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CLT-GROUPS WITH HALL S-QUASINORMALLY EMBEDDED SUBGROUPS*
CLT-ГРУПИ З S-КВАЗІНОРМАЛЬНО ВКЛАДЕНИМИ ПІДГРУПАМИ ХОЛЛА

A subgroup H of a finite group G is said to be Hall S-quasinormally embedded in G if H is a Hall subgroup of the S-quasinormal closure H^{SQG}. We study finite groups G containing a Hall S-quasinormally embedded subgroup of index p^n for each prime power divisor p^n of the order of G.

Підгрупа H скінченної групи G називається підгрупою Холла, S-квазінормально вкладеною в G, якщо H — підгрупа Холла S-квазінормального замикання H^{SQG}. Вивчаються скінченні групи G, що містять S-квазінормально вкладені підгрупи Холла індексу p^n для кожного простого степеневого дільника p^n порядку G.

1. Introduction. All groups considered in this paper are finite, our notation and terminology are standard (see, for example, Robinson [25]).

A CLT-group is a group G of order n, say, having the property that for each divisor d of n, there exists a subgroup in G of order d. Clearly, a CLT-group has Hall p'-subgroups for all primes p, and hence it is solvable, but the converse is not true in general, the alternating group of degree 4 is an example of a non-CLT-group. Several years later, a nice extension was gave by T. M. Gagen [13], which every solvable group can be embedded in a directly indecomposable CLT-group. Adding requirements to the location or structure of the subgroup of order d yields various subclasses of CLT-groups. In this aspect, C.V. Holmes first proved the following result.

Theorem 1.1 ([15], Theorem 1)**.** *A group G is nilpotent if and only if for each divisor d of the order of G there exists a normal subgroup of order d.*

Recently, S. R. Li, J. He, G. P. Nong and L. Q. Zhou [20] studied a new class of CLT-groups. They introduced the following definition:

Definition 1.1 ([20], Definition 1)**.** *A subgroup H of a group G is called Hall normally embedded in G if H is a Hall subgroup of the normal closure H^G.*

They studied the structure of a group G under the assumption that, for every factor d of the order of G there exists a Hall normally embedded subgroup H of G of order d.

Some related topics can be found in [1, 3 – 11, 13 – 17, 19 – 24, 26 – 28, 30, 31] and [29] (Chapters 1, 4 and 6).

Recall that a subgroup H of a group G is S-quasinormal in G if $HP = PH$ for all Sylow subgroups P of G.

In this paper we analyze some results on the base of the following concept.

Definition 1.2. *A subgroup H of a group G is called a Hall S-quasinormally embedded subgroup of G if H is a Hall subgroup of the S-quasinormal closure H^{SQG}, the intersection of all the S-quasinormal subgroups of G which contain H.*

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By definition, all S-quasinormal subgroups and all Hall subgroups (particularly Sylow subgroups) of G are Hall S-quasinormally embedded in G. Clearly, a Hall normally embedded subgroup is certainly a Hall S-quasinormally embedded subgroup, but the converse is not true in general, as the following example shows:

***Example* 1.1.** Let $G = \langle a, b, c | a^8 = b^2 = c^3 = 1, b^{-1}ab = a^{-1}, [a, c] = [b, c] = 1\rangle$, then the group G is a direct product of a dihedral group of order 16 and a cyclic group of order 3. We write $H = \langle b\rangle$, generated by b. It is clear that $H\langle c\rangle = \langle c\rangle H$ and so H is S-quasinormal in G. This implies that $H = H^{SQG}$ and hence H is Hall S-quasinormally embedded in G. We can see that $H^G = \langle a^6, b\rangle$. It means that H is not a Hall subgroup of H^G. Thus H is not Hall normally embedded in G.

In this paper, it is proved that, for each prime power divisor p^n of the order of a group G, there exists a Hall S-quasinormal embedded subgroup of index p^n if and only if the nilpotent residual of G is cyclic of square-free order.

