UDC 517.9

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FRACTIONAL CALCULUS OF A UNIFIED MITTAG-LEFFLER FUNCTION
ДРОБОВЕ ЧИСЛЕННЯ УНІФІКОВАНОЇ ФУНКЦІЇ МІТТАГ-ЛЕФФЛЕРА

The main aim of the paper is to introduce an operator in the space of Lebesgue measurable real or complex functions $L(a,b)$. Certain properties of the Riemann – Liouville fractional integrals and differential operators associated with the function $E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p}(cz;s,r)$ are studied and the integral representations are obtained. Some properties of a special case of this function are also studied by means of fractional calculus.

Головною метою роботи є введення оператора у просторі $L(a,b)$ дійсних або комплексних функцій, вимірних відносно міри Лебега. Вивчено деякі властивості дробових інтегралів Рімана – Ліувілля та диференціальних операторів, що відповідають функції $E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p}(cz;s,r)$. Отримано відповідні інтегральні зображення. Деякі властивості частинного випадку цієї функції також вивчено за допомогою дробового числення.

1. Introduction, definitions and preliminaries. The Mittag-Leffler function has been studied by many researchers either in context with obtaining new properties or by introducing a new generalization and then deriving its properties [9, 11, 13]. Recently, we [7] have also studied various properties of our newly introduced generalization of Mittag-Leffler function in the form

$$E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p}(cz;s,r)=\sum_{n=0}^{\infty}\frac{[(\gamma)_{\delta n}]^{s}\,(cz)^{(pn+\rho-1)}}{\Gamma(\alpha(pn+\rho-1)+\beta)\,[(\lambda)_{\mu n}]^{r}\,(\rho)_{pn}},\tag{1.1}$$

where $\alpha,\beta,\gamma,\lambda,\rho\in\mathbb{C}$, $\mathrm{Re}(\alpha,\beta,\gamma,\lambda,\rho)>0$; $\delta,\,\mu,p,c>0$ and $(\gamma)_{q\,n}=\dfrac{\Gamma(\gamma+q\,n)}{\Gamma(\gamma)}$ is the generalized Pochhammer symbol [8]. In particular, if $q\in\mathbb{N}$. it takes the form

$$(\gamma)_{q\,n}=q^{qn}\prod_{r=1}^{q}\left(\frac{\gamma+r-1}{q}\right)_{n}.$$

If $p=1$, $\rho=1$, $r=0$, $s=1$, $\delta=q$, $s=1$, $c=1$, then (1.1) yields the generalization due to Shukla and Prajapati [11]. Here, we also introduce an operator denoted and defined by

$$\left(\mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p,\omega;a+}f\right)(x)=\int\limits_{a}^{x}(x-t)^{\beta-1}\,E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p}\big(\omega(x-t)^{\alpha};s,r\big)f(t)\,dt,\tag{1.2}$$

where $\alpha,\beta,\gamma,\lambda,\rho,\omega\in\mathbb{C}$; $\mathrm{Re}(\alpha,\beta,\gamma,\lambda,\rho)>0$; $\delta,\mu,p>0$, and $x>a$.

We enlist the following definitions and well-known formulas for studying the properties of the Riemann – Liouville (R–L) fractional integrals and differential operators associated with our generalization (1.1) as well as as the operator (1.2).

The space $L(a,b)$ of (real or complex valued) Lebesgue measurable functions [4, 10] is given by

$$L(a,b)=\left\{f:\|f\|_1=\int\limits_{a}^{b}|f(t)|dt<\infty\right\}.\tag{1.3}$$

For $f(x) \in L(a,b)$, $\mu \in \mathbb{C}$, and $\mathrm{Re}(\mu) > 0$, the R–L fractional integrals of order μ [10] are defined as follows.

The left-sided R–L fractional integral operator of order μ is defined as

$$ {}_aI_x^{\mu} f(x) = I_{a+}^{\mu} f(x) = \frac{1}{\Gamma(\mu)} \int\limits_a^x \frac{f(t)}{(x-t)^{1-\mu}} dt, \quad x > a, \tag{1.4}$$

where the right-sided R–L fractional integral operator of order μ is defined as

$$ {}_xI_b^{\mu} f(x) := I_{b-}^{\mu} f(x) = \frac{1}{\Gamma(\mu)} \int\limits_x^b \frac{f(t)}{(x-t)^{1-\mu}} dt, \quad x < b. \tag{1.5}$$

Further, if $\mu, \beta \in \mathbb{C}$, $\mathrm{Re}(\mu,\beta) > 0$, then [6, 10]

$$ I_{a+}^{\mu}[(t-a)^{\beta-1}](x) = \frac{\Gamma(\beta)}{\Gamma(\mu+\beta)}(x-a)^{\mu+\beta-1}. \tag{1.6}$$

For $\mu \in \mathbb{C}$, $\mathrm{Re}(\mu) > 0$; $n = [\mathrm{Re}(\mu)] + 1$, the R–L fractional derivative is

$$ (D_{a+}^{\alpha} f)(x) = \left(\frac{d}{dx}\right)^n (I_{a+}^{n-\alpha} f)(x). \tag{1.7}$$

Then for $\alpha, \beta, \gamma, \lambda, \rho, \in \mathbb{C}$, $\mathrm{Re}(\alpha, \beta, \gamma, \lambda, \rho, (\beta - m)) > 0$, and $\delta,\ \mu, p, m \in \mathbb{N}$, we have shown that [7]

$$ \left(\frac{d}{dz}\right)^m \left[z^{\beta-1}\, E_{\alpha,\beta,\lambda,\mu,\rho,p}^{\gamma,\delta}\big(\omega\ (cz)^{\alpha}; s, r\big)\right] = $$

$$ = z^{\beta-m-1}\, E_{\alpha,\beta-m,\lambda,\mu,\rho,p}^{\gamma,\delta}\big(\omega\ (cz)^{\alpha}; s, r\big). \tag{1.8}$$

The fractional integral operator investigated by Erdélyi – Kober is defined and represented as

$$ I_x^{\eta,\nu}\{f(x)\} = \frac{x^{-\eta-\nu+1}}{\Gamma(\nu)} \int\limits_0^x (x-t)^{\nu-1} f(t)\, dt, \qquad \mathrm{Re}(\nu) > 0, \quad \eta > 0, \tag{1.9}$$

which is a generalization of the R–L fractional integral operator (1.5).

