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s-CONDITIONALLY PERMUTABLE SUBGROUPS AND p-NILPOTENCY OF FINITE GROUPS*

s-УМОВНО ПЕРЕСТАВНІ ПІДГРУПИ ТА p-НІЛЬПОТЕНТНІСТЬ СКІНЧЕННИХ ГРУП

We study the p-nilpotency of a group such that every maximal subgroup of its Sylow p-subgroups is s-conditionally permutable for some prime p. By using the classification of finite simple groups, we get interesting new results and generalize some earlier results.

Вивчено p-нільпотентність групи, для якої кожна максимальна підгрупа її силовських p-підгруп є s-умовно переставною для деякого простого p. За допомогою класифікації скінченних простих груп отримано цікаві нові результати та узагальнено деякі результати, що отримані раніше.

1. Notation and introduction. In this paper, all groups are finite and G stands for a finite group. Let $\pi(G)$ be the set of all prime divisors of $|G|$. Let G_p and $\mathrm{Syl}_p(G)$ be a Sylow p-subgroup and the set of Sylow p-subgroups of G respectively. Let $\mathcal{F}$ denote a formation, $\mathcal{U}$ the class of supersolvable groups. Let n_p be the p-part of a nature number n, that is, $n_p = p^a$ such that $p^a \mid n$ but $p^{a+1} \nmid n$. Let G be a Lie-type simple group over the finite field F_q. To collect some useful information and for convenience in narrating, we define $n(G)$ in Table 1.1. The other notation and terminology are standard (see [11, 13]).

Table 1.1

G	$n(G)$	G	$n(G)$
$A_n(q)$	$(n+1)f$	${}^2A_{2k}(q)(k \geq 2)$	$(4k+2)f$
$B_n(q)(p \neq 2)$	$2nf$	$B_n(2^f)$	$2nf$
$C_n(q)(p \neq 2)$	$2nf$	${}^2A_{2k+1}(q)(k \geq 2)$	$2(k+1)f$
${}^2D_n(q)$	$2nf$	$D_n(q)$	$2(n-1)f$
$E_8(q)$	$30f$	$E_7(q)$	$18f$
$E_6(q)$	$12f$	${}^2E_6(q)$	$18f$
$F_4(q)$	$12f$	${}^2F_4(q)'$	$12f$
$G_2(q)$	$6f$	${}^3D_4(q)$	$12f$
${}^2G_2(q)$	$6f$	${}^2B_2(q)$	$4f$

Many authors have investigated the structure of a group when maximal subgroups of Sylow subgroups of the group are well situated in the group. Srinivasan [28] showed that a group G is supersolvable if all maximal subgroups of every Sylow subgroup of G are normal. Later, several authors obtain the same conclusion if normality is replaced by some weaker property (see [25, 27]).

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In particular, these results indicate that the generalized normality of some maximal subgroups of Sylow subgroups give a lot of useful information on the structure of groups.

In this paper, we obtain some sufficient conditions on p-nilpotency and supersolvability of groups by using the s-conditional permutability of maximal subgroups of Sylow subgroups. Some earlier results on this topic are generalized.

2. Basic definitions and preliminary results. Let H and K be two subgroups of G. We say that H permutes with K if $HK = KH$. Recently, Huang and Guo [10] introduced a new embedding property, namely, the s-conditional permutability of subgroups of a group.

Definition. *A subgroup H of G is s-conditionally permutable if for every prime $p \in \pi(G)$, there exists a Sylow p-subgroup P of G such that $HP = PH$.*

For the sake of convenience, we list here some known results which will be useful in the sequel.

