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ON SOME RAMANUJAN IDENTITIES FOR THE RATIOS OF ETA-FUNCTIONS

ПРО ДЕЯКІ ТОТОЖНОСТІ РАМАНУДЖАНА ДЛЯ ВІДНОШЕНЬ ЕТА-ФУНКЦІЙ

The purpose of this paper is to provide direct proofs of some of Ramanujan's P-Q modular equations based on simply proved elementary identities of Chapter 16 of his Second Notebook.

Наведено прямі доведення деяких P-Q модульних рівнянь Рамануджана на підставі елементарних тотожностей з глави 16 його Другого зошита, що просто доводяться.

1. Introduction. In the unorganized pages of his second notebook [11], Ramanujan recorded 23 identities involving ratios of Dedekind's eta-function all of which have been proved by B. C. Berndt and L.-C. Zhang [7] by employing Ramanujan's modular identities of various degrees, or via his mixed modular equations, or via the theory of modular forms. Similar 14 identities involving ratios of Dedekind's eta-function found on page 55 of Ramanujan's lost notebook [12] have been proved by Berndt [6] employing the above methods.

The purpose of this paper, consistent with Berndt's often made call for continued efforts to discern Ramanujan's thinking (see, for example, his book [5, p. 1]), is to demonstrate amenability of 10 of the above mentioned identities proved in [3, 4] via modular and mixed modular equations, to more direct proof based on simply proved identities of Chapter 16 of Ramanujan's second notebook [11], including his celebrated, so called, "remarkable identity with several parameters" [9], or "${}_1\psi_1$ summation". In the remainder of this section, we find it convenient, for our later use, to give a brief account of relevant definitions and results of Chapter 16 of the second notebook [11] as well as some results easily deducible there from. Significantly, the ${}_1\psi_1$ summation, stated below as (1.1), is not only the first of the entries (Entry 17, Chapter 16, Second Notebook [11]) with which Ramanujan begins his development of classical theory of theta and elliptic functions but also a very important tool all through his work in the classical theory as well as his own alternative theories:

$$1+\sum_{n=1}^{\infty}\frac{(1/\alpha;q^2)_n(-\alpha q)^n}{(\beta q^2;q^2)_n}z^n+\sum_{n=1}^{\infty}\frac{(1/\beta;q^2)_n(-\beta q)^n}{(\alpha q^2;q^2)_n}z^{-n}=$$

$$=\frac{(-qz;q^2)_\infty(-q/z;q^2)_\infty(q^2;q^2)_\infty(\alpha\beta q^2;q^2)_\infty}{(-\alpha qz;q^2)_\infty(-\beta q/z;q^2)_\infty(\alpha q^2;q^2)_\infty(\beta q^2;q^2)_\infty}, \tag{1.1}$$

where $|q|<1$, $|\beta q|<|z|<1/|\alpha q|$ and, as is customary,

$$(a;q)_n:=\prod_{k=0}^{n-1}(1-aq^k)$$

and

$$(a;q)_\infty := \prod_{k=0}^{\infty}(1-aq^k).$$

For instance, (1.1) contains as its special cases the well known Jacobi's triple product identity and the q-binomial theorem of Euler and Cauchy which are ubiquitous in theory of numbers and in theory of special functions. More over, K. Venkatachaliengar [13] has given an elementary and self contained proof of (1.1) giving it its pride of place in literature above its aforementioned special cases. Venkatachalingar's novel proof consists in first making the simple observation that the product side, say $f(z)$, satisfies a certain functional relationship which then readily yields a recurrence relation for the coefficients in the power series expansion of $f(z)$ in neighbourhood of $z=0$. The recurrence relation, in turn, gives all the coefficients except the constant term. Lastly, the constant term is determined by an application of Abel's theorem. Ramanujan's equivalent of Jacobi's theta function is

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2}b^{n(n-1)/2}, \qquad |ab|<1. \tag{1.2}$$

Note that Jacobi's $\theta_3(q,z)$is same as $f(qz,q/z)$ and that Jacob's triple product identity

$$f(qz,q/z) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-qz;q^2)_\infty(-q/z;q^2)_\infty(q^2;q^2)_\infty, \tag{1.3}$$

is the special case $\alpha=0=\beta$ of (1.1). Also, the Euler – Cauchy q-binomial theorem, in the form

$$\frac{(-qz;q^2)_\infty}{(-\alpha qz;q^2)_\infty} = 1+\sum_{n=1}^{\infty}\frac{(1/\alpha;q^2)}{(q^2;q^2)_\infty}(-\alpha qz)^n \tag{1.4}$$

is the special case $\beta=1$ of (1.1). All through his work Ramanujan employs the following restrictions $\varphi(q)$, $\psi(q)$ and $f(-q)$ of (1.2):

$$\varphi(-q) := f(-q,-q) = 1+2\sum_{k=1}^{\infty}(-1)^k q^{k^2} = \frac{(q;q)_\infty}{(-q;q)_\infty}, \tag{1.5}$$

$$\psi(q) := f(q,q^3) = \sum_{k=0}^{\infty} q^{k(k+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{1.6}$$

and

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n-1)/2} = (q;q)_\infty. \tag{1.7}$$

He also employs the functions

$$\chi(q) := (-q;q^2)_\infty \tag{1.8}$$

and

$$v(q) = q^{1/3}\frac{\chi(-q)}{\chi^3(-q^3)}. \tag{1.9}$$

The series representations in (1.5) – (1.7) including the first equation in (1.6) follow by simple manipulations of the terms in the respective defining series. Similarly, the product forms in (1.5) – (1.7) are obtained on employing (1.3) followed by manipulations of the factors involved. A simple but often used identity obtained by such manipulation of factors is due to Euler, namely:

$$(-q;q)_\infty = \frac{1}{(q;q^2)_\infty}. \tag{1.10}$$

We find it convenient to gather in the the following Lemmas some of the elementary results of Chapters 16 – 20 in Ramanujan's Second Notebook [11] and briefly sketch their proofs. The proofs are elementary and follow from some simple manipulations of series and products. One may see C. Adiga's doctoral thesis [1] for many of these proofs in the spirit of Ramanujan. Though Lemmas 1.2 and 1.3 are more general than we need, we feel that it is desirable to record them here because they seem new and because the proofs are as elementary as those of their special cases.

