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SOME APPROXIMATION PROPERTIES OF SZASZ – MIRAKYAN – BERNSTEIN OPERATORS OF CHLODOVSKY-TYPE

ДЕЯКІ АПРОКСИМАЦІЙНІ ВЛАСТИВОСТІ ОПЕРАТОРІВ САСА – МІРАКЯНА – БЕРНШТЕЙНА ТИПУ ХЛОДОВСЬКОГО

The aim of this paper is to motivate a new sequence of positive linear operators by means of Chlodovsky-type Szasz – Mirakyan – Bernstein operators and to investigate some approximation properties of these operators in the space of continuous functions defined on the right semiaxis. We also find the order of this approximation by using the modulus of continuity and give the Voronovskaya-type theorem.

Метою даної статті є обгрунтування нової послідовності додатних лінійних операторів за допомогою операторів Саса – Міракяна – Бернштейна типу Хлодовського та дослідження деяких апроксимаційних властивостей цих операторів у просторі неперервних функцій, заданих на правій півосі. Крім того, встановлено порядок таких наближень за допомогою модуля неперервності та наведено теорему типу Вороновської.

1. Introduction. Let $\mathbb{N}$ denotes the set of natural numbers and let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let f be real-valued function defined on the closed interval $[0,1]$. The nth Bernstein operator of f, $B_n(f)$ is defined as

$$B_n(f;x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right), \qquad x \in [0,1], \quad n \in \mathbb{N},$$

where

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \qquad 0 \le k \le n. \tag{1}$$

The Bernstein polynomials $B_n(f)$ was introduced to prove the Weierstrass approximation theorem by S. N. Bernstein [1] in 1912. They have been studied intensively and their connection with different branches of analysis, such as convex and numerical analysis, total positivity and the theory of monotone operators have been investigated. Basic facts on Bernstein polynomials and their generalizations can be found in [2, 7 – 9] and references therein.

In 1937, I. Chlodovsky [3] introduced a generalization of the Bernstein polynomials for unbounded intervals. This generalization is named as the Bernstein – Chlodovsky polynomials in the literature and have the following form:

$$C_n(f;x) = \sum_{k=0}^{n} p_{n,k}\left(\frac{x}{b_n}\right) f\left(\frac{k}{n} b_n\right), \qquad x \in [0, b_n], \quad n \in \mathbb{N}_0, \tag{2}$$

where $p_{n,k}$ is defined in (1) and (b_n) is a increasing sequence of positive real numbers such that $\lim_{n\to\infty} b_n = \infty$, $\lim_{n\to\infty} b_n/n = 0$. If we take the case $b_n = 1$, $n \in \mathbb{N}_0$, these polynomials become the classical Bernstein polynomials. The approximation properties of the Bernstein – Chlodovsky polynomials can be found in [2, 3, 5, 6].

For the function f which is continuous on $[0,\infty)$, the Szasz – Mirakyan operators which are introduced by G. M. Mirakyan [10] in 1941 and then, are investigated by J. Favard [11] and O. Szasz [4], are defined as

$$S_n(f;x)=\sum_{m=0}^{\infty} q_{n,m}(x) f\left(\frac{m}{n}\right), \qquad x\in[0,\infty), \quad n\in\mathbb{N},$$

where

$$q_{n,m}(x)=e^{-nx}\frac{(nx)^m}{m!}, \qquad m\in\mathbb{N}_0. \tag{3}$$

Let I is a fixed interval (bounded or not) in $\mathbb{R}$ and μ_m be a sequence of density functions on the interval I, that is, the functions μ_m have the following properties:

(i) μ_m nonnegative for all $x\in I$ and $m\in\mathbb{N}_0$,

(ii) $\sum_{m=0}^{\infty}\mu_m(x)=1$ for all $x\in I$.

Let (L_n) be a sequence of positive linear operators defined on the set of the continuous functions on the interval I, say $C(I)$. Now we defined the new operators $\mathcal{L}_n$ on $C(I)$ by

$$\mathcal{L}_n(f;x)=\sum_{m=0}^{\infty}\mu_m(nx)L_{\varphi}(f;x), \qquad x\in I, \quad m\in\mathbb{N}_0, \quad n\in\mathbb{N}, \tag{4}$$

where μ_n are density functions on I and $\varphi:=\varphi(n,m)=\alpha_n\beta_m$ where (α_n) is a nondecreasing and (β_m) is a strictly increasing natural sequence. It is easy to check that the operators $(\mathcal{L}_n)$ are positive and linear on $C(I)$.

