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SOME APPLICATIONS OF THE OPEN MAPPING THEOREM IN LOCALLY CONVEX CONES

ДЕЯКІ ЗАСТОСУВАННЯ ТЕОРЕМИ ПРО ВІДКРИТЕ ВІДОБРАЖЕННЯ У ЛОКАЛЬНО-ОПУКЛИХ КОНУСАХ

We show that a continuous open linear operator preserves the completeness and barreledness in locally convex cones. Specially, we prove some relations between an open linear operator and its adjoint in uc-cones (locally convex cones which their convex quasi-uniform structures are generated by one element).

Показано, що неперервний відкритий лінійний оператор зберігає повноту та бочкуватість у локально-опуклих конусах. Зокрема, доведено деякі співвідношення між відкритим лінійним оператором та його суміжним у uc-конусах (локально-опуклих конусах, у яких опуклі квазірівномірні структури генеруються одним елементом).

1. Introduction. The theory of locally convex cones deals with ordered cones that are not necessarily embeddable in vector spaces. A topological structure is introduced using an order theoretical concept or a convex quasiuniform structure. In this paper we use the latter. These cones developed in [4, 8]. For recent researches see [1, 9]. We shall review some of the concepts and refer to [4, 8] for details.

A cone is defined to be a commutative monoid $\mathcal{P}$ together with a scalar multiplication by nonnegative real numbers satisfying the same axioms as for vector spaces; that is, $\mathcal{P}$ is endowed with an addition $(a, b) \mapsto a + b\colon \mathcal{P} \times \mathcal{P} \to \mathcal{P}$ which is associative, commutative and admits a neutral element $0 \in \mathcal{P}$, and with a scalar multiplication $(r, a) \mapsto r \cdot a\colon \mathbb{R}_+ \times \mathcal{P} \to \mathcal{P}$ satisfying the usual associative and distributive properties, where $\mathbb{R}_+$ is the set of nonnegative real numbers. We have $1 \cdot a = a$ and $0 \cdot a = 0$ for all $a \in \mathcal{P}$.

Let $\mathcal{P}$ be a cone. A convex quasiuniform structure on $\mathcal{P}$ is a collection $\mathfrak{U}$ of convex subsets $U \subseteq \mathcal{P}^2 = \mathcal{P} \times \mathcal{P}$ such that the following properties hold:

(U_1) $\Delta \subseteq U$ for every $U \in \mathfrak{U}$, where $\Delta = \big\{(a, a) : a \in \mathcal{P}\big\}$;

(U_2) for all $U, V \in \mathfrak{U}$ there is a $W \in \mathfrak{U}$ such that $W \subseteq U \cap V$;

(U_3) $\lambda U \circ \mu U \subseteq (\lambda + \mu)U$ for all $U \in \mathfrak{U}$ and $\lambda, \mu > 0$;

(U_4) $\alpha U \in \mathfrak{U}$ for all $U \in \mathfrak{U}$ and $\alpha > 0$.

Here, for $U, V \subseteq \mathcal{P}^2$, by $U \circ V$ we mean the set of all $(a, b) \in \mathcal{P}^2$ such that there is some $c \in \mathcal{P}$ with $(a, c) \in U$ and $(c, b) \in V$.

Let $\mathcal{P}$ be a cone and $\mathfrak{U}$ be a convex quasiuniform structure on $\mathcal{P}$. We shall say $(\mathcal{P}, \mathfrak{U})$ is a locally convex cone if

(U_5) for each $a \in \mathcal{P}$ and $U \in \mathfrak{U}$ there is some $\lambda > 0$ such that $(0, a) \in \lambda U$.

With every convex quasiuniform structure $\mathfrak{U}$ on $\mathcal{P}$ we associate two topologies on $\mathcal{P}$: the neighborhood bases for an element a in the upper and lower topologies are given by the sets

$$U(a) = \{b \in \mathcal{P} : (b, a) \in U\}, \quad \text{resp.,} \quad (a)U = \{b \in \mathcal{P} : (a, b) \in U\}, \quad U \in \mathfrak{U}.$$