Lemma 2.1 ([10], Lemma 2.3)**.** *Let H and K be subgroups of G. Then the following hold:*

(1) *If H is s-conditionally permutable in G and K is normal in G, then HK/K is s-conditionally permutable in G.*

(2) *If $H \leq K \lhd G$ and H is s-conditionally permutable in G, then H is s-conditionally permutable in K.*

Lemma 2.2 ([24], Lemma 6)**.** *Suppose that G is a non-Abelian simple group. Then there exists an odd prime $r \in \pi(G)$ such that G has no Hall $\{2, r\}$-subgroup.*

Lemma 2.3 ([29], Theorem 3.1)**.** *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, and G a group with a normal subgroup N such that $G/N \in \mathcal{F}$. If all Sylow subgroups of $F^*(N)$ are cyclic, then $G \in \mathcal{F}$.*

Lemma 2.4 ([26], Lemma 1.6)**.** *Let P be a nilpotent normal subgroup of a group G. If $P \cap \cap \Phi(G) = 1$, then P is the direct product of some minimal normal subgroups of G.*

Recall that a prime divisor d of $a^m - 1$ is called primitive, if d does not divide $a^i - 1$ for $1 \leq i \leq m - 1$. For primitive prime divisors, an important property is due to Zsigmondy, refer to [8].

Lemma 2.5 [8]**.** *Let b and n be positive integers.*

(1) *There are primitive prime divisors of $b^n - 1$ unless $(b, n) = (2, 6)$ or b is a Mersenne prime and $n = 2$.*

(2) *Each primitive prime divisor p of $b^n - 1$ is at least $n + 1$. Moreover, if $p = n + 1$, then p^2 divides $b^n - 1$ except for the following cases*:

(i) $n = 2$ *and* $b = 2^s - 1$ *or* $3 \cdot 2^s - 1$;

(ii) $b = 2$ *and* $n = 4, 6, 10, 12$ *or* 18;

(iii) $b = 3$ *and* $n = 4$ *or* 6;

(iv) $b = 5$ *and* $n = 6$.

(3) *For a positive integer s, if a primitive prime divisor of $b^s - 1$ divides $b^n - 1$, then s divides n.*

3. Main results and their proofs.

Theorem 3.1. *Let G be a non-Abelian simple group and $|G|_2 = 2^t$. If G has a subgroup of order $2^{t-1}|G|_r$ for every $r \in \pi(G) \setminus \{2\}$, then $G \cong PSL_2(q)$, where q is a power of an odd prime and $t = 2$.*

Proof. Let $r \in \pi(G) \setminus \{2\}$, H be a subgroup of G of order $2^{t-1}|G|_r$, $A \in \mathrm{Syl}_2(H)$ and $R \in \mathrm{Syl}_r(H)$. Then $|A| = 2^{t-1}$ and $R \in \mathrm{Syl}_r(G)$ and $H = AR$. Let M be a maximal subgroup of G containing H. Then $|M|_2 = 2^t$ or $|M|_2 = 2^{t-1}$. If $|M|_2 = 2^{t-1}$, then $A \in \mathrm{Syl}_2(M)$ and H is a Hall $\{2, r\}$-subgroup of M; if $|M|_2 = 2^t$, then $M_2 \in \mathrm{Syl}_2(G)$, $|G : M|$ is odd and so G has a faithful primitive permutation representation of odd degree and M is listed in [20] (Theorem). By the classification of finite simple groups, we divide the argument into the following cases.

(1) G is a sporadic simple group.

Let $r = \max \pi(G)$. Then by [5] and http://brauer.maths.qmul.ac.uk/Atlas/v3, $2^{t-1} \nmid |M|$, a contradiction.

(2) G is an alternating A_n.

We have $2^t = \left(\dfrac{1}{2}n!\right)_2$. Let $r = \max \pi(G)$. By [3], $R^A = R$ and $2^{t-1} \,\Big|\, \dfrac{1}{2}(r-1)(n-r)!$, this is impossible.

(3) G is a Lie-type simple group over $GF(q)$, where $q = p^f$ and p is a prime.

Suppose that $G = PSL_2(q)$ and $|G|_2 > 4$. If $q = 2^f$, then G has no subgroup of order $\dfrac{1}{2}|G|_2|R|$ by [14], a contradiction. Hence $q = p^f$ with p odd. Thus $(q-1)_2 = 2$ or $(q+1)_2 = 2$. If $(q+1)_2 = 2$, let $t = \max \pi(q+1)$ and $V \in \mathrm{Syl}_t(G)$, then G has no subgroup of order $\dfrac{1}{2}|G|_2|V|$ by [14]; if $(q-1)_2 = 2$, let $u = \max \pi(q-1)$ and $U \in \mathrm{Syl}_u(G)$, then G has no subgroup of order $\dfrac{1}{2}|G|_2|U|$ by [14], a contradiction. Hence $|G|_2 = 2^2$, the result holds. From now, we assume that $n(G) > 2f$.