Taking $\beta_m=m+1$, $\mu_m=q_{n,m}$ and $L_\varphi=C_\varphi$, where C_φ and $q_{n,m}$ defined in (2) and (3) respectively, we can rewrite (4) as

$$E_n(f;x)=\sum_{m=1}^{\infty}\frac{e^{-nx}(nx)^{m-1}}{(m-1)!}\sum_{k=0}^{\alpha_n m}\binom{\alpha_n m}{k}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{\alpha_n m-k} f\left(\frac{kb_n}{m\alpha_n}\right). \tag{5}$$

The operators E_n defined in (5) is called the Szasz – Mirakyan – Bernstein operators of Chlodovsky-type (SMBC). In this study, we investigate some approximation properties of these operators and find Voronovskya-type theorem and the order of this approximation by using modulus of continuity.

2. Notations and auxiliary facts. Let $I=[0,\infty)$, and let $C(I)$ be the space of real-valued continuous function on I equipped with the uniform norm:

$$\|f\|_I:=\sup\{|f(x)|: x\in I\}$$

and $C^r(I), r\in\mathbb{N}_0$, be the set all r-times continuously differentiable functions $f\in C(I)$.

For the real-valued function f defined on I and $\delta\geq 0$, the modulus of continuity $\omega(f,\delta)$ of f with argument δ is defined by

$$\omega(f,\delta):=\sup\{|f(x+h)-f(x)|: x, x+h\in I, |h|<\delta\}.$$

For $M>0$ and $0<\mu\leq 1$, the class of the function $C(I)$ satisfying the relation $\omega(f,\delta)\leq M\delta^\mu$ for all $\delta\geq 0$, is called Lipschitz class and denoted by $\mathrm{Lip}_M\,\mu$.

Let e_r, $r\in\mathbb{N}_0$, denote the test functions defined by $e_r(x)=x^r$ and x_r denote the functions defined by $x_r(t)=(t-x)^r$ for the fixed real numbers x. By the simple calculations we have the following lemma.

Lemma 2.1. *For all $x \in [0, b_n]$, $n \in \mathbb{N}$, we have*

$$E_n(e_0; x) = 1, \qquad E_n(e_1; x) = x, \qquad E_n(e_2; x) = x^2 + \frac{(b_n - x)(1 - e^{-nx})}{n\alpha_n}$$

and for the central moments, we have

$$E_n(x_0; x) = 1, \qquad E_n(x_1; x) = 0, \qquad E_n(x_2; x) = \frac{(b_n - x)(1 - e^{-nx})}{n\alpha_n},$$

$$E_n(x_3; x) = \frac{(b_n - x)(b_n - 2x)}{n\alpha_n^2} \sum_{m=1}^{\infty} \frac{e^{-nx}(nx)^m}{m.m!},$$

$$E_n(x_4; x) = \frac{(b_n - x)(6x^2 - 6b_n x + b_n^2)}{n\alpha_n^3} \sum_{m=1}^{\infty} \frac{e^{-nx}(nx)^m}{m^2 m!} + \frac{3x(b_n - x)^2}{n\alpha_n^2} \sum_{m=1}^{\infty} \frac{e^{-nx}(nx)^m}{m.m!}.$$

By using Lemma 2.1, we have the following estimation for $E_n(x_4)$ at the point $x \in (0, b_n]$:

$$E_n(x_4; x) \le c_n(x) \left(\frac{b_n}{n\alpha_n} \right)^2, \tag{6}$$

where $\lim_{n\to\infty} c_n(x) = 6$.

3. Convergence of sequence of the operators $\boldsymbol{E_n}$. In this section, we assume that f is a function defined on the semiaxis $[0, \infty)$. The aim of this section is to preserve the relation

$$\lim_{n\to\infty} E_n(f; x) = f(x), \qquad x \in [0, \infty),$$

for the reasonably general classes of functions.

Theorem 3.1. *If $b_n = o(n)$, and the function f is bounded on the semiaxis $[0, \infty)$, then*

$$\lim_{n\to\infty} E_n(f; x) = f(x) \tag{7}$$

holds at any continuity point x of the function f.