The common refinement of the upper and lower topologies is called symmetric topology. A neighborhood base for $a \in \mathcal{P}$ in this topology is given by the sets

$$U(a)U = U(a) \cap (a)U, \quad U \in \mathfrak{U}.$$

The extended real numbers system $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is a cone endowed with the usual algebraic operations, in particular $a + (+\infty) = +\infty$ for all $a \in \overline{\mathbb{R}}$, $\alpha \cdot (+\infty) = +\infty$ for all $\alpha > 0$ and $0 \cdot (+\infty) = 0$. We set $\widetilde{\mathcal{V}} = \{\tilde{\varepsilon} : \varepsilon > 0\}$, where

$$\tilde{\varepsilon} = \left\{(a, b) \in \overline{\mathbb{R}}^2 : a \leq b + \varepsilon\right\}.$$

Then $\widetilde{\mathcal{V}}$ is a convex quasiuniform structure on $\overline{\mathbb{R}}$ and $(\overline{\mathbb{R}}, \widetilde{\mathcal{V}})$ is a locally convex cone. For $a \in \mathbb{R}$ the intervals $(-\infty, a + \varepsilon]$ are the upper and the intervals $[a - \varepsilon, +\infty]$ are the lower neighborhoods, while for $a = +\infty$ the entire cone $\overline{\mathbb{R}}$ is the only upper neighborhood, and $\{+\infty\}$ is open in the lower topology. The symmetric topology is the usual topology on $\mathbb{R}$ with as an isolated point $+\infty$.

For cones $\mathcal{P}$ and $\mathcal{Q}$, a mapping $T : \mathcal{P} \to \mathcal{Q}$ is called a *linear operator* if $T(a + b) = T(a) + T(b)$ and $T(\alpha a) = \alpha T(a)$ hold for all $a, b \in \mathcal{P}$ and $\alpha \geq 0$. If both $(\mathcal{P}, \mathfrak{U})$ and $(\mathcal{Q}, \mathcal{W})$ are locally convex cones, the operator T is called (*uniformly*) *continuous* if for every $W \in \mathcal{W}$ one can find $U \in \mathfrak{U}$ such that $(T \times T)(U) \subseteq W$. Uniform continuity implies continuity for the operator $T : \mathcal{P} \to \mathcal{Q}$ with respect to the upper, lower and symmetric topologies on $\mathcal{P}$ and $\mathcal{Q}$, respectively.

A *linear functional* on $\mathcal{P}$ is a linear operator $\mu : \mathcal{P} \to \overline{\mathbb{R}}$. We note that $\mu : \mathcal{P} \to \overline{\mathbb{R}}$ is continuous if and only if there is $U \in \mathfrak{U}$ such that $\mu(a) \leq \mu(b) + 1$ for all $(a, b) \in U$. We denote the set of all linear functional on $\mathcal{P}$ by $\mathcal{L}(\mathcal{P})$ (the algebraic dual of $\mathcal{P}$). For a subset F of $\mathcal{P}^2$, we define polar F° as follows:

$$F^\circ = \left\{\mu \in \mathcal{L}(\mathcal{P}) : \mu(a) \leq \mu(b) + 1 \quad \text{for all} \quad (a, b) \in F\right\}.$$

The dual cone $\mathcal{P}^*$ of a locally convex cone $(\mathcal{P}, \mathfrak{U})$ consists of all continuous linear functionals on $\mathcal{P}$ and is the union of all polars U° of neighborhoods $U \in \mathfrak{U}$.

2. Some applications of the open mapping theorem. An open linear operator was defined in [2] as follows:

Definition 2.1. *Let $(\mathcal{P}, \mathfrak{U})$ and $(\mathcal{Q}, \mathcal{W})$ be locally convex cones. A linear operator $T : (\mathcal{P}, \mathfrak{U}) \to (\mathcal{Q}, \mathcal{W})$ is called* (*uniformly*) *open if for every $U \in \mathfrak{U}$ one can find $W \in \mathcal{W}$ such that $W \subseteq (T \times T)(U)$.*

If $T : (\mathcal{P}, \mathfrak{U}) \to (\mathcal{Q}, \mathcal{W})$ is open, then it is open under the upper, lower and symmetric topologies. Also if $T : \mathcal{P} \to \mathcal{Q}$ is open, then T is surjective (see [2]).

A Cauchy net in locally convex cones was defined in [5] as follows.