Lemma 1.1. *If* $|q| < 1$ *and* $|ab| < 1$, *then*

(i)
$$\varphi(-q) = \frac{f^2(-q)}{f(-q^2)}, \qquad \psi(q) = \frac{f^2(-q^2)}{f(-q)}, \qquad \chi(-q) = \frac{f(-q)}{f(-q^2)},$$
$$\varphi(q)\psi(q^2) = \psi^2(q), \qquad \varphi(q)\varphi(-q) = \varphi^2(-q^2); \tag{1.11}$$

(ii)
$$\psi(q) = f(q^3, q^6) + q\psi(q^9) \qquad \textit{and} \qquad \varphi(q) = 2qf(q^3, q^{15}) + \varphi(q^9); \tag{1.12}$$

(iii)
$$1 + \frac{1}{v(q)} = \frac{\psi(q^{1/3})}{q^{1/3}\psi(q^3)} \qquad \textit{and} \qquad 1 - 2v(q) = \frac{\varphi(-q^{1/3})}{\varphi(-q^3)}; \tag{1.13}$$

(iv)
$$1 + \frac{1}{v^3(q)} = \frac{\psi^4(q)}{q\psi^4(q^3)} \qquad \textit{or} \qquad v^3(q) = \frac{\chi^3(-q)\psi^4(q)}{\chi^9(-q^3)\psi^4(q^3)} - 1, \tag{1.14}$$

$$1 - 8v^3(q) = \frac{\varphi^4(-q)}{\varphi^4(-q^3)} \qquad \textit{or} \qquad \frac{1}{v^3(q)} = \frac{\chi^9(-q^3)\varphi^4(-q)}{q\chi^3(-q)\varphi^4(-q^3)} + 8; \tag{1.15}$$

(v)
$$f(a,b) = af(a(ab), b(ab)^{-1}); \tag{1.16}$$

(vi)
$$f(a, ab^2)f(b, a^2b) = f(a,b)\psi(ab),$$
$$f(a,b)f(-a,-b) = f^2(-a^2,-b^2)\varphi(-ab); \tag{1.17}$$

(vii)
$$f^2(a,b) - f^2(-a,-b) = 4af\left(\frac{b}{a}, \frac{a}{b}a^2b^2\right)\psi(a^2b^2); \tag{1.18}$$

(viii) *if* $ab = cd$, *then*

$$f(a,b)f(c,d) + f(-a,-b)f(-c,-d) = 2f(ac,bd)f(ad,bc) \tag{1.19}$$

and

$$f(a,b)f(c,d) - f(-a,-b)f(-c,-d) = 2af\left(\frac{b}{c}, \frac{c}{b}abcd\right) f\left(\frac{b}{d}, \frac{d}{b}abcd\right). \tag{1.20}$$

Proof. The identities in (1.11) and (1.16) – (1.20) merely follow from (1.5) – (1.8) on simple manipulations of the series or products involved. For instance, the factors in the product representation for $\varphi(-q)$ in (1.5) can be regrouped as $(q;q^2)_\infty(q^2;q^2)_\infty/(-q;q)_\infty$ which becomes $(q^2;q^2)_\infty/(q;q)_\infty(-q;q)_\infty$ on using (1.10). Recombining the factors in the denominator and employing the product representation in (1.7) twice we have the first of (1.11). Identity (1.16) is the well known quasiperiodicity. To simply establish it, we have, from (1.2), the right-hand side of (1.16) to be equal to

$$a\sum_{-\infty}^{\infty}\{a(ab)\}^{n(n+1)/2}\{b(ab)\}^{n(n-1)/2}$$

or, what is the same

$$\sum_{-\infty}^{\infty} a^{(n+1)(n+2)/2} b^{n(n+1)/2},$$

which is $f(a,b)$, as per (1.2) again, with $n+1$ change to n. The identity (1.17) is true, since its right-hand side can be written as, with $p = ab$,

$$(-a;p)_\infty(-b;p)_\infty(p;p)_\infty(p^2;p^2)_\infty/(p;p^2)_\infty$$

or, what is the same, on regrouping various factors,

$$(-a;p^2)_\infty(-ap;p^2)_\infty(-b;p^2)_\infty(-bp;p^2)_\infty(p^2;p^2)^2_\infty$$

or

$$f(a,bp)f(b,ap).$$

But this is the same as the left-hand side of (1.17). For (1.18), we need realize firstly that, by virtue of (1.2), the left-hand side equals $\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty}(ab)^{(m^2+n^2)/2}(a/b)^{(m+n)/2}$ with $m+n =$ odd. Transforming the indices to (s,t) by means of $m+n = 2s+1, m-n = 2t+1$, the double sum can be rewritten as the product $2a\sum_{-\infty}^{\infty}(ab)^{s^2}(a)^{2s}\sum_{-\infty}^{\infty}(ab)^{t^2}(ab)^t$ or, $2af\left(\frac{b}{a}, \frac{a}{b}a^2b^2\right)f(1,a^2b^2)$. But this is the same as right-hand side of (1.18). Proofs of (1.19) and (1.20) follow in the same vein. For instance, with $p = ab = cd$, the left-hand side of (1.19) equals $2\sum_{-\infty}^{\infty} p^{\frac{m^2+n^2}{2}}\left(\frac{a}{b}\right)^{\frac{m}{2}}\left(\frac{c}{d}\right)^{\frac{n}{2}}$, with $m+n =$even. Transforming (m,n) to (s,t) by means of $m+n = 2s$ and $m-n = 2t$, the double sum becomes the product $2\sum_{-\infty}^{\infty} p^{s^2}\left(\frac{ac}{bd}\right)^{\frac{s}{2}}\sum_{-\infty}^{\infty} p^{t^2}\left(\frac{ad}{bc}\right)^{\frac{t}{2}}$, or $2f(ac,bd)f(ad,bc)$. But this is the right-hand side of (1.19).

Proofs of (1.12) – (1.15) are slightly different, but equally simple. For the first of (1.12), we may start with the defining series $2\psi(q) = f(1,q) = \sum_{-\infty}^{\infty} q^{\frac{n(n+1)}{2}}$ and regroup the terms according as

$n \equiv 0, +1, -1 \pmod 3$. Similarly for the other identity of (1.12), we need only regroup the terms in the series $\varphi(q) = \sum_{-\infty}^{\infty} q^{n^2}$ according as $n \equiv 0, +1, -1 \pmod 3$.

Manipulating the factors appearing in the right-hand side of the definition (1.9) and employing the product forms appearing in (1.5)–(1.7) and the first of (1.12) (respectively the second of (1.12)) one gets the first of (1.13) (respectively the second). Now, following Berndt [3, p. 346], since changing $q^{1/3}$ to $q^{1/3}\omega$, $q^{1/3}\omega^2$ changes $v(q)$ respectively to $\omega^2 v(q)$ and $\omega v(q)$, as is clear from (1.8) and (1.9), we have from (1.13)

$$1 + \frac{1}{v^3} = \left(1 + \frac{1}{v}\right)\left(1 + \frac{\omega}{v}\right)\left(1 + \frac{\omega^2}{v^3}\right) = \frac{\psi(q^{1/3})\psi(q^{1/3}\omega)\psi(q^{1/3}\omega^2)}{q^3\omega(q^3)}.$$

This reduces to the first of (1.14), on repeated application of (1.6). Similarly, the second of (1.13) and (1.5) yield the first of (1.15). That the second form of (1.14) is equivalent to the first form can be seen by employing the definition (1.9) to eliminate the χ-ratio involved. The second of (1.15) follows similarly from the first.