2. Preliminaries. In this section we show some lemmas, which are required in Section 3.

Lemma 2.1 ([2], Theorems 1.2.14 and 1.2.19). *Let G be a group. Then the following statements hold*:

(1) *If K is a subgroup of G and H is S-quasinormal in G, then $H \cap K$ is S-quasinormal in K.*

(2) *If H_1 and H_2 are two S-quasinormal subgroups of G, then $H_1 \cap H_2$ is S-quasinormal in G.*

(3) *If H is S-quasinormal in G, then H is subnormal in G.*

Lemma 2.2. *Let H be an S-quasinormal subgroup of a group G. Then H^g is also an S-quasinormal subgroup of G, where $g \in G$.*

Proof. This is obtained by direct checking.

Lemma 2.3. *Let H be a subgroup of a group G. If there exists an S-quasinormal subgroup K of G containing H such that H is a Hall subgroup of K, then H is Hall S-quasinormally embedded in G.*

Proof. According to our hypothesis and Lemma 2.1, we can see that H^{SQG} is a subgroup of K. Hence H is a Hall subgroup of H^{SQG}, as desired.

Lemma 2.4. *Let H be a Hall S-quasinormally embedded subgroup of a group G. Then the following statements hold*:

(a) *If $H \le K \le G$, then H is Hall S-quasinormally embedded in K.*

(b) *If $N \trianglelefteq G$, then HN/N is Hall S-quasinormally embedded in G/N.*

(c) *If N is S-quasinormal in G, then $H \cap N$ is Hall S-quasinormally embedded in G.*

However,

(d) *If $N \trianglelefteq G$, then HN may not be Hall S-quasinormally embedded in G.*

(e) *If $N \trianglelefteq G$ and $N \le K$, then K/N is Hall S-quasinormally embedded in G/N does not imply that K is Hall S-quasinormally embedded in G.*

Proof. (a) Since H is a Hall subgroup of H^{SQG}, H is a Hall subgroup of $H^{SQG} \cap K$. Furthermore, $H^{SQK} \le H^{SQG} \cap K$ and $H^{SQG} \cap K$ is S-quasinormal in K by Lemma 2.1. It follows from Lemma 2.3 that H is a Hall S-quasinormally embedded subgroup of K.

(b) Let π denote the set of prime factors of the order of H. Then H is a π-group and $|H^{SQG} : H|$ is a π'-number. As $(HN)^{SQG} \le H^{SQG}N$, we can see that $|(HN)^{SQG} : HN| \leqslant |H^{SQG}N : HN| = |H^{SQG} : H| / |H^{SQG} \cap N : H \cap N|$, which is a π'-number. Hence HN/N is a Hall subgroup of $(HN/N)^{SQG}$ and therefore HN/N is Hall S-quasinormally embedded in G/N.

(c) It is clear that $H \cap N \leq (H \cap N)^{SQG} \leq H^{SQG} \cap N$ and $H^{SQG} \cap N$ is S-quasinormal in G by Lemma 2.1. On the other hand, H is a Hall subgroup of H^{SQG}. It follows that $H \cap N$ is a Hall subgroup of $H^{SQG} \cap N$. Applying Lemma 2.3, we conclude that $H \cap N$ is Hall S-quasinormally embedded in G.

(d) Let $G = \langle a, b, c, d \mid a^3 = b^2 = c^3 = d^2 = 1,\ b^{-1}ab = a^{-1},\ d^{-1}cd = c^{-1},\ [a, c] = [a, d] = = [b, c] = [b, d] = 1\rangle$, then the group G is a direct product of two symmetric groups of degree 3. We write $H = \langle b\rangle$, generated by b, then $H^{SQG} = \langle a, b\rangle$. It is clear that H is a Hall subgroup of H^{SQG}. That is, H is Hall S-quasinormally embedded in G. Take $N = \langle c, d\rangle$ to be the subgroup of G generated by c and d, then N is normal in G. We can see that $HN = \langle b, c, d\rangle$ and $(HN)^{SQG} = G$, which means that HN is not a Hall subgroup of $(HN)^{SQG}$. Thus HN is not Hall S-quasinormally embedded in G.

(e) Consider a group $K = HN$ as in the proof of (d). We can identify that K/N is a Sylow 2-subgroup of G/N and hence K/N is a Hall S-quasinormally embedded subgroup of G/N. However, K is not Hall S-quasinormally embedded in G.