Definition 2.2. *Let $(\mathcal{P}, \mathfrak{U})$ be a locally convex cone. A net $(x_\alpha)_{\alpha \in \mathcal{I}}$ in $\mathcal{P}$ is called lower* (*upper*) *Cauchy if for every $U \in \mathfrak{U}$ there is some $\gamma_U \in \mathcal{I}$ such that $(x_\beta, x_\alpha) \in U$* (*resp.*, *$(x_\alpha, x_\beta) \in U$*) *for all $\alpha, \beta \in \mathcal{I}$ with $\beta \geq \alpha \geq \gamma_U$. Also $(x_\alpha)_{\alpha \in \mathcal{I}}$ is called symmetric Cauchy if for each $U \in \mathfrak{U}$ there is some $\gamma_U \in \mathcal{I}$ such that $(x_\beta, x_\alpha) \in U$ for all $\alpha, \beta \in \mathcal{I}$ with $\alpha, \beta \geq \gamma_U$.*

We call that a net $(x_i)_{i\in\mathcal{I}}$ in $(\mathcal{P},\mathfrak{U})$ is lower (upper) convergent to $x\in\mathcal{P}$ if for every $U\in\mathfrak{U}$ there is some $\gamma_U\in\mathcal{I}$ such that $(x,x_i)\in U$ (resp., $(x_i,x)\in U$) for all $i\geq\gamma_U$. Also $(x_i)_{i\in\mathcal{I}}$ is called symmetric convergent to x if for each $U\in\mathfrak{U}$ there is some $\gamma_U\in\mathcal{I}$ such that $(x_i,x)\in U$ and $(x,x_i)\in U$ for all $i\geq\gamma_U$.

A locally convex cone $(\mathcal{P},\mathfrak{U})$ is called lower (upper or symmetric) complete if every lower (resp., upper or symmetric) Cauchy net in $\mathcal{P}$ converges in the lower (resp., upper or symmetric) topology to some element of $\mathcal{P}$.

Proposition 2.1. *Let $(\mathcal{P},\mathfrak{U})$ and $(\mathcal{Q},\mathcal{W})$ be locally convex cones. If there is a one-to-one open continuous linear mapping of $\mathcal{P}$ into $\mathcal{Q}$, and $\mathcal{P}$ is lower (upper or symmetric) complete, then so is $\mathcal{Q}$.*

Proof. We prove for the symmetric case. Let the mapping be T and let $(y_i)_{i\in\mathcal{I}}$ be a symmetric Cauchy net in $\mathcal{Q}$. Since every open linear mapping is surjective (see [2]), for every $i\in\mathcal{I}$, there exists $x_i\in\mathcal{P}$ such that $y_i=T(x_i)$. We show that $(x_i)_{i\in\mathcal{I}}$ is a Cauchy net in $\mathcal{P}$. Let $U\in\mathfrak{U}$ be arbitrary. By the openness of T, there exists $W\in\mathcal{W}$ such that $W\subseteq(T\times T)(U)$. Since $(y_i)_{i\in\mathcal{I}}$ is symmetric Cauchy, there is γ_W such that $(y_i,y_j)\in W$ for all $i,j\geq\gamma_W$. Hence

$$(y_i,y_j)=\big(T(x_i),T(x_j)\big)\in(T\times T)(U)$$

for all $i,j\geq\gamma_W$. Since T is one-to-one, $(x_i,x_j)\in U$ for all $i,j\geq\gamma_W$, i.e., $(x_i)_{i\in\mathcal{I}}$ is symmetric Cauchy. Since $\mathcal{P}$ is symmetric complete, there is $x\in\mathcal{P}$ such that $(x_i)_{i\in\mathcal{I}}$ converges to x in the symmetric topology. The continuity of T renders that $(T(x_i))_{i\in\mathcal{I}}$ is convergent to $T(x)\in\mathcal{Q}$. This shows that $(y_i)_{i\in\mathcal{I}}$ is convergent to $T(x)$ in the symmetric topology.

The notions of a barrel and a barreled locally convex cone were introduced in [10] as follows: a *barrel* is a convex subset B of $\mathcal{P}^2$ with the following properties:

$(B1)$ For every $b\in\mathcal{P}$ there is $U\in\mathfrak{U}$ such that for every $a\in U(b)U$ there is $\lambda>0$ such that $(a,b)\in\lambda B$.

