/T\right) \cong \prod \mathrm{Tor}_1(N_\alpha, R/T)$ *for any family $\{N_\alpha\}$ of right R-modules and any n-generated left ideal T in I.*

(4) *Any direct product of copies of R_R is I-n-flat.*

(5) *Any direct product of I-n-flat right R-modules is I-n-flat.*

(6) *Any direct limit of I-n-injective left R-modules is I-n-injective.*

(7) *Any direct limit of injective left R-modules is I-n-injective.*

(8) *A left R-module M is I-n-injective if and only if M^+ is I-n-flat.*

(9) *A left R-module M is I-n-injective if and only if M^{++} is I-n-injective.*

(10) *A right R-module M is I-n-flat if and only if M^{++} is I-n-flat.*

(11) *For any ring S, $\mathrm{Tor}_1(\mathrm{Hom}_S(B,C),R/T) \cong \mathrm{Hom}_S(\mathrm{Ext}^1(R/T,B),C)$ for the situation $({}_R(R/T), {}_RB_S, C_S)$ with T n-generated left ideal in I and C_S injective.*

(12) *Every right R-module has an I-n-flat preenvelope.*

(13) *For any $U \in I_n, U(R)$ is a finitely generated left ideal, where $U(R) = \{r \in R\colon (r, r_2, \ldots \ldots, r_n)U = 0$ for some $r_2, \ldots, r_n \in R\}$.*

Proof. (1) ⇒ (2) follows from [5] (Lemma 2.9(2)).

(1) ⇒ (3) follows from [5] (Lemma 2.10(2)).

(2) ⇒ (6) ⇒ (7); (3) ⇒ (5) ⇒ (4) are trivial.

(7) ⇒ (1). Let T be an n-generated left ideal in I and $(M_\alpha)_{\alpha\in A}$ a direct system of injective left R-modules (with A directed). Then $\varinjlim M_\alpha$ is I-n-injective by (7), and so $\mathrm{Ext}^1(R/T, \varinjlim M_\alpha) = 0$. Thus we have a commutative diagram with exact rows:

$$\begin{array}{ccccccc}
\varinjlim \mathrm{Hom}(R/T, M_\alpha) & \longrightarrow & \varinjlim \mathrm{Hom}(R, M_\alpha) & \longrightarrow & \varinjlim \mathrm{Hom}(T, M_\alpha) & \longrightarrow & 0 \\
\downarrow f & & \downarrow g & & \downarrow h & & \\
\mathrm{Hom}(R/T, \varinjlim M_\alpha) & \longrightarrow & \mathrm{Hom}(R, \varinjlim M_\alpha) & \longrightarrow & \mathrm{Hom}(T, \varinjlim M_\alpha) & \longrightarrow & 0.
\end{array}$$

Since f and g are isomorphism by [21] (25.4(d)), h is an isomorphism by the Five lemma. So T is finitely presented by [21] (25.4(e)) again. Hence R is left I-n-coherent.

(4) ⇒ (1). Let T be an n-generated left ideal in I. By (4), $\mathrm{Tor}_1(\Pi R, R/T) = 0$. Thus we have a commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (\Pi R)\otimes T & \longrightarrow & (\Pi R)\otimes R & \longrightarrow & (\Pi R)\otimes R/T & \longrightarrow & 0 \\
 & & \downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 & & \\
0 & \longrightarrow & \Pi T & \longrightarrow & \Pi R & \longrightarrow & \Pi(R/T) & \longrightarrow & 0
\end{array}$$

Since f_3 and f_2 are isomorphism by [10] (Theorem 3.2.22), f_1 is an isomorphism by the Five lemma. So T is finitely presented by [10] (Theorem 3.2.22) again. Hence R is left I-n-coherent.

(5) ⇒ (12). Let N be any right R-module. By [10] (Lemma 5.3.12), there is a cardinal number $\aleph_\alpha$ dependent on $\mathrm{Card}(N)$ and $\mathrm{Card}(R)$ such that for any homomorphism $f\colon N \to F$ with F I-n-flat, there is a pure submodule S of F such that $f(N) \subseteq S$ and $\mathrm{Card}\, S \le \aleph_\alpha$. Thus f has a factorization $N \to S \to F$ with S I-n-flat by Theorem 3.2. Now let $\{\varphi_\beta\}_{\beta\in B}$ be all such homomorphisms $\varphi_\beta\colon N \to S_\beta$ with $\mathrm{Card}\, S_\beta \le \aleph_\alpha$ and S_β I-n-flat. Then any homomorphism $N \to F$ with F I-n-flat has a factorization $N \to S_i \to F$ for some $i \in B$. Thus the homomorphism $N \to \Pi_{\beta\in B} S_\beta$ induced by all φ_β is an I-n-flat preenvelope since $\Pi_{\beta\in B} S_\beta$ is I-n-flat by (5).

(12) ⇒ (5) follows from [4] (Lemma 1).

(1) ⇒ (11). For any n-generated left ideal T in I, since R is left I-n-coherent, R/T is 2-presented. And so (11) follows from [5] (Lemma 2.7(2)).