Assume that $(n(G), p) = (6, 2)$. Then $(n(G)/f, f)$ is one of $(3, 2)$ and $(6, 1)$, and so G is one of the groups $PSL_3(2^2)$, $PSU_4(2)$, $PSL_6(2)$, $D_4(2)$. Suppose that $G \in \{PSL_3(2^2)$, $PSU_4(2)$, $D_4(2)\}$. Let $r = 3$. Since $M_r \in \mathrm{Syl}_r(G)$, by [5, p. 23, 26, and 85], $M \in \{A_6,\ 3^2 \cdot Q_8\}$ if $G = PSL_3(2^2)$, $M \in \{3_+^{1+2} : 2A_4,\ 3^3 \cdot S_4\}$ if $G = PSU_4(2)$ and $M = 3^4 : 2^3 \cdot S_4$ if $G = D_4(2)$, hence $4 \mid |G : M|$, a contradiction. Suppose that $G = PSL_6(2)$. Let $r = 7$. By http://brauer.maths.qmul.ac.uk/Atlas/lin/L62, $M \in \{2^9 : (L_3(2) \times L_3(2)),\ (L_3(2) \times L_3(2)) : 2,$ $(L_2(8) \times 7) : 3\}$. If $M \in \{(L_3(2) \times L_3(2)) : 2,\ (L_2(8) \times 7) : 3\}$, then $4 \mid |G : M|$; if $M = 2^9 : (L_3(2) \times L_3(2))$, since the maximal subgroup A of $L_3(2)$ satisfying $7 \mid |A|$ is isomorphic to $7 : 3$, M has no the maximal subgroup of order $2^{14} \cdot 7^2$, a contradiction. Hence $(n(G), p) \neq (6, 2)$. By Lemma 2.5, $p^{n(G)} - 1$ has at least one primitive prime divisor. Let r be the largest primitive prime divisor of $p^{n(G)} - 1$ and M a maximal subgroup of G of order $2^{t-1}|G|_r$. Then M is not a parabolic subgroup of G.

Suppose that $G \in \{PSL_3(q),\ PSU_3(q),\ {}^2F_4(2^{2m+1}),\ Sz(q),\ {}^3D_4(q),\ D_4(2^f),\ {}^2G_2(q),\ G_2(q)\}$. The maximal subgroups or orders of maximal subgroups of ${}^2B_2(2^{2m+1})$, $PSL_3(q)$ and $PSU_3(q)$ are listed in the proof of Lemmas 1 – 4 in [7]; the maximal subgroups of ${}^2F_4(2^{2m+1})$, ${}^2G_2(q)$, $G_2(2^f)$, ${}^3D_4(q)$ and $D_4(2^f)$ are listed in [6, 15 – 17, 23]. A simple checking shows that $4 \mid |G : M|$, a contradiction. Suppose that $G = G_2(q)$ with q odd. Since $|M_r| = |G|_r$, by [16], the possibilities of M are $SL_3(q) : 2$, $SU_3(q) : 2$, $L_2(13)$, $G_2(2)$ and J_1. It is easy to prove that if $M \in \{SL_3(q) : 2,$ $SU_3(q) : 2,\ L_2(13),\ G_2(2),\ J_1\}$, then M has no the subgroup of order $2^{t-1}|G|_r$.

Next, we deal with the remaining Lie-type simple group G in the previous argument. Let H be maximal subgroups of G containing a subgroup of G of order $2^{t-1}|G|_p$. Then H is a parabolic subgroup of G.

Suppose that G is an exceptional Lie-type simple group and the notation $K(G)$ is defined in [21] (Theorem). Suppose that $p = 2$. It is easy to see that the maximal subgroup in Table 1 [21] don't contain a subgroup of order $2^{t-1}|G|_r$. Thus by [21] (Theorem), $|M| < 2^{K(G)f}$. On the other hand, by [12], $|M| > (|M|_2)^2 \geq 2^{2(K(G)-1)f}$ if $G \neq E_8(2^f)$ or $|M| > (|M|_2)^2 \geq 2^{2(K(G)-10)f}$ if $G = E_8(2^f)$, a contradiction. Suppose that $p > 2$ and G is one of simple groups $F_4(q)$, $E_6(q)$, ${}^2E_6(q)$, $E_7(q)$, $E_8(q)$. Then $4 \mid |G : H|$, this is impossible. Thus we have proved that there is no exceptional Lie-type simple group satisfying the condition of Theorem 3.1.