Hilfer [2, 3] generalized the R–L fractional derivative operator D_{a+}^{μ} in (1.6) by introducing a right-sided fractional derivative operator $D_{a+}^{\mu,\nu}$ of order $0 < \mu < 1$ and type $0 \le \nu \le 1$ with respect to x as follows:

$$ (D_{a+}^{\mu,\nu} f)(x) = \left(I_{a+}^{\nu(1-\mu)} \frac{d}{dx}(I_{a+}^{(1-\nu)(1-\mu)} f)\right)(x). \tag{1.10}$$

The difference between the fractional derivatives of various types becomes apparent from the following formula involving the Laplace transformation [2, 3]:

$$ \mathcal{L}[D_0^{\mu,\nu} f(x)](s) = s^{\mu}\, \mathcal{L}[f(x)](s) - s^{\nu(1-mu)}\big(I_0^{(1-\nu)(1-\mu)} f\big)(0+), \tag{1.11}$$

where $0 < \mu < 1$, and the initial-value term: $\big(I_0^{(1-\nu)(1-\mu)} f\big)(0+)$ involves the R–L fractional integral operator of order $(1-\nu)(1-\mu)$ evaluated in the limit as $t \to 0+$. Here, as usual

$$\mathcal{L}[f(x)](s) = \int\limits_0^\infty e^{-sx}\, f(x)\, dx, \tag{1.12}$$

provided that the defining integral exists.

Prajapati, Dave and Nathwani [7] has shown that the Mellin–Barnes integral for the function defined by (1.1) is given by

$$E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p}\,(z; s, r) = \frac{[\Gamma(\lambda)]^r\, \Gamma(\rho)\, p\, z^{\rho-1}}{2\pi i\, [\Gamma(\gamma)]^s} \times$$

$$\times \int\limits_L \frac{\Gamma(-p\xi)\, \Gamma(1+p\xi)\, [\Gamma(\gamma+\delta\xi)]^s\, (-z)^{p\xi}}{\Gamma(\beta+\alpha\rho-\alpha+\alpha p\xi)\, [\Gamma(\lambda+\mu\xi)]^r\, \Gamma(\rho+p\xi)} d\xi. \tag{1.13}$$

Wright generalized hypergeometric function [1] is defined as

$${}_p\psi_q \begin{bmatrix} (a_1, A_1), \ldots,\ (a_p, A_p);\ \ z \\ (b_1, B_1), \ldots,\ (b_q, B_q); \end{bmatrix} = \sum_{r=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma(a_j + rA_j)}{\prod_{j=1}^{q} \Gamma(b_j + rB_j)} \frac{z^r}{r!} \tag{1.14}$$

$$= H^{1,p}_{p,q+1} \left[-z \,\middle|\, \begin{matrix} (1-a_1, A_1), \ldots, (1-a_p, A_p) \\ (0,1), (1-b_1, B_1), \ldots, (1-b_q, B_q) \end{matrix} \right], \tag{1.15}$$

where $H^{m,n}_{p,q} \left[-z \,\middle|\, \begin{matrix} (a_1, A_1), \ldots, (a_p, A_p) \\ (b_1, B_1), \ldots, (b_q, B_q) \end{matrix} \right]$ denotes the Fox H-function and $a_i, b_j \in \mathbb{C}$, $A_i, B_j \in \mathbb{R}$, $i = 1, 2, \ldots, p$; $j = 1, 2, \ldots, q$, $1 + \sum_{j=1}^{q} B_j - \sum_{i=1}^{p} A_i > 0$.

2. Main results. We prove in this section the following results.

Theorem 2.1. *Let* $a \in \mathbb{R}_+ = [0, \infty)$, $\alpha, \beta, \gamma, \lambda, \rho, \eta \in \mathbb{C}$, $\mathrm{Re}(\alpha, \beta, \gamma, \lambda, \rho, \eta) > 0$; δ, $\mu, p > 0$ *for* $x > a$, *then*

$$\Big(I^{\eta}_{a+}\, (t-a)^{\beta-1}\, E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p}(\omega(c(t-a))^{\alpha}; s, r\Big)(x) =$$

$$= (x-a)^{(\eta+\beta-1)}\, E^{\gamma,\delta}_{\alpha,\beta+\eta,\lambda,\mu,\rho,p}(\omega(c(x-a))^{\alpha}; s, r) \tag{2.1}$$

and

$$\Big(D^{\eta}_{a+}\, (t-a)^{\beta-1}\, E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p}(\omega(c(t-a))^{\alpha}; s, r)\Big)(x) =$$

$$= (x-a)^{(\beta-\eta-1)}\, E^{\gamma,\delta}_{\alpha,\beta-\eta,\lambda,\mu,\rho,p}(\omega(c(x-a))^{\alpha}; s, r). \tag{2.2}$$

Proof. Applying (1.1) to the left-hand side of (1.13) and then using (1.6), it yields

$$\Big(I^{\eta}_{a+}\, (t-a)^{\beta-1}\, E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p}(\omega(c(t-a))^{\alpha}; s, r\Big)(x) =$$

$$= (x-a)^{\beta+\eta-1} \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta n}]^s (\omega\ (c\ (x-a))^{\alpha})^{(pn+\rho-1)}}{\Gamma(\alpha(pn+\rho-1)+\beta+\eta)\ [(\lambda)_{\mu n}]^r\ (\rho)_{pn}}.$$

Here using (1.1) once again leads us to (2.1).