(11) ⇒ (8). Let $S = \mathbb{Z}$, $C = \mathbb{Q}/\mathbb{Z}$ and $B = M$. Then $\mathrm{Tor}_1(M^+, R/T) \cong \mathrm{Ext}^1(R/T, M)^+$ for any n-generated left ideal T in I by (11), and hence (8) holds.

(8) $\Rightarrow$ (9). Let M be a left R-module. If M is I-n-injective, then M^+ is I-n-flat by (8), and so M^{++} is I-n-injective by Theorem 3.1. Conversely, if M^{++} is I-n-injective, then M, being a pure submodule of M^{++} (see [20, p. 48], Exercise 41), is I-n-injective by Theorem 2.3.

(9) $\Rightarrow$ (10). If M is an I-n-flat right R-module, then M^+ is an I-n-injective left R-module by Theorem 3.1, and so M^{+++} is I-n-injective by (9). Thus M^{++} is I-n-flat by Theorem 3.1 again. Conversely, if M^{++} is I-n-flat, then M is I-n-flat by Theorem 3.2 as M is a pure submodule of M^{++}.

(10) $\Rightarrow$ (5). Let $\{N_\alpha\}_{\alpha\in A}$ be a family of I-n-flat right R-modules. Then by Corollary 3.4(1), $\bigoplus\limits_{\alpha\in A} N_\alpha$ is I-n-flat, and so $\left(\prod_{\alpha\in A} N_\alpha^+\right)^+ \cong \left(\bigoplus\limits_{\alpha\in A} N_\alpha\right)^{++}$ is I-n-flat by (10). Since $\bigoplus\limits_{\alpha\in A} N_\alpha^+$ is a pure submodule of $\prod_{\alpha\in A} N_\alpha^+$ by [3] (Lemma 1(1)), $\left(\prod_{\alpha\in A} N_\alpha^+\right)^+ \to \left(\bigoplus_{\alpha\in A} N_\alpha^+\right)^+ \to 0$ split, and hence $\left(\bigoplus\limits_{\alpha\in A} N_\alpha^+\right)^+$ is I-n-flat. Thus $\prod_{\alpha\in A} N_\alpha^{++} \cong \left(\bigoplus\limits_{\alpha\in A} N_\alpha^+\right)^+$ is I-n-flat. Since $\prod_{\alpha\in A} N_\alpha$ is a pure submodule of $\prod_{\alpha\in A} N_\alpha^{++}$ by [3] (Lemma 1(2)), $\prod_{\alpha\in A} N_\alpha$ is I-n-flat by Theorem 3.2.

(1) $\Rightarrow$ (13). Let $U = (u_1, u_2, \ldots, u_n)' \in I_n$. Write $T_1 = Ru_1 + Ru_2 + \ldots + Ru_n$ and $T_2 = Ru_2 + \ldots + Ru_n$. Then $R/U(R) \cong T_1/T_2$. By (1), T_1 is finitely presented, and so T_1/T_2 is finitely presented. Therefore $U(R)$ is finitely generated.

(13) $\Rightarrow$ (1). Let $T_1 = Ru_1 + Ru_2 + \ldots + Ru_n$ be an n-generated left ideal in I. Let $T_2 = Ru_2 + \ldots + Ru_n$, $T_3 = Ru_3 + \ldots + Ru_n, \ldots, T_n = Ru_n$. Then $T_1/T_2 \cong R/U(R)$ is finitely presented by (13). Similarly, $T_2/T_3, \ldots, T_{n-1}/T_n, T_n$ are finitely presented. Hence T_1 is finitely presented, and (1) follows.

Theorem 4.2 is proved.

***Corollary* 4.1.** *The following statements are equivalent for a ring* R:

(1) *R is left n-coherent.*

(2) $\varinjlim \operatorname{Ext}^1(R/T, M_\alpha) \cong \operatorname{Ext}^1(R/T, \varinjlim M_\alpha)$ *for every n-generated left ideal T and direct system $(M_\alpha)_{\alpha\in A}$ of left R-modules.*

(3) $\operatorname{Tor}_1(\prod N_\alpha, R/T) \cong \prod \operatorname{Tor}_1(N_\alpha, R/T)$ *for any family $\{N_\alpha\}$ of right R-modules and any n-generated left ideal T.*

(4) *Any direct product of copies of R_R is n-flat.*

(5) *Any direct product of n-flat right R-modules is n-flat.*

(6) *Any direct limit of n-injective left R-modules is n-injective.*

(7) *Any direct limit of injective left R-modules is n-injective.*

(8) *A left R-module M is n-injective if and only if M^+ is n-flat.*

(9) *A left R-module M is n-injective if and only if M^{++} is n-injective.*

(10) *A right R-module M is n-flat if and only if M^{++} is n-flat.*

(11) *For any ring S,* $\operatorname{Tor}_1(\operatorname{Hom}_S(B, C), R/T) \cong \operatorname{Hom}_S(\operatorname{Ext}^1(R/T, B), C)$ *for the situation $({}_R(R/T), {}_RB_S, C_S)$ with T n-generated left ideal and C_S injective.*