Proof. Let $\epsilon > 0$ and let $x \in [0, \infty)$ be a continuity point of the function f, then there exist a $\delta > 0$ such that $|f(t) - f(x)| < \epsilon$ holds for all $t \in [0, \infty)$ satisfying the inequality $|t - x| < \delta$. Since the relation (7) is clear for $x = 0$, we assume that $x > 0$. Let $N \in \mathbb{N}$ such that $b_N \ge x$ so that for all $n \ge N$, $b_n \ge x$. For $n \ge N$, by using Lemma 2.1 we have

$$|E_n(f; x) - f(x)| \le \sum_{m=1}^{\infty} q_{n,m-1}(x) \sum_{k=0}^{\alpha_n m} p_{m\alpha_n,k}\left(\frac{x}{b_n}\right) \left| f\left(\frac{kb_n}{m\alpha_n}\right) - f(x) \right| \le$$

$$\le \sum_{m=1}^{\infty} q_{n,m-1}(x) \sum_{\left|\frac{kb_n}{m\alpha_n} - x\right| < \delta} p_{m\alpha_n,k}\left(\frac{x}{b_n}\right) \left| f\left(\frac{kb_n}{m\alpha_n}\right) - f(x) \right| +$$

$$+ \sum_{m=1}^{\infty} q_{n,m-1}(x) \sum_{\left|\frac{kb_n}{m\alpha_n} - x\right| \ge \delta} p_{m\alpha_n,k}\left(\frac{x}{b_n}\right) \left| f\left(\frac{kb_n}{m\alpha_n}\right) - f(x) \right|.$$

Since f is bounded on $[0, \infty)$ there is a number $M > 0$ such that $|f(t)| \leq M$, for all $t \in [0, \infty)$. Therefore, putting $t = xb_n^{-1}$ and using Lemma 2.1 and the inequality (7) in [2, p. 6], we obtain

$$|E_n(f;x) - f(x)| \leq \epsilon E_n(e_0;x) + 2M \sum_{m=1}^{\infty} q_{n,m-1}(x) \sum_{\left|\frac{kb_n}{\alpha_n m} - x\right| \geq \delta} p_{\alpha_n m,k}\left(\frac{x}{b_n}\right) =$$

$$= \epsilon + 2M \sum_{m=1}^{\infty} \frac{e^{-ntb_n}(ntb_n)^{m-1}}{(m-1)!} \sum_{\left|\frac{k}{\alpha_n m} - t\right| \geq \frac{\delta}{b_n}} p_{\alpha_n m,k}(t) \leq$$

$$\leq \epsilon + 2M \frac{t(1-t)}{\alpha_n(\delta b_n^{-1})^2} \sum_{m=1}^{\infty} \frac{e^{-ntb_n}(ntb_n)^{m-1}}{(m-1)!m} \leq$$

$$\leq \epsilon + 2M \frac{(b_n - x)}{n\alpha_n \delta^2}(1 - e^{-nx}) \leq$$

$$\leq \epsilon + \frac{2Mb_n}{\delta^2 n\alpha_n}$$

since $b_n = o(n)$, we get the desired assertion.

The Theorem 3.1 also remains true for unbounded functions which do not grow too rapidly for $x \to \infty$. Let $M(b_n) = \|f\|_{[0,b_n]}$. We will use the following lemma due to Albrycht and Radecki [12] for the proof of the following theorem.

Lemma 3.1 [12]. *For $0 < \delta \leq x < b_n$ and sufficiently large n, we have*

$$\sum_{\left|\frac{kb_n}{n} - x\right| \geq \delta} p_{n,k}\left(\frac{x}{b_n}\right) \leq 2\exp\left(-\frac{\delta^2 n}{4xb_n}\right).$$

Theorem 3.2. *Let $x \in (0, \infty)$ be a continuity point of the function f. If $b_n = o(n)$ and*

$$\lim_{n\to\infty} M(b_n) e^{-\frac{\delta^2\alpha_n}{4xb_n} - nx\left(1 - \exp\left(-\frac{\delta^2\alpha_n}{4xb_n}\right)\right)} = 0, \qquad (8)$$

then (7) *holds.*

Proof. Using the inequality in the proof of Theorem 3.1 with $M(b_n)$ instead of M and the Lemma 3.1, we have

$$|E_n(f;x) - f(x)| \leq \epsilon + 2M(b_n) \sum_{m=1}^{\infty} q_{n,m-1}(x) \sum_{\left|\frac{kb_n}{m\alpha_n} - x\right| \geq \delta} p_{\alpha_n m,k}\left(\frac{x}{b_n}\right) =$$