Now, using (1.7) to the the left-hand side of (2.2) and then applying (2.1), we get

$$\Big(D^{\eta}_{a+}\,(t-a)^{\beta-1}\,E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p}(\omega(c(t-a))^{\alpha};s,r)\Big)(x)=$$

$$=\left(\frac{d}{dx}\right)^{n}\big[(x-a)^{\beta-\eta-1}\,E^{\gamma,\delta}_{\alpha,\beta-\eta,\lambda,\,\mu,\rho,p}(\omega(c(x-a))^{\alpha};s,r)\big].$$

Further use of (1.7), gives the proof of (2.2).

Theorem 2.2. *Let* $\alpha,\gamma,\lambda,\rho,\eta\in\mathbb{C}$; $\mathrm{Re}(\alpha),\mathrm{Re}(\gamma),\mathrm{Re}(\lambda),\mathrm{Re}(\rho),\mathrm{Re}(\eta)>0$; $\delta,\mu,p>0$, *then*

$$ {}_{0}I^{\eta}_{x}\big[\nu\,E^{1,\delta}_{\alpha,1,\lambda,\mu,\rho,p}(\nu(cx)^{\alpha};s,r)\big]=\nu\,x^{\eta}\,E^{1,\delta}_{\alpha,\eta+1,\,\lambda,\mu,\rho,p}(\nu(cx)^{\alpha};s,r). \tag{2.3}$$

Proof. Applying (1.4) to the left-hand side of (2.3) and then using (1.1), we get

$$ {}_{0}I^{\eta}_{x}\Big[\nu\,E^{1,\delta}_{\alpha,1,\lambda,\mu,\rho,p}(\nu(cx)^{\alpha};s,r)\Big]=$$

$$=\frac{1}{\Gamma(\eta)}\sum_{n=0}^{\infty}\frac{[(1)_{\delta n}]^{s}\,c^{\alpha(pn+\rho-1)}\nu^{(pn+\rho)}}{\Gamma(\alpha(pn+\rho-1)+1)\,[(\lambda)_{\mu n}]^{r}\,(\rho)_{pn}}\int\limits_{0}^{x}t^{\alpha(pn+\rho-1)}(x-t)^{(\eta-1)}\,dt.$$

After some simplification and further use of (1.1), gives the proof of (2.3).

Theorem 2.3. *Let* $\alpha,\beta,\gamma,\lambda,\rho,\nu,\omega\in\mathbb{C}$; $\mathrm{Re}(\alpha,\beta,\gamma,\lambda,\rho,\nu)>0$; $\delta,\mu,\,p>0$, *then*

$$\big(\mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p,\omega;a+}\,(t-a)^{\nu-1}\big)(x)=(x-a)^{\beta+\nu-1}\,\Gamma(\nu)E^{\gamma,\delta}_{\alpha,\beta+\nu,\,\lambda,\mu,\rho,p}\big(\omega(x-a)^{\alpha}\big). \tag{2.4}$$

Proof. Putting $f(t)=(t-a)^{\nu-1}$ in (1.2), we get

$$\Big(\mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p,\omega;a+}\,(t-a)^{\nu-1}\Big)(x)=\int\limits_{a}^{x}(x-t)^{\beta-1}\,E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p,}\big(\omega(x-t)^{\alpha}\big)\,(t-a)^{\nu-1}\,dt,$$

and using (1.1), this reduced to

$$=\sum_{n=0}^{\infty}\frac{[(\gamma)_{\delta\,n}]^{s}\,\omega^{(p\,n+\rho-1)}}{\Gamma(\alpha(p\,n+\rho-1)+\beta)\,[(\lambda)_{\mu\,n}]^{r}\,(\rho)_{pn}}\times$$

$$\times\int\limits_{a}^{x}(x-t)^{\alpha(p\,n+\rho-1)+\beta-1}\,(t-a)^{\nu-1}\,dt$$

and simplifying the above equation, it becomes

$$=\sum_{n=0}^{\infty}\frac{[(\gamma)_{\delta\,n}]^{s}\,((\rho)_{pn})^{-1}\,\omega^{(p\,n+\rho-1)}}{\Gamma(\alpha(p\,n+\rho-1)+\beta)[(\lambda)_{\mu\,n}]^{r}}B(\alpha(p\,n+\rho)+\beta-1,\nu)$$

and further simplification of the above equation gives the proof of (2.4).

We show that the operator defined (1.2) is in fact bounded; whose proof is given below in the form of the following theorem.

Theorem 2.4. *Let the function ϕ be in the space $L(a,b)$ of Lebesgue measurable functions on a finite interval $[a,b]$ of the real line $\mathbb{R}$ given by*

$$L(a,b)=\left\{f:\|f\|_1=\int\limits_a^b|f(t)|dt<\infty\right\}.$$

Then the integral operator $\mathcal{E}^{\omega;\gamma,\delta}_{a+;\,\alpha,\beta,\lambda,\mu,\rho,p}$ is bounded on $L(a,b)$ and

$$\left\|\mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p,\omega;a+}\phi\right\|_1\leq\mathfrak{M}\|\phi\|_1, \tag{2.5}$$

where the constant $\mathfrak{M}$, $0<\mathfrak{M}<\infty$, given by

$$\mathfrak{M}=(b-a)^{\mathrm{Re}(\beta)}\sum_{k=0}^{\infty}\frac{|(\gamma)_{\delta\,k}|^s}{|\Gamma(\alpha\,(pk+\rho-1)+\beta)|\,(\mathrm{Re}\,(\alpha(p\,k+\rho-1)+\beta))}\times$$

$$\times\frac{|\omega\,((b-a)c)^{\alpha}|^{\mathrm{Re}(pk+\rho-1)}}{|(\lambda)_{\mu\,k}|^r\,|(\rho)_{pk}|}. \tag{2.6}$$