(12) *Every right R-module has an n-flat preenvelope.*

(13) *For any $U \in R_n, U(R)$ is a finitely generated left ideal, where*

$$U(R) = \{r \in R\colon (r, r_2, \ldots, r_n)U = 0 \ \textit{for some} \ r_2, \ldots, r_n \in R\}.$$

***Corollary* 4.2.** *The following statements are equivalent for a ring* R:

(1) *R is left coherent.*

(2) $\varinjlim \mathrm{Ext}^1(R/T, M_\alpha) \cong \mathrm{Ext}^1(R/T, \varinjlim M_\alpha)$ *for every finitely generated left ideal T and direct system $(M_\alpha)_{\alpha\in A}$ of left R-modules.*

(3) $\mathrm{Tor}_1\left(\prod N_\alpha, R/T\right) \cong \prod \mathrm{Tor}_1(N_\alpha, R/T)$ *for any family $\{N_\alpha\}$ of right R-modules and any finitely generated left ideal T.*

(4) *Any direct product of copies of R_R is flat.*

(5) *Any direct product of flat right R-modules is flat.*

(6) *Any direct limit of F-injective left R-modules is F-injective.*

(7) *Any direct limit of injective left R-modules is F-injective.*

(8) *A left R-module M is F-injective if and only if M^+ is flat.*

(9) *A left R-module M is F-injective if and only if M^{++} is F-injective.*

(10) *A right R-module M is flat if and only if M^{++} is flat.*

(11) *For any ring S,* $\mathrm{Tor}_1(\mathrm{Hom}_S(B, C), R/T) \cong \mathrm{Hom}_S(\mathrm{Ext}^1(R/T, B), C)$ *for the situation* $({}_R(R/T), {}_RB_S, C_S)$ *with T finitely generated left ideal and C_S injective.*

(12) *For any positive integer n and any $U \in R_n$, $U(R)$ is a finitely generated left ideal, where*

$$U(R) = \{r \in R\colon (r, r_2, \ldots, r_n)U = 0 \ \text{for some} \ r_2, \ldots, r_n \in R\}.$$

(13) *Every right R-module has a flat preenvelope.*

Proof. The equivalence of (1) – (12) is a consequence of Corollary 4.1. The proof of (5) ⇔ (13) is similar to that of (5) ⇔ (12) in the proof of Theorem 4.2.

***Corollary* 4.3.** *Let R be a left I-n-coherent ring. Then every left R-module has an I-n-injective cover.*

Proof. Let $0 \to A \to B \to C \to 0$ be a pure exact sequence of left R-modules with B I-n-injective. Then $0 \to C^+ \to B^+ \to A^+ \to 0$ is split. Since R is left I-n-coherent, B^+ is I-n-flat by Theorem 4.2, so C^+ is I-n-flat, and hence C is I-n-injective by Remark 3.1. Thus, the class of I-n-injective modules is closed under pure quotients. By [12] (Theorem 2.5), every left R-module has an I-n-injective cover.

***Corollary* 4.4.** *Let R be a left n-coherent ring. Then every left R-module has an n-injective cover.*

Proposition 4.1. *Let R be a left coherent ring. Then every left R-module has a F-injective cover.*

Proof. It is similar to the proof of Corollary 4.3.

***Corollary* 4.5.** *The following are equivalent for a left I-n-coherent ring R:*

(1) *Every I-n-flat right R-module is n-flat.*

(2) *Every I-n-injective left R-module is n-injective.*

In this case, R is left n-coherent.

Proof. (1) ⇒ (2). Let M be any I-n-injective left R-module. Then M^+ is I-n-flat by Theorem 4.2, and so M^+ is n-flat by (1). Thus M^{++} is n-injective by Corollary 3.1. Since M is a pure submodule of M^{++}, and pure submodule of an n-injective module is n-injective by Theorem 2.3, so M is n-injective.

(2) ⇒ (1). Let M be any I-n-flat right R-module. Then M^+ is I-n-injective left R-module by Theorem 3.1, and so M^+ is n-injective by (2). Thus M is n-flat by Corollary 3.1.

In this case, any direct product of n-flat right R-modules is n-flat by Theorem 4.2, and so R is left n-coherent by Corollary 4.1.

***Corollary* 4.6.** *Left I-semiregular left I-n-coherent ring is left n-coherent.*

Proof. By Corollaries 3.2(1) and 4.5.

***Corollary* 4.7.** *Semiregular left J-coherent ring is left coherent.*

Proposition 4.2. *The following statements are equivalent for a left I-n-coherent ring R:*

(1) *${}_RR$ is I-n-injective.*

(2) *Every right R-module has a monic I-n-flat preenvelope.*

(3) *Every left R-module has an epic I-n-injective cover.*

(4) *Every injective right R-module is I-n-flat.*

Proof. (1) $\Rightarrow$ (2). Let M be any right R-module. Then M has an I-n-flat preenvelope $f\colon M \to F$ by Theorem 4.2. Since $({}_RR)^+$ is a cogenerator, there exists an exact sequence $0 \to M \overset{g}{\to} \prod({}_RR)^+$. Since ${}_RR$ is I-n-injective, by Theorem 4.2, $\prod({}_RR)^+$ is I-n-flat, and so there exists a right R-homomorphism $h\colon F \to \prod({}_RR)^+$ such that $g = hf$, which shows that f is monic.