Following lemma seems new to literature and simply follows from a few applications of (1.1) and series and product manipulations. It contains Entry 10 (iv) of Chapter 19 of [11] as special case $(z = 1)$.

Lemma 1.2.

$$\varphi^2(-q^2)\frac{f(qz, q/z)}{f(-qz, -q/z)} - \varphi^2(-q^{10})\frac{f(q^5 z, q^5/z)}{f(-q^5 z, -q^5/z)} = 2q\varphi^2(-q^{10})f(-q^2, -q^{18})\times$$

$$\times \left[\frac{zf(-q^{14}z^2, -q^6/z^2)}{f(-qz, -q^9/z)f(-q^3 z, -q^7/z)} + \frac{z^{-1}f(-q^6 z^2, -q^{14}/z^2)}{f(-q^9 z, -q/z)f(-q^7 z, -q^3/z)}\right]. \tag{1.21}$$

Proof. Let

$$P(z, q) := \varphi^2(-q^2)\frac{f(qz, q/z)}{f(-qz, -q/z)} \tag{1.22}$$

which is the right-hand side of (1.1) when $\alpha = \beta = -1$, and let

$$P^*(z, q) := \varphi^2(-q^{10})\frac{2qzf(-q^2, -q^{18})f(-q^{14}z^2, -q^6/z^2)}{f(-qz, -q^9/z)f(-q^3 z, -q^7/z)}. \tag{1.23}$$

Then (1.21) is the same as

$$P(z, q) - P(z, q^5) = P^*(z, q) + P^*(z^{-1}, q). \tag{1.24}$$

On putting $\alpha = \beta = -1$ in (1.1), we have

$$P(z, q) = S(z, q) := 1 + 2\sum_1^{\infty} \frac{q^n z^n}{1 + q^{2n}} + 2\sum_1^{\infty} \frac{q^n/z^n}{1 + q^{2n}}. \tag{1.25}$$

Converting each series in this into double series by expanding each of the summands by geometric series, interchanging the order of summation and summing the inner geometric series, we obtain

$$P(z, q) = S(z, q) = S^*(z, q) := 1 + 2\sum_0^{\infty} \frac{(-1)^n zq^{2n+1}}{1 - zq^{2n+1}} + 2\sum_0^{\infty} \frac{(-1)^n z^{-1}q^{2n+1}}{1 - z^{-1}q^{2n+1}}. \tag{1.26}$$

Repeating the same procedure on each of the series obtained by grouping the terms in (1.26) according as $n \equiv 0, 1, 2, 3, 4 \pmod 5$ we get

$$P(z,q) - P(z,q^5) = S^*(z,q) - S^*(z,q^5) =$$

$$= \sum_1^\infty \frac{(q^n + q^{9n})(z^n + z^{-n})}{1+q^{10n}} - \sum_1^\infty \frac{(q^{3n} + q^{7n})(z^n + z^{-n})}{1+q^{10n}} =$$

$$= \left[S(q^{-4}z, q^5) - S(q^{-2}z, q^5)\right] - \left[S(q^{-4}z^{-1}, q^5) - S(q^{-2}z^{-1}, q^5)\right]. \tag{1.27}$$

Now, from (1.25) and (1.22),

$$S(q^{-4}z, q^5) - S(q^{-2}z, q^5) = P(zq^{-4}, q^5) - P(q^{-2}z, q^5) =$$

$$= \varphi^2(-q^{10}) \left[\frac{f(qz, q^9/z)}{f(-qz, -q^9/z)} - \frac{f(q^3z, q^7/z)}{f(-q^3z, -q^7/z)}\right] = P^*(z,q), \tag{1.28}$$

on using the identity (1.20) with $a = qz$, $b = q^9 z$, $c = -q^3 z$ and $d = -q^7/z$; P^* being as in (1.23). Now substituting (1.28) in (1.27) we have (1.24) which is the same as (1.21) by virtue of (1.22) and (1.23).

***Corollary* 1.1** (Entry 10 (iv), Chapter 19, [11]).

$$\varphi^2(q) - \varphi^2(q^5) = 4qf(q, q^9)f(q^3, q^7). \tag{1.29}$$

Proof. Putting $z = 1$ in (1.21), we obtain

$$\frac{\varphi^2(-q^2)\varphi(q)}{\varphi(-q)} - \frac{\varphi^2(-q^{10})\varphi(q^5)}{\varphi(-q^5)} = \frac{4q\varphi^2(-q^{10})f(-q^2, -q^{18})f(-q^6, -q^{14})}{f(-q, -q^9)f(-q^3, -q^7)}.$$

This reduces to (1.29) on applying the last of the identities in (1.11) twice and also the second of (1.17) twice.

Following lemma also seems new to literature and follows from few applications of (1.1). Its proof is similar to that of pervious lemma but slightly more involved. It contains Entry 10 (v) of Chapter 19 of [11] as special case $(z = 1)$.

Lemma 1.3.

$$\psi^2(q)\frac{f(z, q^2/z)}{f(qz, q/z)} - q^2\psi^2(q^5)\frac{f(z, q^{10}/z)}{f(q^5z, q^5/z)} = \psi^2(q^5)f(q, q^4)\times$$

$$\times \left[\frac{f(q^2/z, q^3z)}{f(q/z, q^9z)f(q^3/z, q^7z)} + \frac{zf(q^2z, q^3/z)}{f(qz, q^9/z)f(q^3z, q^7/z)}\right]. \tag{1.30}$$

Proof. Let

$$\hat{P}(q,z) := \psi^2(q)\frac{f(z, q^2/z)}{f(qz, q/z)}, \tag{1.31}$$

which is indeed $(1-q)^{-1}$ times the right-hand side of (1.1) with $\alpha = q = 1/\beta$ and z changed to z/q, and let

$$\hat{P}^*(z,q) := \frac{\psi^2(q^5)f(q,q^4)f(q^2/z,q^3z)}{f(q/z,q^9z)f(q^3/z,q^7z)}. \tag{1.32}$$

Then (1.30) is the same as

$$\hat{P}(z,q) - q^2\hat{P}(z,q^5) = \hat{P}^*(z,q) + z\hat{P}^*(1/z,q). \tag{1.33}$$