Proof. Using (1.2) and (1.3) and interchanging the order of integration by applying the Dirichlet formula [9], we have

$$\left\|\mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p,\omega;a+}\phi\right\|_1=$$

$$=\int\limits_a^b\left|\int\limits_a^x(x-t)^{\beta-1}\,E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p}(\,\omega(c(x-t))^{\alpha};s,r))\;\phi(t)\;dt\right|dx\leq$$

$$\leq\int\limits_a^b\left[\int\limits_t^b(x-t)^{\mathrm{Re}(\beta)-1}\left|E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p}(\omega(c(x-t))^{\alpha};s,r)\right|dx\right]|\phi(t)|\,dt.$$

On substituting $x-t=u$, using (1.1) and simplification of the above equation yields

$$=\int\limits_a^b\left[\int\limits_0^{b-t}u^{\mathrm{Re}(\beta)-1}\;\left|E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p}(\omega(cu)^{\alpha};s,r)\right|du\right]|\phi(t)|\;dt\leq$$

$$\leq\int\limits_a^b\sum_{k=0}^{\infty}\frac{|(\gamma)_{\delta\,k}|^s\,(\omega\,c^{\alpha})^{pk+\rho-1}}{\left|\Gamma(\alpha\,(pk+\rho-1)+\beta)\right|\,|(\lambda)_{\mu k}|^r\,|(\rho)_{pk}|}\times$$

$$\times\left[\int\limits_0^{b-a}u^{\mathrm{Re}(\alpha(pk+\rho-1)+\beta-1)}\right]|\phi(t)|\,dt=$$

$$= \int_a^b \sum_{k=0}^{\infty} \frac{|(\gamma)_{\delta\, k}|^s\, (\omega\, c^{\alpha})^{pk+\rho-1} |b-a|^{\mathrm{Re}(\alpha(pk+\rho-1)+\beta)}\, |\phi(t)|}{\left|\Gamma(\alpha\, (pk+\rho-1)+\beta)\right| |(\lambda)_{\mu k}|^r\, |(\rho)_{pk}| \,\mathrm{Re}\,(\alpha(pk+\rho-1)+\beta)}\, dt =$$

$$= (b-a)^{\mathrm{Re}(\beta)} \sum_{k=0}^{\infty} \frac{|(\gamma)_{\delta\, k}|^s}{|\Gamma(\alpha\, (pk+\rho-1)+\beta)|\, |(\lambda)_{\mu\, k}|^r\, |(\rho)_{pk}|} \times$$

$$\times \frac{|\omega\, (c(b-a))^{\mathrm{Re}\,(\alpha)}|^{pk+\rho-1}}{\mathrm{Re}\,(\alpha(pk+\rho-1)+\beta)} \int_a^b |\phi(t)|\, dt =$$

$$= (b-a)^{\mathrm{Re}(\beta)} \sum_{k=0}^{\infty} \frac{|(\gamma)_{\delta\, k}|^s\, |\omega\, ((b-a)c)^{\alpha}|^{\mathrm{Re}(pk+\rho-1)}}{|\Gamma(\alpha\, (pk+\rho-1)+\beta)|\, \mathrm{Re}\,(\alpha(pk+\rho-1)+\beta)} \frac{\|\phi\|_1}{|(\lambda)_{\mu\, k}|^r\, |(\rho)_{pk}|} =$$

$$= \mathfrak{M}\, \|\phi\|_1,$$

where $\mathfrak{M}$ is finite and given by (2.6). This completes proof of the boundedness property of the integral operator $\mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p,\omega;a+}$ as asserted by Theorem 2.4.

The following theorem incorporates the fractional differential equation for (1.2).

Theorem 2.5. *If* $0 < \eta < 1$, $0 \le \nu \le 1$, $\omega, \xi \in \mathbb{C}$, $R(\alpha) = R(\delta) - 1 > 0$ *and* $\min\{\mathrm{Re}(\beta, \gamma, \lambda, \mu, \rho)\} > 0$, *then*

$$\left(D^{\eta;\nu}_{0+}\, y\right)(x) = \xi \left(\mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p,\omega;0+}\right)(x) + f(x) \tag{2.7}$$

with the initial condition

$$\left(I_0 +^{(1-\nu)(1-\eta)}\ y\right)(0+) = C,$$

has solution in the space $L(0,\infty)$ *given by*

$$y(x) = C\, \frac{x^{\eta-\nu(1-\eta)-1}}{\Gamma(\eta-\nu+\eta\nu)} + \xi\, x^{\eta+\beta}\, E_{\alpha,\beta+\eta+1,\lambda,\mu,\rho,p}\big(\omega(a\, x^{\alpha})\big) +$$

$$+ \frac{1}{\Gamma(\eta)} \int_0^x (x-t)^{\eta-1}\, f(t)\, dt, \tag{2.8}$$

where C is arbitrary constant.

Proof. Applying the Laplace transform of each side of (2.7), and using the formulas (1.2) and (1.11), we find by means of the Laplace convolution theorem that

$$s^{\eta}\, Y(s) - C\, s^{\nu\,(1-\eta)} = \xi\, \mathcal{L}\big[x^{\beta-1}\, E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p}(\omega\, x^{\alpha})\big](s)\mathcal{L}(1)(s) + F(s) =$$

$$= \xi\, s^{-\beta-1} \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta\, n}]^s\, (\omega\, (as)^{\alpha})^{pn+\rho-1}}{[(\lambda)_{\mu\, n}]^r (\rho)_{pn}} + F(s)$$

which readily yields

$$Y(s) = C\, s^{\nu\,(1-\eta)-\eta} + \xi\, s^{-\beta-\eta-1} \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta\, n}]^s\, (\omega\, (as)^{\alpha})^{pn+\rho-1}}{[(\lambda)_{\mu\, n}]^r\, (\rho)_{pn}} + F(s)\, s^{-\eta}. \tag{2.9}$$