(2) $\Rightarrow$ (4). Assume (2). Then for every injective right R-module E, E has a monic I-n-flat preenvelope F, so E is isomorphism to a direct summand of F, and thus E is I-n-flat.

(4) $\Rightarrow$ (1). Since $({}_RR)^+$ is injective, by (4), it is I-n-flat. Thus ${}_RR$ is I-n-injective by Theorem 4.2.

(1) $\Rightarrow$ (3). Let M be a left R-module. Then M has an I-n-injective cover $\varphi\colon C \to M$ by Corollary 4.3. On the other hand, there is an exact sequence $F \overset{\alpha}{\to} M \to 0$ with F free. Since F is I-n-injective by (1), there exists a homomorphism $\beta\colon F \to C$ such that $\alpha = \varphi\beta$. This follows that φ is epic.

(3) $\Rightarrow$ (1). Let $f\colon N \to {}_RR$ be an epic I-n-injective cover. Then the projectivity of ${}_RR$ implies that ${}_RR$ is isomorphism to a direct summand of N, and so ${}_RR$ is I-n-injective.

Proposition 4.2 is proved.

***Corollary* 4.8.** *The following statements are equivalent for a left n-coherent ring R:*

(1) *${}_RR$ is n-injective.*

(2) *Every right R-module has a monic n-flat preenvelope.*

(3) *Every left R-module has an epic n-injective cover.*

(4) *Every injective right R-module is n-flat.*

Proposition 4.3. *The following statements are equivalent for a left coherent ring R:*

(1) *${}_RR$ is F-injective.*

(2) *Every right R-module has a monic flat preenvelope.*

(3) *Every left R-module has an epic F-injective cover.*

(4) *Every injective right R-module is flat.*

Proof. It is similar to the proof of Proposition 4.2.

5. I-n-semihereditary rings.

Definition 5.1. *A ring R is called left I-n-semihereditary if every n-generated left ideal in I is projective. A ring R is called left I-semihereditary if every finitely generated left ideal in I is projective. A ring R is called left IPP if every principal left ideal in I is projective. A ring R is called left JPP if every principal left ideal in J is projective.*

Recall that a ring R is called left PP [13] if every principal left ideal is projective. It is easy to see that a ring R is left PP if and only if R is left R-1-semihereditary, a ring R is left JPP if and only if R is left J-1-semihereditary, a ring R is left n-semihereditary if and only if R is left R-n-semihereditary, a ring R is left J-semihereditary if and only if R is left J-n-semihereditary for every positive integer n.

Theorem 5.1. *The following statements are equivalent for a ring R:*

(1) *R is a left I-n-semihereditary ring.*

(2) *R is left I-n-coherent and submodules of I-n-flat right R-modules are I-n-flat.*

(3) *R is left I-n-coherent and every right ideal is I-n-flat.*

(4) *R is left I-n-coherent and every finitely generated right ideal is I-n-flat.*

(5) *Every quotient module of an I-n-injective left R-module is I-n-injective.*

(6) *Every quotient module of an injective left R-module is I-n-injective.*

(7) *Every left R-module has a monic I-n-injective cover.*

(8) *Every right R-module has an epic I-n-flat envelope.*

(9) *For every left R-module A, the sum of an arbitrary family of I-n-injective submodules of A is I-n-injective.*

Proof. (2) $\Rightarrow$ (3) $\Rightarrow$ (4), and (5) $\Rightarrow$ (6) are trivial.

(1) $\Rightarrow$ (2). R is clearly left I-n-coherent. Let A be a submodule of an I-n-flat right R-module B and T an n-generated left ideal in I. Then T is projective by (1) and hence flat. Then the exactness of $0 = \mathrm{Tor}_2(B/A, R) \to \mathrm{Tor}_2(B/A, R/T) \to \mathrm{Tor}_1(B/A, T) = 0$ implies that $\mathrm{Tor}_2(B/A, R/T) = 0$. And thus from the exactness of the sequence $0 = \mathrm{Tor}_2(B/A, R/T) \to \mathrm{Tor}_1(A, R/T) \to \mathrm{Tor}_1(B, R/T) = 0$ we have $\mathrm{Tor}_1(A, R/T) = 0$, this follows that A is I-n-flat.

(4) $\Rightarrow$ (1). Let T be an n-generated left ideal in I. Then for any finitely generated right ideal K of R, the exact sequence $0 \to K \to R \to R/K \to 0$ implies the exact sequence $0 \to \mathrm{Tor}_2(R/K, R/T) \to \mathrm{Tor}_1(K, R/T) = 0$ since K is I-n-flat. So $\mathrm{Tor}_2(R/K, R/T) = 0$, and hence we obtain an exact sequence $0 = \mathrm{Tor}_2(R/K, R/T) \to \mathrm{Tor}_1(R/K, T) \to 0$. Thus, $\mathrm{Tor}_1(R/K, T) = 0$, and so T is a finitely presented flat left R-module. Therefore, T is projective.