$(B2)$ For all a,b such that $(a,b)\notin B$ there is $\mu\in\mathcal{P}^*$ such that $\mu(c)\leq\mu(d)+1$ for all $(c,d)\in B$ and $\mu(a)>\mu(b)+1$.

A locally convex cone $(\mathcal{P},\mathfrak{U})$ is said to be *barreled* if for every barrel $B\subseteq\mathcal{P}^2$ and every element $b\in\mathcal{P}$ there are neighborhood $U\in\mathfrak{U}$ and $\lambda>0$ such that $(a,b)\in\lambda B$ for all $a\in U(b)U$.

Theorem 2.1. *Let $(\mathcal{P},\mathfrak{U})$ and $(\mathcal{Q},\mathcal{W})$ be two locally convex cones. Let T be a linear continuous and open mapping of $\mathcal{P}$ into $\mathcal{Q}$. If $(\mathcal{P},\mathfrak{U})$ is barrelled, then $(\mathcal{Q},\mathcal{W})$ is barrelled too.*

Proof. Let B be a barrel in $\mathcal{Q}^2$ and $y\in\mathcal{Q}$. We show that there are $W\in\mathcal{W}$ and $\lambda>0$ such that $(x,y)\in\lambda B$ for all $x\in W(y)W$. There is an element $b\in\mathcal{P}$ such that $T(b)=y$, because T is surjective (see [2]). Since T is continuous, $(T\times T)^{-1}(B)$ is a barrel in $\mathcal{P}^2$ (see [6]). Since $\mathcal{P}$ is barrelled, there are $U\in\mathfrak{U}$ and $\lambda>0$ such that $(a,b)\in\lambda(T\times T)^{-1}(B)$ for all $a\in U(b)U$. There is $W\in\mathcal{W}$ such that

$$W\subseteq(T\times T)(U),\tag{2.1}$$

because T is open by the hypothesis. Now let $x\in W(y)W$. We have, by (2.1),

$$W(y)W=W(T(b))W\subseteq T(U(b)U).$$

Hence there is $a'\in U(b)U$ such that $T(a')=x$. Therefore $(a',b)\in\lambda(T\times T)^{-1}(B)$ and so $(x,y)\in\lambda B$.

Theorem 2.1 is proved.

A locally convex cone $(\mathcal{P}, \mathfrak{U})$ is called *upper-barreled* if for every barrel $B \subseteq \mathcal{P}^2$, there is $U \in \mathfrak{U}$ such that $U \subseteq B$ (see [6]). Proposition 2.11 from [2] yields that, every open continuous linear operator preserves the upper-barreledness between locally convex cones.

Let $(\mathcal{P}, \mathfrak{U})$ and $(\mathcal{Q}, \mathcal{W})$ be locally convex cones and $T : \mathcal{P} \to \mathcal{Q}$ be a linear operator. The *adjoint operator* $T^* : \mathcal{Q}^* \to \mathcal{L}(\mathcal{P})$ is defined as follows: for any $\mu \in \mathcal{Q}^*$ define the linear functional $T^*(\mu)$ on $\mathcal{P}$ by $T^*(\mu)(a) = \mu(T(a))$ for all $a \in \mathcal{P}$. If T is continuous, then T^* is a linear operator

Lemma 2.1. *Let $(\mathcal{P}, \mathfrak{U})$ and $(\mathcal{Q}, \mathcal{W})$ be locally convex cones and $T : \mathcal{P} \to \mathcal{Q}$ be a linear operator.*

(1) *If T is surjective, then T^* is one-to-one.*

(2) *If $\mathcal{L}(\mathcal{P})$ separates the elements of $\mathcal{P}$ and T^* is surjective, then T is one-to-one.*

Proof. (1) Suppose $T^*(\mu_1) = T^*(\mu_2)$. Then $T^*(\mu_1)(a) = T^*(\mu_2)(a)$ for all $a \in \mathcal{P}$, i.e., $\mu_1(T(a)) = \mu_2(T(a))$ for all $a \in \mathcal{P}$. Since T is surjective, $\mu_1(q) = \mu_2(q)$ for all $q \in \mathcal{Q}$. Hence $\mu_1 = \mu_2$.