Lemma 2.4 is proved.

Lemma 2.5. *Let H be an S-quasinormal subgroup of a solvable group G. If p is a prime dividing the order of G and H_1 is a Hall p'-subgroup of G containing in H, then $G = N_G(H_1)H$.*

Proof. Applying Lemma 2.1, H is subnormal in G. Therefore, there exists a subgroups series

$$H = G_0 \leq G_1 \leq \ldots \leq G_n = G$$

such that $G_i \trianglelefteq G_{i+1}$, where $0 \leq i \leq n-1$. We prove the lemma by induction on n and suppose that it has already been shown that $G_i = N_{G_i}(H_1)H$ for some $i \in \{1, 2, \ldots, n-1\}$. By the Frattini argument, $G_{i+1} = N_{G_{i+1}}(H_1)G_i = N_{G_{i+1}}(H_1)H$. This completes the induction argument.

Lemma 2.5 is proved.

Let $\mathcal{N}$ denote the class of all nilpotent groups, then $\mathcal{N}$ is a saturated formation. We denote by $G^{\mathcal{N}}$ the nilpotent residual of a group G.

Lemma 2.6. *Let H be a subgroup of a group G. Then $H^{\mathcal{N}} \leq G^{\mathcal{N}}$.*

Proof. Since $H/(H \cap G^{\mathcal{N}}) \cong HG^{\mathcal{N}}/G^{\mathcal{N}} \leq G/G^{\mathcal{N}}$ is nilpotent, we can see that $H^{\mathcal{N}} \leq G^{\mathcal{N}}$.

3. Main results. In this section, we study the structure of a group G when some subgroups are Hall S-quasinormally embedded in G. Our first result is about supersolvability.

Theorem 3.1. *For each prime power divisor p^n of the order of a group G, if there exists a Hall S-quasinormally embedded subgroup of G of index p^n, then G is supersolvable.*

Proof. The proof will follow as a consequence of the following steps.

1. *Every Hall subgroup M of G satisfies the hypothesis of the theorem.* Let $\pi = \pi(M)$, the set of primes of dividing $|M|$. Set $p^n \| |M|$. By hypothesis, there exists a subgroup H of G of index p^n such that H is Hall S-quasinormally embedded in G. Let H_1 is a Hall π-subgroup of H. Then from [25] (Theorem 9.1.7) it follows that $H_1^g \leq M$, for some $g \in G$. We can conclude that $|M : H_1^g| = p^n$. To finish the proof of the statement it is enough to check that H_1^g is Hall S-quasinormally embedded in M. In fact, if $H < H^{SQG}$, then from $(|H^{SQG} : H|, |H|) = 1$ and $|H^{SQG} : H| \big| |G : H|$ we obtain that H is a Hall subgroup of G. It follows that H_1^g is a Hall subgroup of M, as desired. If $H = H^{SQG}$, then from Lemma 2.1 H is S-quasinormal in G and so is H^g by Lemma 2.2. Applying Lemma 2.1 again, $H^g \cap M$ is S-quasinormal in M. Moreover, we can see that H_1^g is a Hall subgroup of $H^g \cap M$, hence H_1^g is Hall S-quasinormally embedded in M by Lemma 2.3.

2. *Let p be the smallest prime dividing the order of G, then G is p-nilpotent.* Let $P \in \mathrm{Syl}_p(G)$. If $p^2 \dagger |G|$, then by a theorem of Burnside [18] (IV, 2.8 Satz), G is p-nilpotent. Hence we can assume

that P is not a cyclic group. By hypothesis, there exists a subgroup H of G of index $|P|/p$ such that H is a Hall subgroup of H^{SQG}. Now the Burnside's theorem [18] (IV, 2.8 Satz) implies that H is p-nilpotent. If $H = H^{SQG}$, then H is subnormal in G by Lemma 2.1, this means that G is p-nilpotent. If $H < H^{SQG}$, then it follows from $(|H^{SQG}\colon H|, |H|) = 1$ and $|H^{SQG}\colon H|\big||G\colon H|$ that H is a Hall p'-subgroup, it is impossible.