$$= \epsilon + 4M(b_n) \sum_{m=1}^{\infty} \frac{e^{-nx}(nx)^{m-1}}{(m-1)!} e^{-\frac{\delta^2 m\alpha_n}{4xb_n}} =$$

$$= \epsilon + 4M(b_n)e^{-nx} e^{nxe^{-\frac{\delta^2\alpha_n}{4xb_n}}} e^{-\frac{\delta^2\alpha_n}{4xb_n}} =$$

$$= \epsilon + 4M(b_n) e^{-\frac{\delta^2 \alpha_n}{4xb_n} - nx\left(1-\exp\left(-\frac{\delta^2\alpha_n}{4xb_n}\right)\right)}.$$

***Remark* 3.1.** If we assume $\alpha_n = O(1)$ in Theorem 3.2, then the condition (8) is equal to the condition

$$\lim_{n\to\infty} M(b_n) e^{-\gamma \frac{n}{b_n}}, \qquad \gamma = \frac{\delta^2}{4x},$$

which is the same with the condition for Bernstein operators of Chlodovsky-type [3].

***Remark* 3.2.** If we assume $\alpha_n = o(b_n)$, the condition

$$\lim_{n\to\infty} M(b_n) e^{-\gamma \frac{n}{b_n}}, \qquad \gamma = \frac{\delta^2}{4x},$$

and if $\alpha_n = O(b_n)$, the condition

$$\lim_{n\to\infty} M(b_n) e^{-\gamma n}, \qquad \gamma = \frac{\delta^2}{4x},$$

is equal to the condition (8).

In view of the remarks, we conclude the following: Under the condition $\alpha_n = O(b_n^{\lambda})$, $\lambda > 0$, for increasing values of λ, the relation (7) is satisfied by larger class of functions.

4. Voronovskya-type theorem.

Theorem 4.1. *Let f be defined on $(0,\infty)$ and satisfies the growth condition*

$$\lim_{n\to\infty} M(b_n) \frac{n\alpha_n}{b_n} e^{-\frac{\delta^2 \alpha_n}{4xb_n} - nx\left(1-\exp\left(-\frac{\delta^2\alpha_n}{4xb_n}\right)\right)} = 0, \qquad \delta > 0, \quad x \in (0,\infty). \tag{9}$$

Then we have

$$\lim_{n\to\infty} \frac{n\alpha_n}{b_n} [E_n(f;x) - f(x)] = \frac{1}{2} f''(x) \tag{10}$$

at each point $x > 0$ for which $f''(x)$ exists.

Proof. Let $x \le b_n$ and f has the second derivative at x. Then, by Taylor's formula, we have

$$f(t) = f(x) + (t-x) f'(x) + (t-x)^2 \left[\frac{f''(x)}{2} + h(t-x) \right] \tag{11}$$

where $h(\xi)$ tends to zero with ξ. Applying E_n to the formula (11), by Lemma 2.1, we obtain

$$E_n(f;x) = f(x) + \frac{1}{2} f''(x) \frac{(b_n - x)(1 - e^{-nx})}{n\alpha_n} + R_n(x),$$

where

$$R_n(x) := \sum_{m=1}^{\infty} q_{n,m-1}(x) \sum_{k=0}^{\alpha_n m} p_{m\alpha_n,k}\left(\frac{x}{b_n}\right) \left(\frac{kb_n}{m\alpha_n} - x\right)^2 h\left(\frac{kb_n}{m\alpha_n} - x\right).$$

To complete the proof, we have to prove that

$$\lim_{n\to\infty} \frac{n\alpha_n}{b_n} R_n(x) = 0.$$

For any $\epsilon > 0$ there exists a $\delta > 0$ such that $|h(\xi)| < \epsilon$ for $|\xi| < \delta$, and we choose δ so small that $\delta \le x$. Let us split the second sum in $R_n(x)$ into two parts as follows:

$$R_n(x) = \sum_{m=1}^{\infty} q_{n,m-1}(x) \left(\sum_{\left|\frac{kb_n}{m\alpha_n} - x\right| < \delta} + \sum_{\left|\frac{kb_n}{m\alpha_n} - x\right| \geq \delta} \right) p_{m\alpha_n,k} \times$$

$$\times \left(\frac{x}{b_n}\right)\left(\frac{kb_n}{m\alpha_n} - x\right)^2 h\left(\frac{kb_n}{m\alpha_n} - x\right) =:$$

$$=: R_{n,1}(x) + R_{n,2}(x).$$

For the sum $R_{n,1}(x)$, we have by Lemma 2.1

$$\frac{n\alpha_n}{b_n} R_{n,1}(x) < \epsilon \frac{(b_n - x)(1 - e^{-nx})}{b_n} < \epsilon.$$