Now, by taking the inverse Laplace transform of each side of equation (2.9), we get

$$y(x) = C\, \mathcal{L}^{-1}\big(s^{\nu(1-\eta)-\eta}\big)(x) +$$

$$+\xi \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta n}]^s\, (\omega\, (a)^{\alpha})^{pn+\rho-1}}{[(\lambda)_{\mu n}]^r\, (\rho)_{pn}}\, \mathcal{L}^{-1}(s^{-\alpha(pn+\rho-1)-\beta-\eta-1})(x) + \mathcal{L}^{-1}(s^{-\eta}\, F(s)) =$$

$$= C\, \frac{x^{\eta-\nu(1-\eta)-1}}{\Gamma(\eta-\nu+\eta\nu)} + \xi\, x^{\eta+\beta} \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta n}]^s\, (\omega\, a^{\alpha})^{pn+\rho-1}\, [(\rho)_{pn}]^{-1}\, x^{\alpha(pn+\rho-1)}}{\Gamma(\alpha(pn+\rho-1)+\beta+\eta+1)[(\lambda)_{\mu n}]^r} +$$

$$+\frac{1}{\Gamma(\eta)} \int_0^x (x-t)^{\eta-1}\, f(t)\, dt =$$

$$= C\, \frac{x^{\eta-\nu(1-\eta)-1}}{\Gamma(\eta-\nu+\eta\,\nu)} + \xi\, x^{\eta+\beta}\, E^{\gamma,\delta}_{\alpha,\beta+\eta+1,\lambda,\mu,\rho,p}\big(\omega(a\, x^{\alpha})\big) +$$

$$+\frac{1}{\Gamma(\eta)} \int_0^x (x-t)^{\eta-1}\, f(t)\, dt$$

which completes the proof of Theorem 2.5 under the various already stated parametric constraints.

Theorem 2.6. *If* $0 < \eta < 1,\ 0 \le \nu \le 1,\ \omega, \xi \in \mathbb{C},\ R(\alpha) = R(\delta) - 1 > 0$ *and* $\min\{R(\beta, \gamma, \lambda, \mu, \rho)\} > 0$, *then*

$$\big(D^{\eta,\nu}_{0+}\, y\big)(x) = \xi\, \Big(\mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p,\omega;0+}\Big)(x) + x^{\beta}\, E^{\gamma,\delta}_{\alpha,\beta+1,\lambda,\mu,\rho,p}\big((\omega(ax)^{\alpha}); s, r\big) \tag{2.10}$$

with the initial condition

$$\Big(I_0 +^{(1-\nu)(1-\eta)}\ y\Big)(0+) = C$$

has solution in the space $L(0,\infty)$ *given by*

$$y(x) = C\, \frac{x^{\eta-\nu(1-\eta)-1}}{\Gamma(\eta-\nu+\eta\nu)} + (\xi+1)\, x^{\eta+\beta}\, E^{\gamma,\delta}_{\alpha,\beta\eta+1,\lambda,\mu,\rho,p}\big((\omega(ax)^{\alpha}); s, r\big), \tag{2.11}$$

where C is arbitrary constant.

Proof. Now, substituting

$$f(t) = t^{\beta}\, E^{\gamma,\delta}_{\alpha,\beta+1,\lambda,\mu,\rho,p}\big((\omega(at)^{\alpha}); s, r\big)$$

in above Theorem 2.5, we get

$$y(x) = C\, \frac{x^{\eta-\nu(1-\eta)-1}}{\Gamma(\eta-\nu+\eta\,\nu)} + \xi\, x^{\eta+\beta}\, E^{\gamma,\delta}_{\alpha,\beta+\eta+1,\lambda,\mu,\rho,p}\big((\omega(ax)^{\alpha}); s, r\big) +$$

$$+\frac{1}{\Gamma(\eta)} \int_0^x (x-t)^{\eta-1}\, t^{\beta}\, E^{\gamma,\delta}_{\alpha,\beta+1,\lambda,\mu,\rho,p}\big((\omega(ax)^{\alpha}); s, r\big)\, dt. \tag{2.12}$$

Here

$$\int\limits_0^x (x-t)^{\eta-1}\, t^{\beta}\, E^{\gamma,\delta}_{\alpha,\beta+1,\lambda,\mu,\rho,p}\big((\omega(ax)^{\alpha}); s, r\big)\, dt =$$

$$= \int\limits_0^x (x-t)^{\eta-1}\, t^{\beta} \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta\, n}]^s (\omega\, (at)^{\alpha})^{pn+\rho-1}}{\Gamma(\alpha(pn+\rho-1)+\beta+1)\; [(\lambda)_{\mu\, n}]^r\; (\rho)_{pn}}\, dt =$$

$$= \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta\, n}]^s (\omega\, (a)^{\alpha})^{pn+\rho-1}}{\Gamma(\alpha(pn+\rho-1)+\beta+1)\; [(\lambda)_{\mu\, n}]^r\; (\rho)_{pn}} \times$$

$$\times \int\limits_0^x (x-t)^{\eta-1}\, t^{\alpha(pn+\rho-1)+\beta}\, dt. \qquad (2.13)$$

Take $t = xu$, then $dt = xdu$ and as $t \to 0$, $u \to 0$ and as $t \to x$, $u \to 1$

$$= \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta n}]^s\, (\omega\, a^{\alpha})^{pn+\rho-1}\, x^{\alpha(pn+\rho-1)+\eta+\beta}}{\Gamma(\alpha(pn+\rho-1)+\beta)\; [(\lambda)_{\mu n}]^r\; (\rho)_{pn}} \int\limits_0^1 (1-u)^{\eta-1}\, u^{\alpha(pn+\rho-1)+\beta} dt =$$

$$= \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta\, n}]^s\, (\omega\, a^{\alpha})^{pn+\rho-1}\, x^{\alpha(pn+\rho-1)+\eta+\beta}\, \Gamma(\eta)\, \Gamma(\alpha(pn+\rho-1)+\beta)}{\Gamma(\alpha(pn+\rho-1)+\beta)\; [(\lambda)_{\mu\, n}]^r\; (\rho)_{pn}\, \Gamma(\alpha(pn+\rho-1)+\beta+\eta+1)} =$$

$$= \Gamma(\eta) \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta\, n}]^s\, (\omega\, (ax)^{\alpha})^{pn+\rho-1}\, x^{(\eta+\beta)}}{\Gamma(\alpha(pn+\rho-1)+\beta+\eta+1)\; [(\lambda)_{\mu\, n}]^r\; (\rho)_{pn}} =$$

$$= x^{(\eta+\beta)}\, \Gamma(\eta)\, E^{\gamma,\delta}_{\alpha,\beta+\eta+1,\lambda,\mu,\rho,p}\, \big(\omega(ax)^{\alpha}; s, r\big)$$

using this in (2.12) we get (2.11).