3. *G possesses Sylow tower of supersolvable type.* Let K be a normal p-complement of G and q^m a prime power divisor of the order of K, where $q \neq p$. By hypothesis, there exists a subgroup H of G of index q^m such that H is Hall S-quasinormally embedded in G. It follows from $(|G\colon K|, |G\colon H|) = 1$ that $G = HK$ and hence $|K : K \cap H| = |G\colon H| = q^m$. By Lemma 2.4, $K \cap H$ is Hall S-quasinormally embedded in K. So K satisfies the hypothesis. By induction, K possesses Sylow tower of supersolvable type. Hence G possesses Sylow tower of supersolvable type, as desired.

4. *Finish the proof.* Let q be the largest prime divisor of the order of G and $Q \in \mathrm{Syl}_q(G)$. Then, by hypothesis, G contains a subgroup H of index $|Q|/q$ such that H is a Hall subgroup of H^{SQG}. We can argue as above to deduce that $H = H^{SQG}$ and hence H is S-quasinormal in G. Let $H = H_1(H \cap Q)$, where H_1 is Hall q'-subgroup of H. In view of Lemma 2.5, $G = N_G(H_1)H = N_G(H_1)(H \cap Q)$. We can see that $H_1^G = H_1^{N_G(H_1)(H\cap Q)} = H_1^{H\cap Q} \leq H$, it is clear that $H_1^G = H$ or H_1. If $H_1^G = H$, then $H \trianglelefteq G$. Since H_1 is a Hall subgroup of G, then, by statement 1 and induction argument, H_1 is supersolvable and so is $N_G(H_1)$. We can conclude that $G\big/(H\cap Q)$ is supersolvable and $|H\cap Q| = q$, this means that G is supersolvable. If the latter is true, then $G = H_1 \times Q$ and hence G is supersolvable.

Theorem 3.1 is proved.

We can now prove the following theorem.

Theorem 3.2. *Let G be a group. Then the following statements are equivalent*:

(a) *For each prime power divisor p^n of the order of G there exists a Hall S-quasinormally embedded subgroup of G of index p^n.*

(b) *For each divisor d of the order of G there exists a Hall S-quasinormally embedded subgroup of G of order d.*

(c) $G = G^{\mathcal{N}}N$ *with* $G^{\mathcal{N}} \cap N = 1$, *where $G^{\mathcal{N}}$ is a cyclic group of square-free order.*

(d) $G^{\mathcal{N}}$ *is cyclic of square-free order.*

Proof. (a) $\Rightarrow$ (b). Let $|G|/d = p_1^{a_1}p_2^{a_2}\ldots p_n^{a_n}$, where $a_i > 0$ and $p_1, p_2, \ldots, p_n$ are distinct primes. According to statement (a), there exists a Hall S-quasinormally embedded subgroup B_i of G of index $p_i^{a_i}$ for all $1 \leq i \leq n$. Set

$$H = B_1 \cap B_2 \cap \ldots \cap B_n.$$

Since all $|G\colon B_i|$ are pairwise coprime, by [18] (I, 2.13 Hilssatz) we have

$$|G\colon H| = |G\colon B_1||G\colon B_2|\ldots|G\colon B_n| = p_1^{a_1}p_2^{a_2}\ldots p_n^{a_n},$$

whence $|H| = d$. To finishing our proof, we only need to show that H is a Hall subgroup of H^{SQG}. It is clear that B_i is a Hall subgroup of B_i^{SQG}. If $B_i < B_i^{SQG}$, then B_i is a Hall p_i'-subgroup of G. If $B_i = B_i^{SQG}$, then B_i is S-quasinormal in G. Hence we may assume without loss of generality that every element of $\{B_1, B_2, \ldots, B_j\}$ is a Hall subgroup of G and every element of $\{B_{j+1}, B_{j+2}, \ldots, B_n\}$ is S-quasinormal in G, where $0 \leq j \leq n$. Then

$$C = B_1 \cap \ldots \cap B_j$$

is a Hall subgroup of G and

$$D = B_{j+1} \cap \ldots \cap B_n$$

is a S-quasinormal subgroup of G. Moreover, $H = C \cap D$. Applying Lemma 2.4, we have H is Hall S-quasinormally embedded in G.