On changing z to z/q and putting $\alpha = q$, $\beta = 1/q$ in (1.1) and then dividing throughout by $(1-q)$, we have

$$\hat{P}(z,q) = \hat{S}(z,q) := \sum_0^\infty \frac{(-1)^n q^n z^n}{1-q^{2n+1}} + \sum_0^\infty \frac{(-1)^n q^n/z^n}{1-q^{2n+1}}. \tag{1.34}$$

Converting each series in this into double series by expanding the summands by geometric series, interchanging the order of summation and summing the inner sums, we obtain

$$\hat{P}(z,q) = \hat{S}(z,q) = \hat{S}^*(z,q) := \sum_0^\infty \frac{zq^n}{1+zq^{2n+1}} + \sum_0^\infty \frac{q^n}{1+z^{-1}q^{2n+1}}. \tag{1.35}$$

Repeating the same procedure on each of the series obtained by regrouping the terms in (1.35) according as $n \equiv 0,1,2,3,4 \pmod 5$ we get

$$\hat{P}(z,q) - q^2\hat{P}(z,q^5) = \hat{S}^*(z,q) - q^2\hat{S}^*(z,q^5) =$$

$$= \sum_0^\infty \frac{(-1)^n(q^n+q^{9n+4})(z^{n+1}+z^{-n})}{1-q^{10n+5}} + \sum_0^\infty \frac{(-1)^n(q^{3n+1}+q^{7n+3})(z^{n+1}+z^{-n})}{1-q^{10n+5}} \tag{1.36}$$

Now, we have from (1.25) and (1.22)

$$P(iz,iq) - P(iz^{-1},iq) = S(iz,iq) - S(iz^{-1},iq),$$

or

$$\varphi^2(q^2)\left[\frac{f(-qz,q/z)}{f(qz,-q/z)} - \frac{f(qz,-q/z)}{f(-qz,q/z)}\right] = 4\sum_0^\infty \frac{q^{2n+1}(z^{-2n-1}-z^{2n+1})}{1-q^{2(2n+1)}},$$

or

$$\varphi(q^2)\psi(q^4)\frac{f(-1/z^2,-z^2q^4)}{f(-q^2z^2,-q^2/z^2)} = \sum_0^\infty \frac{q^{2n}(z^{2n}-z^{2(n+1)})}{1-q^{2(2n+1)}},$$

or, changing q^2 to q^5 and z^2 to $-z^{-1}$, and using fourth of the equations in (1.11)

$$\psi^2(q^5)\frac{f(z,q^{10}/z)}{f(q^5z,q^5/z)} = \sum_0^\infty \frac{(-1)^n q^{5n}(z^{n+1}+z^{-n})}{1-q^{10n+5}}.$$

This gives, on changing z to $q^{-4}z$,

$$\psi^2(q^5)\frac{f(q^{-4}z,q^{14}/z)}{f(qz,q^9/z)} = \sum_0^\infty \frac{(-1)^n(q^n z^{n+1}+q^{9n+1}z^{-n})}{1-q^{10n+5}}.$$

In turn, this yields, on changing z to z^{-1} and then multiplying throughout by z,

$$\psi^2(q^5)\frac{zf(q^{-4}z^{-1},q^{14}z)}{f(qz^{-1},q^9z)}=\sum_0^\infty\frac{(-1)^n(q^nz^{-n}+q^{9n+4}z^{n+1})}{1-q^{10n+5}}.$$

Similarly, we have

$$q^3\psi^2(q^5)\frac{f(q^{-2}z,q^{12}/z)}{f(q^3z,q^7/z)}=\sum_0^\infty\frac{(-1)^n(q^{3n+1}z^{n+1}+q^{7n+3}z^{-n})}{1-q^{10n+5}},$$

$$q^3\psi^2(q^5)\frac{zf(q^{-2}z^{-1},q^{12}z)}{f(q^3z^{-1},q^7z)}=\sum_0^\infty\frac{(-1)^n(q^{3n+1}z^{-n}+q^{7n+3}z^{n+1})}{1-q^{10n+5}}.$$

Adding the last four identities and making four applications of (1.16) to rewrite $q^4f(q^{-4}z,q^{14}/z)$, $q^4f(q^{-4}/z,q^{14}z)$, $q^3f(q^{-2}z,q^{12}/z)$, and $q^3f(q^{-2}/z,q^{12}z)$ respectively as $zf(q^6z,q^4/z)$, $z^{-1}f(q^6/z,q^4z)$, $qzf(q^8z,q^2/z)$ and $qz^{-1}f(q^8/z,q^2z)$, we get

$$\psi^2(q^5)\left[\frac{zf(q^6z,q^4/z)}{f(qz,q^9/z)}+\frac{f(q^6/z,q^4z)}{f(q/z,q^9z)}+\frac{qzf(q^8z,q^2/z)}{f(q^3z,q^7/z)}+\frac{qf(q^8/z,q^2z)}{f(q^3/z,q^7z)}\right]=$$

$$=\sum_0^\infty\frac{(q^n+q^{9n+4}+q^{3n+1}+q^{7n+3})(z^{n+1}+z^{-n})}{1-q^{10n+5}}. \tag{1.37}$$

or, on using the sum of (1.19) twice with $(a,b,c,d)=(q,q^4,q^2/z,q^3z)$ and $(a,b,c,d)=(q,q^4,q^2z,q^3/z)$, we obtain

$$\psi^2(q^5)f(q,q^4)\left[\frac{f(q^2/z,q^3z)}{f(q/z,q^9z)f(q^3/z,q^7z)}+\frac{zf(q^2z,q^3/z)}{f(qz,q^9/z)f(q^3z,q^7/z)}\right]=$$

$$=\sum_0^\infty\frac{(q^n+q^{9n+4}+q^{3n+1}+q^{7n+3})(z^{n+1}+z^{-n})}{1-q^{10n+5}}.$$

Using this in (1.36) gives (1.33) or what is the same (1.30).