Which completes the proof of Theorem 2.6.

Theorem 2.7 (Mellin transform of the operator $\big(\mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p,\omega;\,0_+} f\big)(x)$). *Let* $\alpha, \beta, \gamma, \lambda, \rho, \omega \in \mathbb{C}$, $\mathrm{Re}(\alpha, \beta, \gamma, \lambda, \rho) > 0$; $\delta, \mu > 0$, $p \in \mathbb{N}$, $\mathrm{Re}(1 - S - \alpha\rho + \alpha - \beta) > 0$, *then*

$$M\left\{(\mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p,\omega;\,0_+} f)(x); S\right\} = \frac{[\Gamma(\lambda)]^r \Gamma(\rho)\, p}{2\pi i\, [\Gamma(\gamma)]^s \Gamma(1-S)} H^{r+3,s+1}_{s+1,r+3} \times$$

$$\times \left[-wt^{\alpha} \,\middle|\, \begin{matrix} [(1-\gamma,\delta)]^s, (0,p) \\ (0,1), (1-S-\alpha\rho+\alpha-\beta,\ \alpha\, p), [(1-\lambda,\mu)]^r, (1-\rho,p) \end{matrix} \right] M\big\{t^{\beta} f(t);\ S\big\}. \quad (2.14)$$

Proof. By the definition of the Mellin transform, we have

$$M\left\{(\mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p,\omega;\,0_+} f)(x); S\right\} =$$

$$= \int_0^\infty x^{S-1} \int_0^x (x-t)^{\beta-1}\, E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p}\big(\omega(x-t)^\alpha; s, r\big) f(t)\, dt\, dx.$$

Interchanging the order of integration, which is permissible under the given conditions, we find that

$$M\left\{(E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p,\omega;0_+} f)(x); S\right\} =$$

$$= \int_0^\infty f(t) \int_t^\infty x^{S-1}(x-t)^{\beta-1} E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p}\big(\omega(x-t)^\alpha; s, r\big)\, dx\; dt.$$

If we set $x = t + u$ the above integral takes the form

$$M\left\{(\mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p,\omega;0_+} f)(x); S\right\} = \int_0^\infty f(t) \int_t^\infty (t+u)^{S-1} u^{\beta-1} E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p}(\omega u^\alpha)\, du\; dt.$$

To evaluate the u-integral, we express the Mittag-Leffler function in terms of its Mellin–Barnes contour integral by means of the formula (1.14), then the above expression transforms into the form

$$M\left\{(\mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p,\omega;0_+} f)(x); S\right\} = \int_0^\infty f(t) \frac{[\Gamma(\lambda)]^r \Gamma(\rho) p (-\omega)^{\rho-1}}{2\pi i [\Gamma(\gamma)]^s} \times$$

$$\times \int_{-i\infty}^{i\infty} \frac{\Gamma(-p\xi)\,\Gamma(1+p\xi)\,[\Gamma(\gamma+\delta\xi)]^s\,(-\omega)^{p\xi}}{\Gamma(\beta+\alpha\rho-\alpha+\alpha p\xi)\,[\Gamma(\lambda+\mu\xi)]^r \Gamma(\rho+p\xi)} \times$$

$$\times \int_0^\infty (t+u)^{s-1} u^{\alpha(p\xi+\rho-1)+\beta-1} du\; d\xi\; dt.$$

If we evaluate the u-integral with the help of the formula

$$\int_0^\infty x^{\nu-1}(x+a)^{-\rho} dx = \frac{\Gamma(\nu)\Gamma(\rho-\nu)}{\Gamma(\rho)}, \quad \operatorname{Re}(\rho) > \operatorname{Re}(\nu) > 0,$$

then after some simplification, it is seen that the right-hand side of above equation simplifies to

$$\frac{[\Gamma(\lambda)]^r \Gamma(\rho) p}{2\pi i [\Gamma(\gamma)]^s \Gamma(1-s)} \int_{-i\infty}^{i\infty} \frac{\Gamma(-p\xi)\Gamma(1+p\xi)[\Gamma(\gamma+\delta\xi)]^s \Gamma(1-s-\alpha(p\xi+\rho-1)-\beta)}{[\Gamma(\lambda+\mu\xi)]^r \Gamma(\rho+p\xi)} \times$$

$$\times (-\omega t^\alpha)^{p\xi+\rho-1} d\xi \int_0^\infty t^{\beta+s-1} f(t)\, dt.$$

By using the definition of H-function yields the desired result.

For $s = 1$, $r = 0$, $\rho = 1$, $p = 1$, $\delta = q$ the Theorem 2.7 reduces to the following corollary.

***Corollary* 2.1.**

$$M\Big\{(E^{\gamma,q}_{\alpha,\beta,\omega;0_+}f)(x);S\Big\}=\frac{1}{\Gamma(\gamma)\Gamma(1-S)}\times$$

$$\times H^{2,1}_{1,2}\left[-wt^{\alpha}\,\middle|\,\begin{matrix}(1-\gamma,q)\\ (0,1)(1-S-\beta,\,\alpha)\end{matrix}\right]M\{t^{\beta}f(t);S\},$$

where $\mathrm{Re}(\alpha)>0$, $\mathrm{Re}(\beta)>0$, $\mathrm{Re}(\gamma)>0$; $q\in(0,1)\cup\mathbb{N}$, $\mathrm{Re}(1-S-\beta)>0$ *and* $H^{2,1}_{1,2}(\cdot)$ *is the* H*-function defined by* (1.15).