Suppose that G is a classical simple group on n-dimension vector space V and $n > 3$. We shall use the notation of the book [13] in the following argument. Aschbacher [1] classified maximal subgroups of a classical simple group into 9 types: C_i, where $1 \leq i \leq 8$, and S, see [13] for the description.

Suppose that $p = 2$. If $3 < n < 12$, using [14] and [15], it is easy to see that $4 \mid |G : M|$, G doesn't satisfy the condition of Theorem 3.1. Hence we assume that $n \geq 12$. Assume that M is an almost simple group. Since $2^{t-1} \mid |M|_2$, by [18], $|M| < 2^{3fn} < 2^{\frac{1}{2}n(n-2)f-2} \leq (|M|_2)^2$. On the other hand, by [12], $|M| > (|M|_2)^2$, a contradiction. Suppose that M is a C_i subgroup. By [15] (Table A–E), a simple checking shows that $4 \mid |G : M|$, G doesn't satisfy the hypothesis.

Assume that $p > 2$. Since $4 \nmid |G : K|$, we have $4 \nmid n$ if $G = PSL_n(q)$; $2 \nmid n$ if $G = PSL_n(q)$ with $4 \mid (q+1)$; $4 \nmid (q+1)$ if $G \neq PSL_n(q)$; $4 \nmid n(n-1)$ if $G \in \{U_n(q), PSp_n(q)\}$; $2 \nmid k$ if $G \in \{P\Omega_{2k}^{+}(q), P\Omega_{2k+1}(q)\}$; $2 \nmid (k-1)$ if $G = P\Omega_{2k}^{-}(q)$. Suppose that $2 < n < 12$. From [14] and [15], it is easy to see that either $2^{t-1} \nmid |M|$ or $M_r \notin \mathrm{Syl}_r(G)$, a contradiction. Hence we may assume that $n \geq 12$. By Lemma 2.5, we may assume that $r > n(G)+1$ or $r = n(G)+1$ and $r^2 \mid p^{n(G)} - 1$. By [20], it is easy to see that $|G : M|$ is not odd, hence M has a Hall $\{2, r\}$-subgroup. Suppose that M is a S subgroup of G. Then the covering group of M is a subgroup of $GL_n(q)$ and there is a non-Abelian simple group S such that $S \leq M \leq \mathrm{Sut}(S)$. Moreover, if N is the preimage of S in G, then N is absolutely irreducible on V and N is not a classical group defined over a subfield of $GF(q)$(in its natural representation). All possibilities of S have given in Examples 2.6–2.9 in [9]. For all possible S either $2^{t-1} \nmid |M|$ or $r^2 \nmid |M|$ when $r = n(G)+1$, this is impossible. Suppose that M is not a S subgroup of G. Since $r \mid |M|$, by [14] (Table 3.5.A–F), it is easy to see that M must be one of C_3, C_6 and C_8 subgroups of G. Since $r > n(G)+1$ or $r^2 \mid |M|$ if $r = n(G)+1$, M is not a C_6 subgroup. If M is C_3 and C_8 subgroups, a simple calculation shows that $2^{t-1} \nmid |M|$, a final contradiction.

Theorem 3.1 is proved.

Let $\mathcal{M}$ be a class of groups. If there is no the section in a group G to be isomorphic to a member of $\mathcal{M}$, then G is called $\mathcal{M}$-free. For the convenience, write $\Im$ for the set of all $PSL_2(q)$, where $q = p^f$ is odd and the order of Sylow 2-subgroup of $PSL_2(q)$ is 4.