(2) Suppose $T(x) = T(y)$. Let $\mu \in \mathcal{L}(\mathcal{P})$. There is $\mu_1 \in \mathcal{Q}^*$ such that $T^*(\mu_1) = \mu$. We have $\mu_1(T(x)) = \mu_1(T(y))$ and so $T^*(\mu_1)(x) = T^*(\mu_1)(y)$. Therefore $\mu(x) = \mu(y)$ for all $\mu \in \mathcal{L}(\mathcal{P})$. Hence $x = y$, since $\mathcal{L}(\mathcal{P})$ separates the elements of $\mathcal{P}$ by the hypothesis.

Lemma 2.1 is proved.

Lemma 2.2. *Let $(\mathcal{P}, \mathfrak{U})$ and $(\mathcal{Q}, \mathcal{W})$ be locally convex cones and T from $\mathcal{P}$ onto $\mathcal{Q}$ be a linear operator. Then for the adjoint operator T^* we have*

$$\widetilde{T^*(W^\circ)} = (T^* \times T^*)(\widetilde{W^\circ}),$$

where $\widetilde{W^\circ} = \{(\mu, \nu) \in \mathcal{Q}^* \times \mathcal{Q}^* : \nu \in \mu + W^\circ\}$.

Proof. Let $(\mu, \nu) \in (T^* \times T^*)(\widetilde{W^\circ})$. Then there is $(\mu', \nu') \in \widetilde{W^\circ}$ such that $\mu = T^*(\mu')$ and $\nu = T^*(\nu')$. Hence there is $\Lambda \in W^\circ$ such that $\nu' = \mu' + \Lambda$. Then $T^*(\nu') = T^*(\mu') + T^*(\Lambda)$. So $\nu = \mu + T^*(\Lambda)$, i.e., $(\mu, \nu) \in \widetilde{T^*(W^\circ)}$. Conversely, let $(\mu, \nu) \in \widetilde{T^*(W^\circ)}$. Then $\nu \in \mu + T^*(W^\circ)$. Hence there is $\Lambda \in W^\circ$ such that $\nu = \mu + T^*(\Lambda)$. Thus $T^{*-1}(\nu) = T^{*-1}(\mu) + \Lambda$ (by Lemma 2.1 (1), T^* is invertible). Hence $(T^{*-1}(\mu), T^{*-1}(\nu)) \in \widetilde{W^\circ}$, i.e., $(\mu, \nu) \in (T^* \times T^*)(\widetilde{W^\circ})$.

Lemma 2.2 is proved.

We recall the following results.

Lemma 2.3 ([2], Lemma 2.3). *Let $(\mathcal{P}, \mathfrak{U})$ and $(\mathcal{Q}, \mathcal{W})$ be locally convex cones and $T : \mathcal{P} \to \mathcal{Q}$ a linear operator. Then*:

(1) *for each subset F of $\mathcal{P}^2$, $T^{*-1}(F^\circ) = \big((T \times T)(F)\big)^\circ$,*

(2) *if the polar E° being taken in $\mathcal{Q}^*$, for each subset E of $\mathcal{Q}^2$, $T^*(E^\circ) \subseteq \big((T \times T)^{-1}(E)\big)^\circ$ and if T is invertible, then we have the inverse inclusion, i.e., $T^*(E^\circ) = \big((T \times T)^{-1}(E)\big)^\circ$.*

Theorem 2.2 (Extension theorem [4], II.2.9). *Let $\mathcal{Q}$ be a subcone of a locally convex cone $(\mathcal{P}, \mathfrak{U})$. Then every continuous linear functional on $\mathcal{Q}$ can be extended to a continuous linear functional on $\mathcal{P}$.*

A locally convex cone $(\mathcal{P}, \mathfrak{U})$ is called a uc-cone whenever $\mathfrak{U} = \{\alpha U : \alpha > 0\}$ for some $U \in \mathfrak{U}$. The uc-cones in locally convex cones play the role of normed spaces in topological vector spaces. If $(\mathcal{P}, \mathfrak{U})$ is a uc-cone and $\mathfrak{U} = \{\alpha U : \alpha > 0\}$, then $(\mathcal{P}^*, \mathfrak{U}_\beta(\mathcal{P}^*, \mathcal{P}))$ is a uc-cone, where $\mathfrak{U}_\beta(\mathcal{P}^*, \mathcal{P}) = \{\alpha \widetilde{U^\circ} : \alpha > 0\}$ (see [1]). If $(\mathcal{P}, \mathfrak{U})$ and $(\mathcal{Q}, \mathcal{W})$ are uc-cones, then the definition of an open (continuous) linear operator T can be written as in the following simple case: an operator $T : \mathcal{P} \to \mathcal{Q}$ is open (continuous) if there is $\beta > 0$ such that $\beta W \subseteq (T \times T)(U)$ (resp., $(T \times T)(U) \subseteq \subseteq \beta W$).