Theorem 2.8 (Laplace transform of the operator $\big(\mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p,\omega;0_+}f\big)(x)$).

$$L\left\{(\mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p,\omega;\,0_+}f)(x);P\right\}=$$

$$=\frac{[\Gamma(\lambda)]^{r}\,\Gamma(\rho)\,\omega^{\rho-1}}{[\Gamma(\gamma)]^{s}\,P^{\beta+\alpha\rho-\alpha}}\;{}_{s+1}\psi_{r+1}\left[\begin{matrix}[(\gamma,q)]^{s}, & (1,1); & \omega^{p}/P^{p\alpha}\\ [(\lambda,\mu)]^{r}, & (\rho,p); & \end{matrix}\right]F(P),$$

where $\mathrm{Re}(\alpha)>0$, $\mathrm{Re}(\beta)>0$, $\mathrm{Re}(\gamma)>0$; $\mathrm{Re}(p)>|\omega|^{1/\,\mathrm{Re}(\alpha)}$ *and* $F(P)$ *is the Laplace transform of* $f(t)$, *defined by*

$$L\{f(t);p\}=F(P)=\int\limits_{0}^{\infty}e^{-Pt}f(t)\,dt,$$

where $\mathrm{Re}(p)>0$ *and the integral is convergent.*

Proof. By virtue of the definition of Laplace transform, it follows that

$$L\left\{(\mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p\omega;\,0_+}f)(x);P\right\}=\int\limits_{0}^{\infty}e^{-Pt}\int\limits_{0}^{x}(x-t)^{\beta-1}E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p}[\omega(x-t)^{\alpha}]f(t)\,dt\,dx.$$

Interchanging the order of integration, which is permissible under the conditions given in the theorem, we find that

$$L\left\{(E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p\,\omega;\,0_+}f)(x);P\right\}=\int\limits_{0}^{\infty}f(t)dt\int\limits_{t}^{\infty}e^{-Pt}(x-t)^{\beta-1}E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p}[\omega(x-t)^{\alpha}]\,dx.$$

If we set $x=t+u$ we obtain

$$L\left\{(\mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p\omega;0_+}f)(x);P\right\}=\int\limits_{0}^{\infty}e^{Pt}f(t)dt\int\limits_{0}^{\infty}e^{-Pu}u^{\beta-1}E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p}[\omega u^{\alpha}]\,du.$$

On making use of the series definition (1.1), the above expression becomes

$$=\sum_{k=0}^{\infty}\frac{[(\gamma)_{\delta k}]^{s}\;\omega^{(pk+\rho-1)}}{\Gamma(\alpha k+\beta)\;[(\lambda)_{\mu k}]^{r}\;(\rho)_{pk}}\int\limits_{0}^{\infty}e^{Pt}f(t)dt\int\limits_{0}^{\infty}e^{-Pu}u^{\beta+\alpha(pk+\rho-1)-1}du=$$

$$= \sum_{k=0}^{\infty} \frac{[(\gamma)_{\delta k}]^s \ \omega^{(pk+\rho-1)}}{P^{\beta+\alpha(pk+\rho-1)}[(\lambda)_{\mu k}]^r \ (\rho)_{pk}} \int_0^{\infty} e^{-Pt} f(t)dt =$$

$$= \frac{[\Gamma(\lambda)]^r \ \Gamma(\rho) \ \omega^{\rho-1}}{[\Gamma(\gamma)]^s \ P^{\beta+\alpha\rho-\alpha}} \ {}_{s+1}\psi_{r+1} \left[\begin{matrix} [(\gamma, q)]^s, & (1,1); & \omega^p/P^{p\alpha} \\ [(\lambda, \mu)]^r, & (\rho, p); & \end{matrix} \right] F(p)$$

and $F(P)$ is the Laplace transforms of $f(t)$.

For $s = 1$, $r = 0$, $\rho = 1$, $p = 1$, $\delta = q$ the Theorem 2.8 reduces to the following corollary.

***Corollary* 2.2.**

$$L\left\{(\mathcal{E}^{\gamma,q}_{\alpha,\beta,\omega;0_+} f)(x); P\right\} = \frac{1}{\Gamma(\gamma)} P_1^{-\beta} \psi_0 \left[\begin{matrix} (\gamma, q); & \omega/P^{\alpha} \\ -; & \end{matrix} \right] F(p),$$

where $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$, $\mathrm{Re}(\gamma) > 0$; $\mathrm{Re}(P) > |\omega|^{1/\mathrm{Re}(\alpha)}$ *and* $F(P)$ *is the Laplace transform of* $f(t)$, *defined by*

$$L\{f(t); P\} = F(P) = \int_0^{\infty} e^{-P\ t} f(t)\, dt,$$

where $\mathrm{Re}(P) > 0$ *and the integral is convergent.*

3. Properties. In this section certain properties of the functions $E_t(c, \nu, \gamma, \delta, \lambda, \mu, \rho, p)$ and $E_t(c, -\eta, \gamma, \delta, \lambda, \mu, \rho, p)$ will be obtained. We begin with the function

$$f(t) = \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta n}]^s \ (ct)^{(pn+\rho-1)}}{\Gamma(\rho) \ ((\rho)_{pn})^2 \ [(\lambda)_{\mu n}]^r},$$

where $\gamma \in \mathbb{C}$, $\delta > 0$, c — arbitrary constant.