***Corollary* 1.2** (Entry 10 (v), Chapter 19, [11])**.**

$$\psi^2(q)-q\psi^2(q^5)=f(q,q^4)f(q^2,q^3). \tag{1.38}$$

Proof. Putting $z=1$ in (1.30), we have, on employing the fourth of the equations in (1.11) and (1.3) several times and (1.6),

$$\psi^2(q^2)-q^2\psi^2(q^{10})=\frac{\psi^2(q^5)f(q,q^4)f(q^2,q^3)}{f(q,q^9)f(q^3,q^7)}=$$

$$=\left[\frac{(q^{10};q^{10})^2_\infty(q^5;q^5)^2_\infty}{(q^5;q^{10})^2_\infty(q^{10};q^{10})^2_\infty}\right]\left[\frac{(-q;q^5)_\infty(-q^4;q^5)_\infty(-q^2;q^5)_\infty(-q^3;q^5)_\infty}{(-q;q^{10})_\infty(-q^9;q^{10})_\infty(-q^3;q^{10})_\infty(-q^7;q^{10})_\infty}\right]=$$

$$=(q^{10};q^{10})^2_\infty\frac{(-q;q)_\infty}{(q^5;q^5)_\infty}\times$$

$$\times\left[\frac{(-q^2;q^{10})_\infty(-q^4;q^{10})_\infty(-q^5;q^{10})_\infty(-q^6;q^{10})_\infty(-q^8;q^{10})_\infty(-q^{10};q^{10})_\infty}{(-q;q)_\infty}\right]=$$

$$=(q^{10};q^{10})_\infty^2(-q^2;q^{10})_\infty(-q^4;q^{10})_\infty(-q^6;q^{10})_\infty(-q^8;q^{10})_\infty=$$

$$=f(q^2,q^8)f(q^4,q^6).$$

Changing q^2 to q in this we have (1.38).

The first two identities in the following lemma are due to Ramanujan (Chapter 19, [11] and [3]) which we obtain as special cases of our general results in Lemmas 1.2 and 1.3. The other two results are due to S.-Y. Kang [10] and our proofs are slightly different from hers. In what follows we also employ repeatedly some additional notations, for brevity, without further mention.

Lemma 1.4. *Let*

$$\lambda_n:=\varphi^2(-q^n),\qquad \mu_n:=\psi^2(q^n),\qquad s_n:=\chi(-q^n)\qquad \text{and}\qquad t_n:=f(-q^n). \tag{1.39}$$

Then

$$\lambda_1-\lambda_5=-4qf(-q,-q^9)f(-q^3,-q^7)=-4q\frac{s_1}{s_5}t_{10}^2=-4qs_1s_5\mu_5, \tag{1.40}$$

$$\mu_1-q\mu_5=f(q,q^4)f(q^2,q^3)=\frac{s_5}{s_1}t_5^2=\frac{\lambda_5}{s_1s_5}, \tag{1.41}$$

$$\lambda_1-5\lambda_5=-\frac{4s_5}{s_1}t_2^2=-4s_1s_5\mu_1 \tag{1.42}$$

and

$$\mu_1-5q\mu_5=\frac{s_1}{s_5}t_2^2=\frac{\lambda_1}{s_1s_5}. \tag{1.43}$$

Proof. Identities in (1.40) follow from (1.21) on putting $z=1$, changing q to $-q$, employing (1.11), (1.3) repeatedly and suitably manipulating the factors involved. Similarly, (1.41) follows from (1.30).

Identities in (1.42) follow by simply eliminating μ_5 between (1.40) and (1.41) and then by manipulating factors involved. Similarly, identities in (1.43) follow by eliminating λ_5 between (1.40) and (1.41).

The following lemma seems new.

Lemma 1.5. *We have*

$$\lambda_1-5\lambda_{25}=-4(s_1s_5\mu_1+5q^5s_5s_{25}\mu_5) \tag{1.44}$$

and

$$\mu_1-5q^6\mu_{25}=\frac{\lambda_1}{s_1s_5}+\frac{5q\lambda_{25}}{s_5s_{25}}. \tag{1.45}$$

Proof. Changing q to q^5 in (1.40) we get $\lambda_5 - \lambda_{25} = -4q^5 s_5 s_{25}\mu_{25}$. Adding this to (1.40) and using (1.43) we obtain

$$\lambda_1 - \lambda_{25} = -4q^5 s_5 s_{25}\mu_{25} - \frac{4}{5}s_1 s_5\left(\mu_1 - \frac{\lambda_1}{s_1 s_5}\right).$$

But this reduces to (1.44). Proof of (1.45) is similar. We need change q to q^5 in (1.41) and add the resulting identity to (1.41) and lastly use (1.42).

2. Main results. Theorem 2.1 below establishes four of Ramanujan's P-Q identities easily from the v-identities (1.13), (1.14) and (1.15).

Theorem 2.1. *In the notations of* (1.39) *of Lemma* 1.4 *we have the following*:

(i) ([11, *p.* 327], [4, *p.* 204], *Entry* 51). *Let*

$$P := \frac{t_1^2}{q^{1/6}t_3^2} \qquad \textit{and} \qquad Q := \frac{t_2^2}{q^{1/3}t_6^2}.$$

Then

$$PQ + \frac{9}{PQ} = \left(\frac{Q}{P}\right)^3 + \left(\frac{P}{Q}\right)^3. \tag{2.1}$$

(ii) ([11, *p.* 327], [4, *p.* 205], *Entry* 52). *Let*

$$P := \frac{t_2}{q^{1/24}t_3} \qquad \textit{and} \qquad Q := \frac{t_1}{q^{5/24}t_6}.$$

Then

$$(PQ)^2 - \frac{9}{(PQ)^2} = \left(\frac{Q}{P}\right)^3 - 8\left(\frac{P}{Q}\right)^3. \tag{2.2}$$

(iii) (*Equivalent of Entry* 5 (xii) *of* [11, *p.* 231] *and* [3, *p.* 230]). *Let*

$$P := \frac{t_1}{q^{1/24}t_2} \qquad \textit{and} \qquad Q := \frac{t_3}{q^{1/8}t_6}.$$

Then

$$(PQ)^3 + \frac{8}{(PQ)^3} = \left(\frac{Q}{P}\right)^6 - \left(\frac{P}{Q}\right)^6. \tag{2.3}$$

(iv) ([11, *p.* 327], [4, *p.* 210], *Entry* 56). *If*

$$P := \frac{t_1}{q^{1/3}t_9} \qquad \textit{and} \qquad Q := \frac{t_2}{q^{2/3}t_{18}},$$

then

$$P^3 + Q^3 = P^2Q^2 + 3PQ. \tag{2.4}$$

Proof. (i) Eliminating v between the first of (1.14) and the first of (1.15) we have,

$$\left(\frac{\mu_1^2}{9\mu_3^2} - 1\right)\left(1 - \frac{\lambda_1^2}{\lambda_3^2}\right) = 8,$$

or

$$\frac{\lambda_1^2\mu_1^2}{q\lambda_3^2\mu_3^2} + 9 = \frac{\mu_1^2}{q\mu_3^2} + \frac{\lambda_1^2}{\lambda_3^2},$$

or, identically,

$$(PQ)^2 + 9 = (PQ)\left(\frac{Q}{P}\right)^3 + (PQ)\left(\frac{P}{Q}\right)^3,$$

on routinely employing the definitions of P, Q, λ_1, μ_1, λ_3 and μ_3 and (1.11) and (1.39). This, on dividing throughout by PQ, gives (2.1).