(b) $\Rightarrow$ (a). Clear.

(b) $\Rightarrow$ (c). By Theorem 3.1, G is supersolvable. Let q be the largest prime dividing the order of G and $Q \in \mathrm{Syl}_q(G)$. Then, the theorem of Schur – Zassenhaus gives a complement K of Q in G and $G = KQ$. The statement (c) will follow from the next four steps.

1. *K satisfies the hypothesis of statement* (b), *by induction,* $K = K^{\mathcal{N}} K_1$ *with* $K^{\mathcal{N}} \cap K_1 = 1$, *and the nilpotent residual* $K^{\mathcal{N}}$ *is cyclic of square-free order.* Let d be a divisor of the order of K. By hypothesis, G contains a subgroup H of order d such that H is Hall S-quasinormally embedded in G. In view of [25] (Theorem 9.1.7), $H^g \leq K$ for some $g \in G$. Notice that $H^{SQG} = L$ is S-quasinormal in G, it follows from Lemma 2.2 that L^g is S-quasinormal in G and therefore $L^g \cap K$ is S-quasinormal in K by Lemma 2.1. Now the Lemma 2.3 may be applied to K to show that H^g is Hall S-quasinormal embedded in K, as required.

2. $G = K[K,Q] \times C_Q(K) = KQ_1 \times Q_2$, *where* $Q_1 = [K,Q]$, $Q_2 = C_Q(K)$. In view of [12] (Proposition 12.5), we can see that $Q = [K,Q]C_Q(K)$. By hypothesis, there exists a Hall S-quasinormally embedded subgroup H of G of order $q|K|$. We can conclude that $H = H^{SQG}$. Write $H = KQ_1$, where $Q_1 = H \cap Q$. Applying Lemma 2.5, $G = N_G(K)H = N_G(K)Q_1$. If $Q_1 \leq N_G(K)$, then $K \trianglelefteq G$ and therefore $G = K \times Q = K \times C_Q(K)$, as desired. Hence we must only consider the case that $Q_1 \nleq N_G(K)$. Let $Q_2 = Q \cap N_G(K)$, then $N_G(K) = Q_2 \times K$ and so $G = KQ_2Q_1$. In this case, $[K,Q] = [K,Q_2Q_1] = [K,Q_1] = Q_1$. Since Q_2 is a maximal subgroup of Q and $K \leq C_G(Q_2)$, both Q_2 and KQ_1 are normal in G. This implies that $G = KQ_1Q_2 = K[K,Q] \times Q_2$. Furthermore, we have $C_Q(K) = C_G(K) \cap Q \geq Q_2$. If $C_Q(K) > Q_2$, then $C_Q(K) = Q$ and therefore $G = K \times Q$, in contradiction to the fact that $Q_1 \nleq N_G(K)$. Hence $C_Q(K) = Q_2$, as desired.

3. $G^{\mathcal{N}} = K^{\mathcal{N}}Q_1$. Obviously, K normalizes $K^{\mathcal{N}}$ and Q_1. It follows from $G = KQ_1Q_2$ that $K^{\mathcal{N}}Q_1$ is normal in G. We obtain that

$$G/K^{\mathcal{N}}Q_1 = KQ_1/K^{\mathcal{N}}Q_1 \times K^{\mathcal{N}}Q_1Q_2/K^{\mathcal{N}}Q_1 \cong K/K^{\mathcal{N}} \times K^{\mathcal{N}}Q_1Q_2/K^{\mathcal{N}}Q_1$$

is nilpotent, which shows that $G^{\mathcal{N}} \leq K^{\mathcal{N}}Q_1$. Since $|Q_1| = q$ or 1, we see $Q_1 \cap G^{\mathcal{N}} = Q_1$ or 1. If the latter is true, then, since $G = G/(Q_1 \cap G^{\mathcal{N}}) \lesssim G/Q_1 \times G/G^{\mathcal{N}}$ is q-nilpotent, we have that K is normal in G and thus $Q_1 = [K,Q] = 1$, Consequently, $G^{\mathcal{N}} = K^{\mathcal{N}}$ by Lemma 2.6, as desired. Thus we consider $Q_1 \leq G^{\mathcal{N}}$, in this case, $K^{\mathcal{N}}Q_1 \leq G^{\mathcal{N}}$ and hence $G^{\mathcal{N}} = K^{\mathcal{N}}Q_1$.