(1) $\Rightarrow$ (5). Let M be an I-n-injective left R-module and N be a submodule of M. Then for any n-generated left ideal T in I, since T is projective, the exact sequence $0 = \mathrm{Ext}^1(T, N) \to \mathrm{Ext}^2(R/T, N) \to \mathrm{Ext}^2(R, N) = 0$ implies that $\mathrm{Ext}^2(R/T, N) = 0$. Thus the exact sequence $0 = \mathrm{Ext}^1(R/T, M) \to \mathrm{Ext}^1(R/T, M/N) \to \mathrm{Ext}^2(R/T, N) = 0$ implies that $\mathrm{Ext}^1(R/T, M/N) = 0$. Consequently, M/N is I-n-injective.

(6) $\Rightarrow$ (1). Let T be an n-generated left ideal in I. Then for any left R-module M, by hypothesis, $E(M)/M$ is I-n-injective, and so $\mathrm{Ext}^1(R/T, E(M)/M) = 0$. Thus, the exactness of the sequence $0 = \mathrm{Ext}^1(R/T, E(M)/M) \to \mathrm{Ext}^2(R/T, M) \to \mathrm{Ext}^2(R/T, E(M)) = 0$ implies that $\mathrm{Ext}^2(R/T, M) = 0$. Hence, the exactness of the sequence $0 = \mathrm{Ext}^1(R, M) \to \mathrm{Ext}^1(T, M) \to \mathrm{Ext}^2(R/T, M) = 0$ implies that $\mathrm{Ext}^1(T, M) = 0$, this shows that T is projective, as required.

(2), (5) $\Rightarrow$ (7). Since R is left I-n-coherent by (2), for any left R-module M, there is an I-n-injective cover $f\colon E \to M$ by Corollary 4.3. Note that $\mathrm{im}(f)$ is I-n-injective by (5), and $f\colon E \to M$ is an I-n-injective precover, so for the inclusion map $i\colon \mathrm{im}(f) \to M$, there is a homomorphism $g\colon \mathrm{im}(f) \to E$ such that $i = fg$. Hence $f = f(gf)$. Observing that $f\colon E \to M$ is an I-n-injective cover and gf is an endomorphism of E, so gf is an automorphisms of E, and hence $f\colon E \to M$ is a monic I-n-injective cover.

(7) $\Rightarrow$ (5). Let M be an I-n-injective left R-module and N be a submodule of M. By (7), M/N has a monic I-n-injective cover $f\colon E \to M/N$. Let $\pi\colon M \to M/N$ be the natural epimorphism. Then there exists a homomorphism $g\colon M \to E$ such that $\pi = fg$. Thus f is an isomorphism, and whence $M/N \cong E$ is I-n-injective.

(2) $\Leftrightarrow$ (8). By Theorem 4.2 and [4] (Theorem 2).

(5) $\Rightarrow$ (9). Let A be a left R-module and $\{A_\gamma \mid \gamma \in \Gamma\}$ be an arbitrary family of I-n-injective submodules of A . Since the direct sum of I-n-injective modules is I-n-injective and $\sum_{\gamma\in\Gamma} A_\gamma$ is a homomorphic image of $\oplus_{\gamma\in\Gamma} A_\gamma$, by (5), $\sum_{\gamma\in\Gamma} A_\gamma$ is I-n-injective.

(9) ⇒ (6). Let E be an injective left R-module and $K \leq E$. Take $E_1 = E_2 = E$, $N = E_1 \oplus \oplus E_2, D = \{(x, -x) \mid x \in K\}$. Define $f_1 \colon E_1 \to N/D$ by $x_1 \mapsto (x_1, 0) + D$, $f_2 \colon E_2 \to N/D$ by $x_2 \mapsto (0, x_2) + D$ and write$\overline{E}_i = f_i(E_i)$, $i = 1, 2$. Then $\overline{E}_i \cong E_i$ is injective, $i = 1, 2$, and hence $N/D = \overline{E}_1 + \overline{E}_2$ is I-n-injective. By the injectivity of $\overline{E}_i$, $(N/D)/\overline{E}_i$ is isomorphic to a summand of N/D and thus it is I-n-injective.

Theorem 5.1 is proved.

***Corollary* 5.1.** *The following statements are equivalent for a ring R*:

(1) *R is a left n-semihereditary ring.*

(2) *R is left n-coherent and submodules of n-flat right R-modules are n-flat.*

(3) *R is left n-coherent and every right ideal is n-flat.*

(4) *R is left n-coherent and every finitely generated right ideal is n-flat.*

(5) *Every quotient module of an n-injective left R-module is n-injective.*

(6) *Every quotient module of an injective left R-module is n-injective.*

(7) *Every left R-module has a monic n-injective cover.*

(8) *Every right R-module has an epic n-flat envelope.*

(9) *For every left R-module A, the sum of an arbitrary family of n-injective submodules of A is n-injective.*

Recall that a ring R is called left *P-coherent* [15] if it is left 1-coherent.