Theorem 2.3. *Let* $(\mathcal{P},\mathfrak{U})$ *and* $(\mathcal{Q},\mathcal{W})$ *be uc-cones. Suppose that* T *is a linear operator of* $\mathcal{P}$ *onto a subcone* $\mathcal{Q}_1$ *of* $\mathcal{Q}$ *such that*

$$T(a)=T(b) \quad \textit{implies} \quad a+N=b+N \tag{2.2}$$

for all $a,b\in\mathcal{P}$, *where* $N=\ker T$. *If* T *is open and continuous, then* T^* *is an open continuous mapping of* $(\mathcal{Q}^*,\mathfrak{U}_\beta(\mathcal{Q}^*,\mathcal{Q}))$ *onto* $((N\times N)^\circ,\mathfrak{U}_\beta(\mathcal{P}^*,\mathcal{P}))$. *Hence*:

(1) *if*, *moreover*, T *is one-to-one, then* T^* *maps onto* $\mathcal{P}^*$;

(2) *if* T *maps onto* $\mathcal{Q}$, *then* T^* *is an isomorphism of* $\mathcal{Q}^*$ *onto* $(N\times N)^\circ$;

(3) *if* T *is an isomorphism of* $\mathcal{P}$ *onto* $\mathcal{Q}$, *then* T^* *is an isomorphism of* $\mathcal{Q}^*$ *onto* $\mathcal{P}^*$.

Proof. Let $\mu\in\mathcal{Q}^*$ be arbitrary. If $(x,y)\in N\times N$, then $\mu(T(x))=\mu(T(y))=0$. Thus

$$T^*(\mu)(x)\le T^*(\mu)(y)+1.$$

Hence $T^*(\mu)$ is in $(N\times N)^\circ$. Conversely, choose an element f of $(N\times N)^\circ$. We must find an element μ of $\mathcal{Q}^*$ such that $\mu\circ T=f$, that is, the values of μ on $\mathcal{Q}_1$ must be given by $\mu(T(x))=f(x)$. It is, in fact, possible to use this formula to define μ on $\mathcal{Q}_1$. Observe that if $T(x)=T(y)$, then by the hypothesis we have $x+N=y+N$. So $x+n=y+n'$ for some $n,n'\in N$. Since f vanishes on N, $f(x)=f(y)$. (We note that N is a subcone of $\mathcal{P}$ and $(N\times N)^\circ=\{\mu\in\mathcal{P}^*: \mu(n)=0 \text{ for all } n\in N\}$.) Hence μ is well defined. We show next that the functional μ is continuous on $\mathcal{Q}_1$. Since $f\in(N\times N)^\circ\subseteq\mathcal{P}^*$, there exists $\alpha>0$ such that $f\in(\alpha U)^\circ$, that is, $(x,y)\in\alpha U$ implies $f(x)\le f(y)+1$. By openness of T, there is $\beta>0$ such that $\beta W\subseteq(T\times T)(\alpha U)$. Now if $a,b\in\mathcal{Q}_1$ and $(a,b)\in\beta W$, then there is $(a',b')\in\alpha U$ such that $T(a')=a$ and $T(b')=b$. We have $f(a')\le f(b')+1$, that is,