(ii) Eliminating v between the second of (1.14) and the second of (1.15) we get

$$\left(\frac{s_1^3\mu_1^2}{s_3^9\mu_3^2} - 1\right)\left(\frac{s_3^9\lambda_1^2}{qs_1^3\lambda_3^2} + 8\right) = 1,$$

or

$$\frac{1}{q}\left(\frac{\lambda_1\mu_1}{\lambda_3\mu_3}\right)^2 - 9 = \frac{s_3^9\lambda_1^2}{qs_1^3\lambda_3^2} - 8\frac{s_1^3\mu_1^2}{s_3^9\mu_3^2},$$

or, identically,

$$(PQ)^4 - 9 = (PQ)^2\left(\frac{Q}{P}\right)^3 - 8(PQ)^2\left(\frac{P}{Q}\right)^3,$$

on routinely employing the definitions of P, Q, λ_1, μ_1, λ_3 and μ_3 and (1.11) and (1.39). This becomes (2.2) on dividing throughout by $(PQ)^2$.

(iii) From the first of (1.14) and the first of (1.15), we immediately have

$$\left(1 + \frac{1}{v^3}\right)\frac{\varphi^4(-q)}{\varphi^4(-q^3)} - (1 - 8v^3)\frac{\psi^4(q)}{q\psi^2(q^3)} = 0,$$

or, on using the definition (1.9) of v,

$$\frac{s_3^9\lambda_1^2}{qs_1^3\lambda_3^2} + 8\frac{s_1^3\mu_1^2}{s_3^9\mu_3^2} = \frac{\mu_1^2}{q\mu_3^2} - \frac{\lambda_1^2}{\lambda_3^2},$$

or

$$\frac{s_3^{18}\lambda_1^2\mu_3^2}{qs_1^6\lambda_3^2\mu_1^2} + 8 = \frac{s_3^9}{qs_1^3} - \frac{s_3^9\lambda_1^2\mu_3^2}{s_1^3\lambda_3^2\mu_1^2},$$

or, identically,

$$(PQ)^6 + 8 = (PQ)^3\left(\frac{P}{Q}\right)^3 - (PQ)^3\left(\frac{P}{Q}\right)^6,$$

on using the definitions P, Q, λ_1, μ_1, λ_3 and μ_3 and (1.11) and (1.39).

(iv) Eliminating $v(q^3)$ from the two equations in (1.13) and expanding we obtain

$$\frac{\psi(q)\varphi(-q)}{q\psi(q^9)\varphi(-q^9)} + 3 = \frac{\psi(q)}{q\psi(q^9)} - \frac{\varphi(-q)}{\varphi(-q^9},$$

or, identically,

$$PQ + 3 = \frac{1}{(PQ)}P^2 + \frac{1}{(PQ)}Q^2,$$

on using the definitions of P and Q and (1.11). This becomes (2.4) on multiplying throughout by (PQ).

Theorem 2.1 is proved.

The following corollary is needed later in the proof of Theorem 2.4.

***Corollary* 2.1.** *If*

$$u := \frac{q^{1/3}t_1t_{15}}{t_3t_5} \qquad \textit{and} \qquad v := \frac{q^{2/3}t_2t_{30}}{t_6t_{10}},$$

$$x := \left(\frac{v}{u}\right)^3 + \left(\frac{u}{v}\right)^3 \qquad \textit{and} \qquad w := uv + \frac{1}{uv},$$

$$B_n := \frac{t_{2n}t_{3n}}{q^{n/12}t_nt_{6n}} \qquad \textit{and} \qquad D_n := \frac{t_nt_{3n}}{q^{n/6}t_{2n}t_{6n}},$$

$$y_1 := (B_1B_5)^2 + \left(\frac{1}{B_1B_5}\right)^2 \qquad \textit{and} \qquad y_2 := (D_1D_5)^2 + \left(\frac{4}{D_1D_5}\right)^2,$$

then

$$(y_1^3 - 3y_1 - 9w^2 + 18)^2 = (y_2^3 - 48y_2 + 128).$$

Proof. Setting

$$C_n := \frac{t_nt_{2n}}{q^{n/4}t_{3n}t_{6n}}, \qquad y := (C_1C_5)^2 + \left(\frac{9}{C_1C_5}\right)^2, \qquad y_4 := (D_1D_5)^3 + \left(\frac{4}{D_1D_5}\right)^3$$

and

$$y_5 := \left(\frac{D_1}{D_5}\right)^3 + \left(\frac{D_5}{D_1}\right)^3,$$

we can write (2.1) as

$$C_1^2 + \frac{9}{C_1^2} = B_1^6 + \frac{1}{B_1^6},$$

from which follows, on changing q to q^5,

$$C_5^2 + \frac{9}{C_5^2} = B_5^6 + \frac{1}{B_5^6}.$$

Multiplying the last two equations, we obtain

$$y_3 = y_1^3 - 3y_1 - 9w^2 + x^2 + 16, \tag{2.5}$$

since, as can be easily shown,

$$x = \left(\frac{B_1}{B_5}\right)^3 + \left(\frac{B_5}{B_1}\right)^3 \qquad \text{and} \qquad w = \frac{C_1}{C_5} + \frac{C_5}{C_1}.$$

Similarly (2.2) and (2.3) respectively yield

$$y_4 - 8y_5 = y_3 - 9w^2 + 18 \tag{2.6}$$

and

$$y_4 + 8y_5 = y_1^3 - 3y_1 - x^2 + 2. \tag{2.7}$$

Adding (2.5), (2.6) and (2.7), we have

$$y_4 = y_1^3 - 3y_1 - 9w^2 + 18.$$

From our definitions of y_2 and y_4 we get

$$y_4^2 = y_2^3 - 48y_2 + 128.$$

Eliminating y_4 between the last two equations, we have the required result.

The following theorem establishes some $P - Q$ identities of Ramanujan that simply follow from the results of Ramanujan [12] and Kang [10], which we have recollected in Lemma 1.4.