***Corollary* 5.2.** *The following statements are equivalent for a ring R*:

(1) *R is a left PP ring.*

(2) *R is left P-coherent and submodules of P-flat right R-modules are P-flat.*

(3) *R is left P-coherent and every right ideal is P-flat.*

(4) *R is left P-coherent and every finitely generated right ideal is P-flat.*

(5) *Every quotient module of a P-injective left R-module is P-injective.*

(6) *Every quotient module of an injective left R-module is P-injective.*

(7) *Every left R-module has a monic P-injective cover.*

(8) *Every right R-module has an epic P-flat envelope.*

(9) *For every left R-module A, the sum of an arbitrary family of P-injective submodules of A is P-injective.*

***Corollary* 5.3.** *The following statements are equivalent for a ring R*:

(1) *R is a left JPP ring.*

(2) *R is left J-P-coherent and submodules of J-P-flat right R-modules are J-P-flat.*

(3) *R is left J-P-coherent and every right ideal is J-P-flat.*

(4) *R is left J-P-coherent and every finitely generated right ideal is J-P-flat.*

(5) *Every quotient module of a J-P-injective left R-module is J-P-injective.*

(6) *Every quotient module of an injective left R-module is J-P-injective.*

(7) *Every left R-module has a monic J-P-injective cover.*

(8) *Every right R-module has an epic J-P-flat envelope.*

(9) *For every left R-module A, the sum of an arbitrary family of J-P-injective submodules of A is J-P-injective.*

Proposition 5.1. *Let R be an left I-semiregular ring. Then*:

(1) *R is left n-semihereditary if and only if it is left I-n-semihereditary.*

(2) *R is left semihereditary if and only if it is left I-semihereditary.*

(3) *R is left PP if and only if it is left IPP.*

Proof. (1). We need only to prove the sufficiency. Suppose R is left I-n-semihereditary, then by Theorem 5.1, every quotient module of an injective left R-module is I-n-injective. Since R is left I-semiregular, every I-n-injective left R-module is n-injective by Theorem 2.2. So every quotient module of an injective left R-module is n-injective, and hence R is left n-semihereditary by Corollary 5.1.

(2), (3) follows from (1).

Proposition 5.1 is proved.

From Proposition 5.1, we have immediately the following results.

***Corollary* 5.4.** *Let R be a semiregular ring. Then*:

(1) *R is left n-semihereditary if and only if it is left J-n-semihereditary.*

(2) *R is left semihereditary if and only if it is left J-semihereditary.*

(3) *R is left PP if and only if it is left JPP.*

6. *I*-*P*-injective rings and *I*-regular rings. In this section we extend the concept of regular rings to I-regular rings, give some characterizations of I-regular rings and I-P-injective modules, and give some properties of left I-P-injective rings.

Definition 6.1. *A ring R is called I-regular if every element in I is regular.*

Clearly, every ring is 0-regular, R is semiprimitive if and only if R is J-regular, R is regular if and only R is R-regular.

We call a module M is absolutely I-P-pure if M is I-P-pure in every module containing M.

Theorem 6.1. *Let M be a left R-module. Then the following statements are equivalent*:

(1) *M is I-P-injective.*

(2) $\mathrm{Ext}^1(R/Ra, M) = 0$ *for all $a \in I$.*

(3) $\mathbf{r}_M\mathbf{l}_R(a) = aM$ *for all $a \in I$.*

(4) $\mathbf{l}_R(a) \subseteq \mathbf{l}_R(x)$, *where $a \in I$, $x \in M$, implies $x \in aM$.*

(5) $\mathbf{r}_M(Rb \cap \mathbf{l}_R(a)) = \mathbf{r}_M(b) + aM$ *for all $a \in I$ and $b \in R$.*

(6) *If $\gamma\colon Ra \to M$, $a \in I$, is R-linear, then $\gamma(a) \in aM$.*

(7) *M is absolutely I-P-pure.*

(8) *M is I-P-pure in its injective envelope $E(M)$.*

(9) *M is an I-P-pure submodule of an I-P-injective module.*

(10) *For each $a \in I$ and any $f \in \mathrm{Hom}(Ra, M)$, if (α, g) is the pushout of (f, i) in the following diagram*:

$$\begin{array}{ccc} aR & \xrightarrow{\ i\ } & R \\ f\downarrow & & \downarrow g \\ M & \xrightarrow{\ \alpha\ } & P \end{array}$$

where i is the inclusion map, then there exists a homomorphism $h\colon P \to M$ such that $h\alpha = 1_M$.

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) $\Leftrightarrow$ (10) are follows from Theorem 2.1. (7) $\Rightarrow$ (8) $\Rightarrow$ (9) are clear.

(4) $\Rightarrow$ (6). Let $\gamma\colon Ra \to M$ be R-linear, where $a \in I$. Then $\mathbf{l}_R(a) \subseteq \mathbf{l}_R(\gamma(a))$. By (4), $\gamma(a) \in aM$.

(6) $\Rightarrow$ (1). Let $\gamma\colon Ra \to M$ be R-linear, where $a \in I$. By (6), write $\gamma(a) = am$, $m \in M$. Then $\gamma = \cdot m$, proving (1).