Theorem 3.2. *Let G be a group and N a normal subgroup of G, $p \in \pi(G)$ and $P \in \mathrm{Syl}_p(N)$. Suppose that $(|G|, p-1) = 1$ and G/N is p-nilpotent. If G is $\Im$-free and all maximal subgroups of P are s-conditionally permutable in G, then G is p-nilpotent.*

Proof. Assume that the result is false. Let (G, N) be a counterexample with $|G| + |N|$ minimal.

(1) G has a unique minimal normal subgroup L contained in N, G/L is p-nilpotent and $L \nleq \Phi(G)$, and so L is not a p'-group.

Let L be a minimal normal subgroup of G contained in N. Consider the quotient group $\overline{G} = G/L$. Clearly $\overline{G}/\overline{N} \cong G/N$ is p-nilpotent and $\overline{P} = PL/L$ is a Sylow p-subgroup of $\overline{N}$, where $\overline{N} = N/L$. Let $\overline{P_1} = P_1L/L$ be a maximal subgroup of $\overline{P}$. We may assume that P_1 is a maximal subgroup of P. By Lemma 2.1(1), $\overline{P_1}$ is s-conditionally permutable in $\overline{G}$. The choice of G implies that $\overline{G}$ is

p-nilpotent. Since the class of p-nilpotent groups is a saturated formation, we may assume that L is a unique minimal normal subgroup of G contained in N and $L \nleq \Phi(G)$, and so L is not a p'-group.

(2) $O_p(N) = 1$.

If not, then by (1), $L \leq O_p(N)$ and, there is a maximal subgroup M of G such that $G = LM$ and $L \cap M = 1$, so $N = G \cap N = L(M \cap N)$ and $L \cap (M \cap N) = 1$. It is clear that $LM_p \in \mathrm{Syl}_p(G)$ and we may let $(M \cap N)_p < P$, where $(M \cap N)_p \in \mathrm{Syl}_p(M \cap N)$. Let P_1 be a maximal subgroup of P containing $(M \cap N)_p$. Then $P = P_1 L$. By the hypothesis, P_1 is a s-conditionally permutable subgroup of G, then there exists a Sylow q-subgroup Q of G such that $P_1 Q = Q P_1$ for any $q \in \pi(G)$, where $q \neq p$. Let $L_1 = L \cap P_1$. Then $|L : L_1| = |L : L \cap P_1| = |LP_1 : P_1| = |P : P_1| = p$. So L_1 is a maximal subgroup of L. If $L \leq P_1 Q$, then $P = LP_1 \leq P_1 Q$, a contradiction. Hence $L \cap P_1 Q < L$ and $L_1 = L \cap P_1 Q$. Consequently, $L_1 = L \cap P_1 Q \lhd P_1 Q$, $P_1 Q \leq N_G(L_1)$. It is clear that $L_1 \lhd L$. So $P = LP_1 \leq N_G(L_1)$. By the arbitrariness of $q \in \pi(G)$, we have $L_1 \lhd G$, hence $L_1 = 1$ by the minimal normality of L in G. This means that L is a cyclic subgroup of prime order. Since $G/C_G(L)$ is isomorphic to a subgroup of $\mathrm{Aut}(L)$ and $|\mathrm{Aut}(L)| = p - 1$, by $(|G|, p - 1) = 1$, we have $C_G(L) = G$, and $L \leq Z(G)$. Hence $G = L \times M$. Since $M \cong G/L$, we get M is p-nilpotent by (1), so G is p-nilpotent, a contradiction.

(3) End of the proof.

By (1) and (2), L is not solvable and so $p = 2$ by the Odd Order Theorem. Let $L = T_1 \times T_2 \times \ldots \ldots \times T_s$, where T_i are non-Abelian simple groups with $T_i \cong T_1$, $1 \leq i \leq s$. Since $P \cap L \in \mathrm{Syl}_2(L)$, we have $P \cap L = K_1 \times K_2 \times \ldots \times K_s$, where $K_i \in \mathrm{Syl}_2(T_i)$. Now we claim that there exists a maximal subgroup P_1 of P and i such that $K_i \leq P_1$. If $P \cap L < P$, it is clear. Assume that $P \cap L = P$. Then (L, L) satisfies the hypothesis by Lemma 2.1(2). If L is a non-Abelian simple group, then every maximal subgroup of P is s-conditionally permutable in L. By the hypothesis and Theorem 3.1, we get $L \in \Im$, a contradiction. Hence L is not a non-Abelian simple group. Therefore, we can choose the maximal subgroup P_1 of P and i such that $K_i \leq P_1$. By the hypothesis, there exists a Sylow q-subgroup Q of G such that $P_1 Q = Q P_1$ for any $q \in \pi(G)$, where $q \neq 2$. Hence $T_i \cap P_1 Q$ is a Hall $\{2, q\}$-subgroup of T_i for any $q \in \pi(T)$ with $q \neq 2$. This contradicts the Lemma 2.2.