Theorem 2.2. *In the notations* (1.39) *we have the following*:

(i) ([11, *p.* 325], [4, *p.* 206], *Entry* 53). *Let*

$$P := \frac{t_1}{q^{1/6}t_5} \qquad \text{and} \qquad Q := \frac{t_2}{q^{1/3}t_{10}}.$$

Then

$$PQ + \frac{5}{PQ} = \left(\frac{Q}{P}\right)^3 + \left(\frac{P}{Q}\right)^3. \tag{2.8}$$

(ii) ([12, *p.* 55], [6]). *Let*

$$P := \frac{t_1}{q^{1/24}t_2} \qquad \text{and} \qquad Q := \frac{t_5}{q^{5/24}t_{10}}.$$

Then

$$(PQ)^2 + \frac{4}{(PQ)^2} = \left(\frac{Q}{P}\right)^3 - \left(\frac{P}{Q}\right)^3. \tag{2.9}$$

(iii) ([11, *p.* 327], [4, *p.* 207], *Entry* 54). *Let*

$$P := \frac{t_2}{q^{1/8}t_5} \qquad \text{and} \qquad Q := \frac{t_1}{q^{3/8}t_{10}}.$$

Then

$$PQ - \frac{5}{PQ} = \left(\frac{Q}{P}\right)^2 - 4\left(\frac{P}{Q}\right)^2. \tag{2.10}$$

Proof. (i) Eliminating $s_1 s_5$ by multiplying (1.40) and (1.41) we get

$$\frac{\lambda_1\mu_1}{q\lambda_5\mu_5} + 5 = \frac{\lambda_1}{\mu_5} + \frac{\mu_1}{q\mu_5}.$$

But this is precisely, identically,

$$(PQ)^2 + 5 = (PQ)\left(\frac{Q}{P}\right)^3 + (PQ)\left(\frac{P}{Q}\right)^3,$$

on routinely using the definitions of P, Q, λ_1, μ_1, λ_5 and μ_5 (1.11) and (1.39). Dividing this throughout by PQ we have (2.8).

Proof. (ii) Dividing (1.42) by λ_5 and (1.43) by $q\mu_5$ and subtracting the resulting equations one from the other, we have

$$\frac{\mu_1}{q\mu_5} - \frac{\lambda_1}{\lambda_5} = \frac{s_1 t_1^2}{q s_5 \mu_5} + 4\frac{s_5 t_2^2}{s_1 \lambda_5},$$

or

$$\frac{1}{q}\left(\frac{s_1t_1}{s_5t_2}\right)+4=\frac{s_1\mu_1\lambda_5}{s_5t_2^2\mu_5}-\frac{s_1\lambda_1}{s_5t_2^2},$$

or, identically,

$$(PQ)^4+4=(PQ)^2\left(\frac{Q}{P}\right)^3-(PQ)^2\left(\frac{P}{Q}\right)^3,$$

on routinely using the definitions of P, Q, λ_1, μ_1, λ_5 and μ_5 (1.11) and (1.39). Dividing this throughout by $(PQ)^2$, we have (2.9).

(iii) Dividing (1.40) by λ_5 and (1.43) by μ_1 and adding the resulting equations, we obtain

$$\frac{\lambda_1}{\lambda_5}-\frac{5q\mu_5}{\mu_1}=\frac{s_1t_1^2}{s_5\mu_1}-\frac{4qs_1s_5\mu_5}{\lambda_5},$$

or

$$\frac{\lambda_1\mu_1}{q\lambda_5\mu_5}-5=\frac{s_1t_1^2}{qs_5\mu_5}-4\frac{s_1s_5\mu_1}{\lambda_5},$$

or, identically,

$$(PQ)^2-5=(PQ)\left(\frac{Q}{P}\right)^2-4PQ\left(\frac{P}{Q}\right)^2,$$

on routinely using the definitions of P, Q, λ_1, μ_1, λ_5 and μ_5 (1.11) and (1.39). Dividing this throughout by (PQ), we have (2.10).

Theorem 2.2 is proved.

The following corollary, along with Corollary 2.1, is useful in the proof of Theorem 2.4.

***Corollary* 2.2.** *In the notations of Corollary* 2.1, *we have*

$$4y_1=5w-x$$

and

$$8y_2=9w^2-40w-x^2-16.$$

Proof. Setting

$$E_n:=\frac{t_nt_{2n}}{q^{n/2}t_{5n}t_{10n}},\qquad F_n:=\frac{t_{2n}t_{5n}}{q^{n/6}t_nt_{10n}}$$

and

$$y_6:=E_1E_3+\frac{25}{E_1E_3},$$

we can rewrite (2.8) as

$$E_1+\frac{5}{E_1}=F_1^3+\frac{1}{F_1^3}.$$

Changing q to q^3 in this, we get

$$E_3+\frac{5}{E_3}=F_3^3+\frac{1}{F_3^3}.$$

Multiplying the last two equations we have, employing the notations of Corollary 2.1,

$$y_6+5w=y_5+x, \tag{2.11}$$

since, as is easily verified,

$$w = \frac{E_1}{E_3} + \frac{E_3}{E_1}, \qquad x = \left(\frac{F_1}{F_3}\right)^3 + \left(\frac{F_3}{F_1}\right)^3 \qquad \text{and} \qquad y_5 = (F_1F_3)^3 + \frac{1}{(F_1F_3)^3}.$$

Similarly, (2.9) and (2.10) respectively yield

$$y_2 + 4y_1 = y_5 - x \tag{2.12}$$

and

$$y_6 + 4y_1 = y_2 + 5w. \tag{2.13}$$

Subtracting (2.11) from the sum of (2.12) and (2.13), we obtain the first of the required results.

Subtracting (2.7) from the sum of (2.5) and (2.6) yields

$$8y_5 = 9w^2 - x^2 - 16.$$

Using this in the sum of (2.11) and (2.12) and then subtracting (2.13) from the resulting equation gives the second of the required results.

The following theorem establishes a $P - Q$ identity of Ramanujan simply from our Lemma 1.5.

Theorem 2.3 ([11, p. 325], [4, p. 212], Entry 58)**.** *Let*

$$P := \frac{f(-q^{1/5})}{q^{1/5}f(-q^5)} \qquad \textit{and} \qquad Q := \frac{f(-q^{2/5})}{q^{2/5}f(-q^{10})}.$$

Then

$$PQ + \frac{25}{PQ} = \left(\frac{Q}{P}\right)^3 - 4\left(\frac{Q}{P}\right)^2 - 4\left(\frac{P}{Q}\right)^2 + \left(\frac{P}{Q}\right)^3. \tag{2.14}$$

Proof. Multiplying (1.44) and (1.45) and then dividing throughout by $q\lambda_{25}\mu_{25}$ and then expanding we have

$$\frac{\lambda_1\mu_1}{q^6\lambda_{25}\mu_{25}} + 25 = \frac{\mu_1}{q^6\mu_{25}} - 4\frac{s_1\mu_1}{q^5 s_{25}\mu_{25}} - 4\frac{s_{25}\lambda_1}{qs_1\lambda_{25}} + \frac{\lambda_1}{\lambda_{25}},$$

or, identically,

$$(PQ)^2 + 25 = PQ\left(\frac{Q}{P}\right)^3 - 4PQ\left(\frac{Q}{P}\right)^2 - 4PQ\left(\frac{P}{Q}\right)^2 + PQ\left(\frac{P}{Q}\right)^3,$$

on using the definitions of P, Q, λ_1, μ_1, λ_{25} and μ_{25} (1.11) repeatedly. This becomes (2.14) on dividing throughout by PQ.