Let us now estimate $R_{n,2}(x)$. If we write $t = \dfrac{kb_n}{m\alpha_n}$ in (11), we get

$$\left(\frac{kb_n}{m\alpha_n} - x\right)^2 h\left(\frac{kb_n}{m\alpha_n} - x\right) =$$

$$= f\left(\frac{kb_n}{m\alpha_n}\right) - f(x) - \left(\frac{kb_n}{m\alpha_n} - x\right) f'(x) - \left(\frac{kb_n}{m\alpha_n} - x\right)^2 \frac{f''(x)}{2}$$

and hence

$$|R_{n,2}(x)| \leq \sum_{m=1}^{\infty} q_{n,m-1}(x) \sum_{\left|\frac{kb_n}{m\alpha_n} - x\right| \geq \delta} p_{m\alpha_n,k}\left(\frac{x}{b_n}\right) \left| f\left(\frac{kb_n}{m\alpha_n}\right) - f(x) \right| +$$

$$+ \left|f'(x)\right| \sum_{m=1}^{\infty} q_{n,m-1}(x) \sum_{\left|\frac{kb_n}{m\alpha_n} - x\right| \geq \delta} p_{m\alpha_n,k}\left(\frac{x}{b_n}\right) \left|\frac{kb_n}{m\alpha_n} - x\right| +$$

$$+ \frac{|f''(x)|}{2} \sum_{m=1}^{\infty} q_{n,m-1}(x) \sum_{\left|\frac{kb_n}{m\alpha_n} - x\right| \geq \delta} p_{m\alpha_n,k}\left(\frac{x}{b_n}\right) \left(\frac{kb_n}{m\alpha_n} - x\right)^2 =:$$

$$=: \Sigma_1 + \Sigma_2 + \Sigma_3.$$

By the Lemma 3.1, we obtain

$$\Sigma_1 \leq 4M(b_n) e^{-\frac{\delta^2\alpha_n}{4xb_n} - nx\left(1 - \exp\left(-\frac{\delta^2\alpha_n}{4xb_n}\right)\right)},$$

thus

$$\frac{n\alpha_n}{b_n}\Sigma_1 \leq 4M(b_n) \frac{n\alpha_n}{b_n} e^{-\frac{\delta^2\alpha_n}{4xb_n} - nx\left(1 - \exp\left(-\frac{\delta^2\alpha_n}{4xb_n}\right)\right)}.$$

If we consider the Cauchy – Schwartz inequality, the inequality (6) and Lemma 3.1 for the second sum Σ_2, we have

$$\Sigma_2 \leq |f'(x)| \left\{ \sum_{m=1}^{\infty} q_{n,m-1}(x) \sum_{\left|\frac{kb_n}{m\alpha_n}-x\right|\geq\delta} p_{m\alpha_n,k}\left(\frac{x}{b_n}\right) \right\}^{1/2} \times$$

$$\times \left\{ \sum_{m=1}^{\infty} q_{n,m-1}(x) \sum_{\left|\frac{kb_n}{m\alpha_n}-x\right|\geq\delta} p_{m\alpha_n,k}\left(\frac{x}{b_n}\right)\left(\frac{kb_n}{m\alpha_n}-x\right)^2 \right\}^{1/2} \leq$$

$$\leq \sqrt{2}|f'(x)| e^{-\frac{\delta^2\alpha_n}{8xb_n}-\frac{nx}{2}\left(1-\exp\left(-\frac{\delta^2\alpha_n}{4xb_n}\right)\right)} \times$$

$$\times \left\{ \delta^{-2} \sum_{m=1}^{\infty} q_{n,m-1}(x) \sum_{\left|\frac{kb_n}{m\alpha_n}-x\right|\geq\delta} p_{m\alpha_n,k}\left(\frac{x}{b_n}\right)\left(\frac{kb_n}{m\alpha_n}-x\right)^4 \right\}^{1/2} \leq$$