(2) $\Rightarrow$ (7). By Theorem 3.3(5).

(9) $\Rightarrow$ (2). Let M be an I-P-pure submodule of an I-P-injective module N. Then (2) follows from the the exact sequence

$$\mathrm{Hom}_R(R/Ra, N) \to \mathrm{Hom}_R(R/Ra, N/M) \to \mathrm{Ext}^1_R(R/Ra, M) \to 0$$

and Theorem 3.3(5).

Theorem 6.1 is proved.

***Corollary* 6.1.** *Let $R = I_1 \oplus I_2$, where I_1, I_2 are ideals of R. Then R is left P-injective if and only if ${}_RR$ is I_1-P-injective and I_2-P-injective.*

Proof. We need only to prove the sufficiency. Let $a = a_1 + a_2 \in R$, where $a_1 \in I_1$, $a_2 \in I_2$. Then by routine computations, we have $\mathbf{r}_R\mathbf{l}_R(a_1) = \mathbf{r}_{I_1}\mathbf{l}_{I_1}(a_1)$, $\mathbf{r}_R\mathbf{l}_R(a_2) = \mathbf{r}_{I_2}\mathbf{l}_{I_2}(a_2)$, $\mathbf{r}_R\mathbf{l}_R(a_1 + a_2) = \mathbf{r}_{I_1}\mathbf{l}_{I_1}(a_1) + \mathbf{r}_{I_2}\mathbf{l}_{I_2}(a_2)$, $a_1R + a_2R = (a_1 + a_2)R$. Since R is left I_1-P-injective and left I_2-P-injective, $\mathbf{r}_R\mathbf{l}_R(a_1) = a_1R$, $\mathbf{r}_R\mathbf{l}_R(a_2) = a_2R$. Hence, $\mathbf{r}_R\mathbf{l}_R(a) = aR$, which shows that R is left P-injective.

Proposition 6.1. *Let R be a left I-P-injective ring. Then*:

(1) *Every left ideal in I that is isomorphic to a direct summand of ${}_RR$ is itself a direct summand of ${}_RR$.*

(2) *If $Re \cap Rf = 0$, $e^2 = e \in R$, $f^2 = f \in I$, then $Re \oplus Rf = Rg$ for some $g^2 = g$.*

(3) *If Rk is a simple left ideal in I, then kR is a simple right ideal.*

(4) $\mathrm{Soc}({}_RI) \subseteq \mathrm{Soc}(I_R)$.

Proof. (1). If T is a left ideal in I and $T \cong Re$, where $e^2 = e \in R$, then $T = Ra$ for some $a \in T$ and T is projective. Hence $\mathbf{l}_R(a) \subseteq^{\oplus} {}_RR$, say $\mathbf{l}_R(a) = Rf$, where $f^2 = f \in R$. Then $aR = \mathbf{r}_R\mathbf{l}_R(a) = (1-f)R \subseteq^{\oplus} R_R$, and so $T = Ra \subseteq^{\oplus} {}_RR$.

(2). Observe that $Re \oplus Rf = Re \oplus Rf(1-e)$, so $Rf(1-e) \cong Rf$. Since R is left I-P-injective, by (1), $Rf(1-e) = Rh$ for some idempotent element $h \in I$. Let $g = e + h - eh$. Then $g^2 = g$ such that $ge = g = eg$ and $gh = h = hg$. It follows that $Re \oplus Rf = Re \oplus Rh = Rg$.

(3). If Rk is a simple left ideal in I, and $0 \neq ka \in kR$, define $\gamma\colon Rk \to Rka$; $rk \mapsto rka$. Then γ is an isomorphism, and so, as R is left I-P-injective, $\gamma^{-1} = \cdot c$ for some $c \in R$. Then $k = \gamma^{-1}(ka) = kac \in kaR$. Therefore, kR is a simple right ideal.

(4). It follows from (3).

Proposition 6.1 is proved.

A ring R is called *left Kasch* if every simple left R-module embeds in ${}_RR$, or equivalently, $\mathbf{r}_R(T) \neq 0$ for every maximal left ideal T of R. Right Kasch, right P-injective rings have been discussed in [19]. Next, we discuss left Kasch left I-P-injective rings.

Proposition 6.2. *Let R be a left I-P-injective left Kasch ring. Then*:

(1) $\mathrm{Soc}(I_R) \subseteq^{ess} I_R$.

(2) $\mathbf{r}_I(J) \subseteq^{ess} I_R$.

Proof. (1). If $0 \neq a \in I$, let $\mathbf{l}_R(a) \subseteq T$, where T is a maximal left ideal. Then $\mathbf{r}_R(T) \subseteq \mathbf{r}_R\mathbf{l}_R(a) = aR$, and (1) follows because $\mathbf{r}_R(T)$ is simple by [19] (Theorem 3.31).