Now, using (1.4), the fractional integral operator of order ν is given as

$$I^{\nu} \ f(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-\xi)^{\nu-1} \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta n}]^s \ (c\xi)^{(pn+\rho-1)}}{\Gamma(\rho) \ ((\rho)_{pn})^2 \ [(\lambda)_{\mu n}]^r} \ d\xi =$$

$$= \frac{1}{\Gamma(\nu)} \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta n}]^s \ c^{(pn+\rho-1)}}{\Gamma(\rho) \ ((\rho)_{pn})^2 \ [(\lambda)_{\mu n}]^r} \int_0^t \xi^{pn+\rho-1}(t-\xi)^{\nu-1} \ d\xi.$$

After some simplification and using (1.1), we can write

$$I^{\nu} \ f(t) = t^{\nu} \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta\ n}]^s \ (ct)^{(pn+\rho-1)}}{\Gamma(1(pn+\rho-1)+\nu+1) \ ((\rho)_{pn}) \ [(\lambda)_{\mu\ n}]^r} = \tag{3.1}$$

$$= t^{\nu} \ E^{\gamma,\delta}_{1,\nu+1,\lambda,\mu,\rho,p}(ct; s, r). \tag{3.2}$$

We denote the function (3.2) as $E_t(c, \nu, \gamma, \delta, \lambda, \mu, \rho, p)$, i.e.,

$$E_t(c,\nu,\gamma,\delta,\lambda,\mu,\rho,p) = t^{\nu}\ E^{\gamma,\delta}_{1,\nu+1,\lambda,\mu,\rho,p}(ct;s,r). \tag{3.3}$$

Now, using (1.7), the fractional differential operator of order η is given as

$$D^{\eta}\ f(t) = D^{k}\left[I^{k-\eta}\sum_{n=0}^{\infty}\frac{[(\gamma)_{\delta\, n}]^{s}\ (ct)^{(pn+\rho-1)}}{\Gamma(\rho)\ ((\rho)_{pn})^{2}}\ [(\lambda)_{\mu\, n}]^{r}\right].$$

Applying (3.1), after some simplification and using (1.1) it yields

$$D^{\eta}\ f(t) = t^{-\eta}\sum_{n=0}^{\infty}\frac{[(\gamma)_{\delta n}]^{s}\ (ct)^{(pn+\rho-1)}}{\Gamma(1(pn+\rho-1)+(1-\eta))\ ((\rho)_{pn})\ [(\lambda)_{\mu n}]^{r}} =$$

$$= t^{-\eta}\ E^{\gamma,\delta}_{1,1-\eta,\lambda,\mu,\rho,p}(ct;s,r). \tag{3.4}$$

We denote the function (3.4) as $E_t(c,-\eta,\gamma,\delta,\lambda,\mu,\rho,p)$, i.e.,

$$E_t(c,-\eta,\gamma,\delta,\lambda,\mu,\rho,p) = t^{-\eta}\ E^{\gamma,\delta}_{1,1-\eta,\lambda,\mu,\rho,p}(ct;s,r). \tag{3.5}$$

Theorem 3.1. *Let* $\gamma\in\mathbb{C}$, $\mathrm{Re}(\gamma)>0$, $\delta>0$, c *is arbitrary constant and fractional integral and differential operator is of order* σ, *then*

$$I^{\sigma}E_t(c,\nu,\gamma,\delta,\lambda,\mu,\rho,p) = E_t(c,\sigma+\nu,\gamma,\delta,\lambda,\mu,\rho,p) \tag{3.6}$$

and

$$D^{\sigma}\big(E_t(c,\nu,\gamma,\delta,\lambda,\mu,\rho,p)\big) = E_t(c,\nu-\sigma,\gamma,\delta,\lambda,\mu,\rho,p). \tag{3.7}$$

Proof. From (1.4), we get

$$I^{\sigma}E_t(c,\nu,\gamma,\delta,\lambda,\mu,\rho,p) = \frac{1}{\Gamma(\sigma)}\int_0^t (t-\xi)^{\sigma-1}\ E_{\xi}(c,\nu,\gamma,\delta,\lambda,\mu,\rho,p)\ d\xi.$$

Using (3.3), above equation becomes,

$$I^{\sigma}E_t(c,\nu,\gamma,\delta,\lambda,\mu,\rho,p) = \frac{1}{\Gamma(\sigma)}\int_0^t (t-\xi)^{\sigma-1}\ \xi^{\nu}\times$$

$$\times\sum_{n=0}^{\infty}\frac{[(\gamma)_{\delta\, n}]^{s}\ (c\xi)^{(pn+\rho-1)}}{\Gamma(1(pn+\rho-1)+\nu+1)\ ((\rho)_{pn})\ [(\lambda)_{\mu\, n}]^{r}}\ d\xi.$$

Now, substituting $\xi = xt$, after some simplification and once again use of (3.3) gives (3.6).

From (1.7) and using (3.6), we get

$$D^{\sigma}\big(E_t(c,\nu,\gamma,\delta,\lambda,\mu,\rho,p)\big) = D^{k}\big\{t^{k-\sigma+\nu}\ E^{\gamma,\delta}_{1,k-\eta+\nu+1,\lambda,\mu,\rho,p}(ct;s,r)\big\}.$$

Using (1.1) and (3.3), we get (3.7).

In the light of the Theorem 2.7, we prove following theorem.

Theorem 3.2. *Let* $\eta \in \mathbb{C}$, $\mathrm{Re}(\eta) > 0$, $\delta > 0$, c *is arbitrary constant and fractional integral and differential operator is of order* σ, *then*

$$I^{\sigma} E_t(c, -\eta, \gamma, \delta, \lambda, \mu, \rho, p) = E_t(c, \sigma - \eta, \gamma, \delta, \lambda, \mu, \rho, p),$$

$$D^{\sigma}\big(E_t(c, -\eta, \gamma, \delta, \lambda, \mu, \rho, p)\big) = E_t(c, -\sigma - \eta, \gamma, \delta, \lambda, \mu, \rho, p).$$

Acknowledgement. Authors are indebted to Dr. B. I. Dave for his guidance and sincere thanks to him for his kind help in LaTeX type setting.

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Received 14.05.12