$$\leq \sqrt{2}|f'(x)|\delta^{-1}\sqrt{c_n(x)}\frac{b_n}{n\alpha_n} e^{-\frac{\delta^2\alpha_n}{8xb_n}-\frac{nx}{2}\left(1-\exp\left(-\frac{\delta^2\alpha_n}{4xb_n}\right)\right)},$$

thus

$$\frac{n\alpha_n}{b_n}\Sigma_2 \leq \sqrt{2}|f'(x)|\delta^{-1}\sqrt{c_n(x)} e^{-\frac{\delta^2\alpha_n}{8xb_n}-\frac{nx}{2}\left(1-\exp\left(-\frac{\delta^2\alpha_n}{4xb_n}\right)\right)}.$$

By the similar arguments, we obtain the estimate

$$\frac{n\alpha_n}{b_n}\Sigma_3 \leq \frac{|f''(x)|\sqrt{2}}{2}\sqrt{c_n(x)} e^{-\frac{\delta^2\alpha_n}{8xb_n}-\frac{nx}{2}\left(1-\exp\left(-\frac{\delta^2\alpha_n}{4xb_n}\right)\right)}.$$

Therefore, we have

$$\lim_{n\to\infty}\frac{n\alpha_n}{b_n}\Sigma_i = 0, \qquad i = 1,\, 2,\, 3,$$

under the condition (9).

***Corollary* 4.1.** *If the function f is bounded on the semiaxis $[0,\infty)$, then* (10) *holds at each point $x > 0$ for which $f''(x)$ exists.*

5. Rates of convergence.

Theorem 5.1. *If f is uniformly continuous on the semiaxis $(0,\infty)$, then*

$$\|E_n(f) - f\|_{[0,b_n]} \leq 2\omega\left(f; \sqrt{\frac{b_n}{n\alpha_n}}\right).$$

Proof. Let $x \in (0, b_n]$. Since

$$E_n(f;x) - f(x) = \sum_{m=1}^{\infty}\frac{e^{-nx}(nx)^{m-1}}{(m-1)!}\sum_{k=0}^{\alpha_n m} p_{m\alpha_n,k}\left(\frac{x}{b_n}\right)\left[f\left(\frac{kb_n}{m\alpha_n}\right) - f(x)\right],$$

then we have

$$|E_n(f;x) - f(x)| \le \sum_{m=1}^{\infty} \frac{e^{-nx}(nx)^{m-1}}{(m-1)!} \sum_{k=0}^{\alpha_n m} p_{m\alpha_n,k}\left(\frac{x}{b_n}\right) \omega\left(f; \left|\frac{kb_n}{m\alpha_n} - x\right|\right).$$

Taking into account that $\omega(f;\lambda\delta) \le (\lambda+1)\omega(f;\delta)$ and using Cauchy – Schwartz inequality, for $\delta > 0$ we obtain

$$|E_n(f;x) - f(x)| \le \omega(f;\delta)\left\{\frac{1}{\delta}\sum_{m=1}^{\infty} \frac{e^{-nx}(nx)^{m-1}}{(m-1)!} \sum_{k=0}^{\alpha_n m} p_{m\alpha_n,k}\left(\frac{x}{b_n}\right)\left|\frac{kb_n}{m\alpha_n} - x\right| + 1\right\} \le$$

$$\le \omega(f;\delta)\left\{\frac{1}{\delta}\sqrt{E_n(x_2;x)} + 1\right\} =$$

$$= \omega(f;\delta)\left\{\frac{1}{\delta}\sqrt{\frac{(b_n - x)(1 - e^{-nx})}{n\alpha_n}} + 1\right\} \le$$

$$\le \omega(f;\delta)\left\{\frac{1}{\delta}\sqrt{\frac{b_n}{n\alpha_n}} + 1\right\}.$$

Choosing $\delta = \sqrt{\dfrac{b_n}{n\alpha_n}}$, we have the inequality

$$|E_n(f;x) - f(x)| \le 2\omega\left(f; \sqrt{\frac{b_n}{n\alpha_n}}\right)$$

which is also trivial for $x = 0$.

Theorem 5.1 is proved.

***Corollary* 5.1.** *Let* $f \in C(0,\infty)$. *If* $f \in \mathrm{Lip}_M\,\mu$, *then*

$$\|E_n(f) - f\|_{[0,b_n]} \le 2M\left(\frac{b_n}{n\alpha_n}\right)^{\mu/2}.$$

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