4. *Finish the proof.* By statement 2, $K^{\mathcal{N}}$ is cyclic of square-free order and Q_1 is of order q or 1, it follows that $G^{\mathcal{N}}$ is of square-free order. As G is supersolvable, we can see that G' is nilpotent by [18] (VI, 9.1 Satz). Moreover, $G^{\mathcal{N}} \leq G'$ and so $G^{\mathcal{N}}$ is cyclic. Let $N = K_1Q_2$, then $G = KQ_1Q_2 = G^{\mathcal{N}}K_1Q_2 = G^{\mathcal{N}}N$, as desired.

(c) $\Rightarrow$ (d). Clear.

(d) $\Rightarrow$ (b). Let $|G|/d = p_1^{a_1}p_2^{a_2}\ldots p_n^{a_n}$. Without generality, we may assume that every element of $\{p_1, p_2, \ldots, p_i\}$ is not a prime divisor of the order of $G^{\mathcal{N}}$, every element of $\{p_{i+1}, p_{i+2}, \ldots, p_n\}$ is a prime divisor of $G^{\mathcal{N}}$, where $0 \leq i \leq n$. We can conclude that G is p_k-nilpotent, where $1 \leq k \leq i$. Hence there exists a normal subgroup H_k of G such that $|G \colon H_k| = p_k^{a_k}$. Let $p_j \big| |G^{\mathcal{N}}|$, where $i+1 \leq j \leq n$. If $p_j^{a_j+1} \dagger |G|$, then, since G is solvable, it follows that G contains a Hall p_j'-subgroup H_j and therefore H_j is Hall S-quasinormally embedded in G. Now we may assume that $p_j^{a_j+1} \big| |G|$.

As $G/G^{\mathcal{N}}$ is nilpotent and $G^{\mathcal{N}}$ is of square-free order, we deduce that there exists a normal subgroup $H_j/G^{\mathcal{N}}$ of $G/G^{\mathcal{N}}$ such that $|G\colon H_j| = p_j^{a_j}$. Thus there exists a subgroup H_j of G of index $p_j^{a_j}$ such that H_j is Hall S-quasinormally embedded in G in these two cases. Put

$$H = H_1 \cap H_2 \cap \ldots \cap H_n.$$

Since all $|G\colon H_i|$ are pairwise coprime, by [18] (I, 2.13 Hilssatz) we have

$$|G\colon H| = |G\colon H_1||G\colon H_2| \ldots |G\colon H_n| = p_1^{a_1} p_2^{a_2} \ldots p_n^{a_n}.$$

It is clear that $|H| = d$. By Lemma 2.4, H is a Hall S-quasinormally embedded in G.

Theorem 3.2 is proved.

For convenience, we can give the following definition:

A group G is called an SEG-group if for each prime power divisor p^n of the order of G, there exists a Hall S-quasinormally embedded subgroup of G of index p^n.

Notice that the class of CLT-groups is not closed under taking subgroups and quotient groups in general. However, we have the following theorem.

Theorem 3.3. *Let G be an SEG-group. Then the following statements are true*:

(1) *every subgroup of G is an SEG-group*;

(2) *every epimorphic image of G is an SEG-group.*

Proof. (1) Let $K \leq G$, then $K^{\mathcal{N}} \leq G^{\mathcal{N}}$ by Lemma 2.6. It follows from Theorem 3.2 that $G^{\mathcal{N}}$ is cyclic of square-free order and so is $K^{\mathcal{N}}$. Applying Theorem 3.2 again, K is an SEG-group.

(2) Let N be a normal subgroup of G. It follows from Theorem 3.2 and $(G/N)^{\mathcal{N}} = G^{\mathcal{N}}N\big/N \cong$ $\cong G^{\mathcal{N}}\big/G^{\mathcal{N}} \cap N$ that $(G/N)^{\mathcal{N}}$ is cyclic of square-free order. Again by Theorem 3.2, we can see that G/N is an SEG-group.

Theorem 3.3 is proved.

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