The proof of the following theorem is elementary and is so devised as to circumvent any temptation to use computer packages.

Theorem 2.4. (i) ([11, *p.* 330], [4, *p.* 314], *Entry* 59). *If*

$$P := \frac{t_3t_5}{q^{1/3}t_1t_{15}} \qquad \textit{and} \qquad Q := \frac{t_6t_{10}}{q^{2/3}t_2t_{30}},$$

then

$$PQ + \frac{1}{PQ} = \left(\frac{Q}{P}\right)^3 + \left(\frac{P}{Q}\right)^3 + 4. \tag{2.15}$$

(ii) ([11, *p.* 330], [4, *p.* 218], *Entry* 61). *If*

$$P := \frac{t_6 t_5}{q^{1/4} t_2 t_{15}} \qquad \textit{and} \qquad Q := \frac{t_3 t_{10}}{q^{3/4} t_1 t_{30}},$$

then

$$PQ + 1 + \frac{1}{PQ} = \left(\frac{Q}{P}\right)^2 + \left(\frac{P}{Q}\right)^2. \tag{2.16}$$

(iii) ([11, *p.* 213], [4, *p.* 230], *Entry* 65). *If*

$$P := \frac{t_1 t_2}{q^{1/2} t_5 t_{10}} \qquad \textit{and} \qquad Q := \frac{t_3 t_6}{q^{3/2} t_{15} t_{30}},$$

then

$$PQ + \frac{25}{PQ} = \left(\frac{Q}{P}\right)^2 + \left(\frac{P}{Q}\right)^2 - 3\left(\frac{Q}{P} + \frac{P}{Q} + 2\right). \tag{2.17}$$

Proof. (i) Substituting in the result of Corollary 2.1, the expressions for y_1 and y_2 obtained in Corollary 2.2 we obtain

$$\Big\{8(9w^2 - 40w - x^2 - 16)^3 - 6.8^4(9w^2 - 40w - x^2 - 16) + 2.8^2\Big\} -$$

$$-\Big\{(5w - x)^3 - 48(5w - x) - 9.8^2 w^2 + 18.8^2\Big\}^2 = 0.$$

Or, in terms of the analytic functions

$$W := qw \qquad \text{and} \qquad X := qx, \qquad |q| < 1,$$

with $W(q) \to 1$ and $X(q) \to 1$, as can be seen from our definitions of w and x in Corollary 2.1, we have

$$F(W, X) := 8(9W^2 - 40Wq - X^2 - 16q^2)^3 - 6.8^4(9W^2 - 40Wq - X^2 - 16q^2)q^4 +$$

$$+2.8^6.q^6 - \Big\{(5W - X)^3 - 48(5W - X)q^2 - 9.8^2 Wq^2 + 18.8^2 q^3\Big\}^2 = 0.$$

Now, for $W = W_0 := X + 4q$, we have $F(W_0, X) = 0$ as can be easily verified. In fact, after slight simplification, we get

$$F(W_0, X) = 8^4\Big[(X^2 + 4Xq - 4q^2)^3 - 48(X^2 + 4Xq - 4q^2)q^4 + 128q^6\Big].$$

This in turn is seen to be identically 0 by further simplification.

Thus we can write

$$F(W, X) = (W - W_0)G(W, X),$$

where $G(W(q), X(q))$ is analytic in $|q| < 1$. In fact, we can realize $G(W, X)$ as, by applying Taylor's formula to $F(W, X)$, or otherwise,

$$G(W, X) = \frac{\partial F}{\partial W_0} + \sum_{k=2}^{6} \frac{(W - W_0)^{k-1}}{k!} \frac{\partial^k F}{\partial W_0^k}.$$

Further, from this form of $G(W,X)$ and the definition of $F(W,X)$, and since $W(q), X(q)$ and $W_(q) \to 1$ as $q \to 0$, we have

$$\lim_{q\to 0} G(W(q),X(q)) = \lim_{q\to 0} \frac{\partial F}{\partial W_0} =$$

$$= \lim_{q\to 0} \Big[3.8(9W^2 - X^2)^2(18W) - 30(5W - X)^5 + \circ(q) \Big] = -3072 \neq 0.$$

This implies, because of continuity of $G(W(q),X(q))$ in $|q|<1$, that there exists a neighborhood N of $q=0$, where $G(W(q),X(q)) \neq 0$. This in turn gives, since $F(W(q),X(q)) = (W(q) - - W_0(q))G(W(q),X(q))$ is identically 0 in $|q|<1$, that

$$W(q) - W_0(q) = 0$$

identically in N. Since $W(q) - W_0(q)$ is analytic in all of $|q|<1$, this implies by analytic continuation

$$0 = W(q) - W_0(q) = q(w(q) - x(q) - 4) \qquad \text{throughout} \quad |q| < 1,$$

or

$$w(q) = x(q) + 4, \qquad \text{in} \quad 0 < q < 1, \tag{2.18}$$

or

$$uv + \frac{1}{uv} = \left(\frac{v}{u}\right)^3 + \left(\frac{u}{v}\right)^3 + 4, \qquad \text{in} \quad 0 < |q| < 1.$$

This is the same as the required result (2.15) since $P = 1/u$ and $Q = 1/v$.

(ii) In the notations of Corollaries 2.1 and 2.2, the required result is easily seen to be the same as

$$\frac{C_5}{C_1} + \frac{C_1}{C_5} + 1 = (B_1B_5)^2 + \frac{1}{(B_1B_5)^2}$$

or

$$w + 1 = y_1.$$

But this at once follows by adding (2.18) to the first of the results of Corollary 2.2.

(iii) In the notations of Corollaries 2.1 and 2.2, the required result is seen to be the same as

$$E_1E_3 + \frac{25}{E_1E_3} = \left(\frac{E_3}{E_1}\right)^3 + \left(\frac{E_1}{E_3}\right)^3 - 3\left(\frac{E_3}{E_1} + \frac{E_1}{E_3}\right) - 6$$

or

$$y_6 = w^2 - 3w - 8.$$

To obtain this, we first rewrite, on using (2.18), the results of Corollary 2.2 as

$$y_1 = w + 1$$

and

$$y_2 = w^2 - 3w - 4.$$

It now suffices to use the last two equations in (2.17).

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