(2). If $0 \neq b \in I$. Choose M maximal in Rb, let $\sigma\colon Rb/M \to {}_RR$ be monic, and define $\gamma\colon Rb \to {}_RR$ by $\gamma(x) = \sigma(x + M)$. Then $\gamma = \cdot c$ for some $c \in R$ by hypothesis. Hence $bc = \sigma(b + M) \neq 0$ because $b \notin M$ and σ is monic. But $Jbc = \gamma(Jb) = 0$ because $Jb \subseteq M$ (if $Jb \nsubseteq M$, then $Jb + M = Rb$. But $Jb << Rb$, so $M = Rb$, a contradition). So $0 \neq bc \in bR \cap \mathbf{r}_I(J)$, as required.

Proposition 6.2 is proved.

Recall that a left R-module M is called *mininjective* [17] if every R-homomorphism from a minimal left ideal to M extends to a homomorphism of R to M.

Proposition 6.3. *If M is a JP-injective left R-module, then it is mininjective.*

Proof. Let Ra be a minimal left ideal of R. If $(Ra)^2 \neq 0$, then exists $k \in Ra$ such that $Rak \neq 0$. Since Ra is minimal, $Rak = Ra$. Thus $k = ek$ for some $0 \neq e \in Ra$, this shows that $e^2 - e \in \mathbf{l}_{Ra}(k)$. But $\mathbf{l}_{Ra}(k) \neq Ra$ because $ek \neq 0$, and note that Ra is simple, we have $\mathbf{l}_{Ra}(k) = 0$, and so $e^2 = e$ and $Ra = Re$. Clearly, in this case, every homomorphism from Ra to M can be extended to a homomorphism of R to M. If $(Ra)^2 = 0$, then $a \in J(R)$. Since M is JP-injective, every homomorphism from Ra to M can be extended to R.

Proposition 6.3 is proved.

Theorem 6.2. *The following statements are equivalent for a ring R:*

(1) *R is an I-regular ring.*

(2) *Every left R-module is I-F-injective.*

(3) *Every left R-module is I-P-injective.*

(4) *Every cyclic left R-module is I-P-injective.*

(5) *Every left R-module is I-flat.*

(6) *Every left R-module is I-P-flat.*

(7) *Every cyclic left R-module is I-P-flat.*

(8) *R is left I-semihereditary and left I-F-injective.*

(9) *R is left IPP and left I-P-injective.*

Proof. (2) $\Rightarrow$ (3) $\Rightarrow$ (4); (5) $\Rightarrow$ (6) $\Rightarrow$ (7); and (8) $\Rightarrow$ (9) are obvious.

(1) $\Rightarrow$ (2), (5), (8). Assume (1). Then it is easy to prove by induction that every finitely generated left ideal in I is a direct summand of ${}_RR$, so (2), (5), (8) hold.

(4) $\Rightarrow$ (1). Let $a \in I$. Then by (4), Ra is I-P-injective, so that Ra is a direct summand of ${}_RR$. And thus (1) follows.

(7) $\Rightarrow$ (1). Let $a \in I$. Then by (5), R/Ra is I-P-flat. This follows that Ra is I-P-pure in R by Proposition 3.1(1). By Theorem 3.3(3), we have $Ra \bigcap aR = aRa$, and hence $a = aba$ for some $b \in R$. Therefore, R is an I-regular ring.

(9) $\Rightarrow$ (1). Let $a \in I$. Since R is left I-P-injective, $\mathbf{r}_R\mathbf{l}_R(a) = aR$ by Theorem 6.1(3). Since R is left IPP, Ra is projective, so $\mathbf{l}_R(a) = Re$ for some $e^2 = e \in R$. Thus, $aR = \mathbf{r}_R(Re) = (1-e)R$ is a direct summand of R_R, and hence a is regular.

Theorem 6.2 is proved.

***Corollary* 6.2.** *The following statements are equivalent for a ring R:*

(1) *R is a semiprimitive ring.*

(2) *Every left R-module is J-F-injective.*

(3) *Every left R-module is J-P-injective.*

(4) *Every cyclic left R-module is J-P-injective.*

(5) *Every left R-module is J-flat.*

(6) *Every left R-module is J-P-flat.*

(7) *Every cyclic left R-module is J-P-flat.*

(8) *R is left J-semihereditary and left J-F-injective.*

(9) *R is left JPP and left J-P-injective.*

***Corollary* 6.3.** *The following statements are equivalent for a ring R:*

(1) *R is a regular ring.*

(2) *Every left R-module is F-injective.*

(3) *Every left R-module is P-injective.*

(4) *Every cyclic left R-module is P-injective.*

(5) *Every left R-module is flat.*

(6) *Every left R-module is P-flat.*

(7) *Every cyclic left R-module is P-flat.*

(8) *R is left semihereditary and left F-injective.*

(9) *R is left PP and left P-injective.*

Theorem 6.3. *The following statements are equivalent for a ring R:*

(1) *R is a regular ring.*

(2) *R is a left I-semiregular I-regular ring.*

Proof. (1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (1). Let M be any left R-module. Since R is I-regular, by Theorem 6.2, M is I-P-injective. But R is left I-semiregular, by Theorem 2.2, M is P-injective. Hence, R is a regular ring by Corollary 6.3.

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