Theorem 3.2 is proved.

***Corollary* 3.1.** *Suppose that G is $\Im$-free. If for every prime p dividing the order of G and $P \in$ $\in \mathrm{Syl}_p(G)$, every maximal subgroup of P is s-conditionally permutable in G, then G is a Sylow tower group of supersolvable type.*

Theorem 3.3. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, and G a group with a normal subgroup N such that $G/N \in \mathcal{F}$. If N is $\Im$-free and all maximal subgroups of every noncyclic Sylow subgroup P of N are s-conditionally permutable in G, then $G \in \mathcal{F}$.*

Proof. Assume that the result is false and let (G, N) be a counterexample with $|G| + |N|$ minimal.

If all Sylow subgroups of N are cyclic, then all Sylow subgroups of $F^*(N)$ are cyclic. By Lemma 2.3, $G \in \mathcal{F}$. Therefore, when we want to prove $\overline{G} \in \mathcal{F}$ in the following arguments, we always assume that $\overline{N}$ has a noncyclic Sylow subgroup if $(\overline{G}, \overline{N})$ satisfies the hypothesis of (G, N) in Theorem 3.3. By Lemma 2.1(2) and Corollary 3.1 N is a Sylow tower group of supersolvable type. Let r be the largest prime in $\pi(N)$ and $R \in \mathrm{Syl}_r(N)$. Then R is normal in G and $(G/R)/(N/R) \cong$ $\cong G/N \in \Im$. By Lemma 2.1(1), every maximal subgroup of any Sylow subgroup of N/R is s-conditionally permutable in G/R. Therefore, G/R satisfies the hypotheses for the normal subgroup N/R. Thus, by induction, $G/R \in \mathcal{F}$, so R is noncyclic by Lemma 2.3. By Lemma 2.1(1), we may assume that G has a unique minimal normal subgroup L which is contained in R and $G/L \in \mathcal{F}$. If $L \leq \Phi(G)$, then it follows that $G \in \mathcal{F}$, a contradiction. Thus, we may further assume that $R \cap \Phi(G) = 1$. Then, by Lemma 2.4, $R = F(R) = L$ is an elementary abelian minimal normal subgroup of G. Since $R = L \nleq \Phi(G)$, we may choose a maximal subgroup M of G such that $R \nleq M$. Let M_r be a Sylow r-subgroup of M. Then $G = RM$, $R \cap M = 1$ and $G_r = RM_r$ is a Sylow r-subgroup of G. Let G_1 be a maximal subgroup of G_r containing M_r. Then $R \cap G_1$ is a maximal subgroup of R. By the hypothesis, $R \cap G_1$ is s-conditionally permutable in G, so there exists a $Q \in \mathrm{Syl}_q(G)$ such that $(R \cap G_1)Q = Q(R \cap G_1)$ with $q \neq r$, thus $R \cap G_1 = (R \cap G_1)(R \cap Q) = R \cap (R \cap G_1)Q \lhd (R \cap G_1)Q$, hence $(R \cap G_1)Q \leq N_G(R \cap G_1)$. Clearly, $R \cap G_1 \lhd G_r$. Therefore, $R \cap G_1 \lhd G$. By the minimal normality of R in G, we have $R \cap G_1 = 1$. Hence $|R| = r$, R is cyclic, a contradiction.

Theorem 3.3 is proved.

***Corollary* 3.2** ([10], Theorem 4.2). *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, and G a group with a solvable normal subgroup N such that $G/N \in \mathcal{F}$. If all maximal subgroups of every noncyclic Sylow subgroup P of N are s-conditionally permutable in G, then $G \in \mathcal{F}$.*

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