$$(\mu\circ T)(a')\le(\mu\circ T)(b')+1.$$

Hence $\mu(a)\le\mu(b)+1$. Therefore $\mu\in(\beta W\cap(\mathcal{Q}_1\times\mathcal{Q}_1))^\circ$ and so μ is continuous on $\mathcal{Q}_1$. By Theorem 2.2, we can extend μ to a continuous linear functional on $\mathcal{Q}$. This shows that T^* is onto $(N\times N)^\circ$. We will now prove that T^* is an open mapping. We show that there exists $\beta>0$ such that $\beta\widetilde{U^\circ}\subseteq(T^*\times T^*)(\widetilde{W^\circ})$. Since T is open, there is $\beta>0$ such that $\beta W\subseteq(T\times T)(U)$. Thus $\beta((T\times T)(U))^\circ\subseteq W^\circ$. By Lemma 2.3 (1), $T^{*-1}(U^\circ)=((T\times T)(U))^\circ$. Hence $\beta T^{*-1}(U^\circ)\subseteq W^\circ$. Then $\beta T^*(T^{*-1}(U^\circ))\subseteq T^*(W^\circ)$. Since T^* is surjective, $\beta U^\circ\subseteq T^*(W^\circ)$ and so $\beta\widetilde{U^\circ}\subseteq\widetilde{T^*(W^\circ)}$. By Lemma 2.2, we have $\beta\widetilde{U^\circ}\subseteq(T^*\times T^*)(\widetilde{W^\circ})$. Thus T^* is open. Now we show that T^* is continuous. We prove that there exists $\gamma>0$ such that $(T^*\times T^*)(\widetilde{W^\circ})\subseteq\gamma\widetilde{U^\circ}$. Since T is continuous, there exists $\gamma>0$ such that $(T\times T)(U)\subseteq\gamma W$. Hence $W^\circ\subseteq\gamma((T\times T)(U))^\circ$. By Lemma 2.3 (1), $W^\circ\subseteq\gamma T^{*-1}(U^\circ)$. Thus $T^*(W^\circ)\subseteq\gamma U^\circ$. Therefore, by Lemma 2.2,

$$(T^*\times T^*)(\widetilde{W^\circ})\subseteq\widetilde{T^*(W^\circ)}\subseteq\gamma\widetilde{U^\circ}.$$

Hence T^* is continuous.

(1) If T is one-to-one, then $N=\{0\}$, so $(N\times N)^\circ$ is the whole of $\mathcal{P}^*$.

(2) If T maps onto $\mathcal{Q}$, then by Lemma 2.1, T^* is one-to-one. Since it is also open and continuous, it is an isomorphism of $\mathcal{Q}^*$ onto $(N\times N)^\circ$.

(3) The result follows from (1) and (2).

Theorem 2.3 is proved.

The above theorem holds for a normed space without condition (2.2) (see, for example, [3], II.24.4). In the following example (similar to Example 2.2 of [7]) we show that if condition (2.2) does not hold, then the functional μ is not necessarily well defined, i.e., the mapping T^* is not necessarily onto $(N \times N)^\circ$.

***Example* 2.1.** Let $\mathcal{P} = \big\{(x,y) \in \overline{\mathbb{R}}^2 | x, y \geq 0\big\}$, endowed with the convex quasiuniform structure $\mathfrak{U} = \{\alpha\widetilde{(1,1)} : \alpha > 0\}$, where

$$\widetilde{(1,1)} = \Big\{\big((a,b),(c,d)\big) \in \overline{\mathbb{R}}^2_+ \times \overline{\mathbb{R}}^2_+ : (a,b) \leq (c,d) + (1,1)\Big\}$$

(the order in $\overline{\mathbb{R}}^2$ is coordinatewise) and let the subcone $\mathcal{Q}_1 = \overline{\mathbb{R}}_+ = [0,+\infty]$ of $\mathcal{Q} = \overline{\mathbb{R}}$, endowed with the convex quasiuniform structure $\widetilde{\mathcal{V}} = \{\tilde{\varepsilon} : \varepsilon > 0\}$, where

$$\tilde{\varepsilon} = \Big\{(a,b) \in \overline{\mathbb{R}}^2_+ : a \leq b + \varepsilon\Big\}.$$

Let T be the linear mapping from $\mathcal{P}$ onto $\mathcal{Q}_1$ defined by $T(x,y) = x + y$. Then we have $N = \ker(T) = \{(0,0)\}$. The mapping $T : (\mathcal{P}, \mathfrak{U}) \to (\mathcal{Q}_1, \widetilde{\mathcal{V}})$ is open and continuous. Let f be the linear functional on $\mathcal{P}$ defined by $f(x,y) = x$. This functional is continuous and is an element of $(N \times N)^\circ$. But there exists no μ in $\mathcal{Q}_1^*$ such that $\mu \circ T = f$ on $\mathcal{Q}_1$. Indeed, the dual cone of $(\mathcal{Q}_1, \widetilde{\mathcal{V}})$ is the positive reals together with $\overline{0}$ which maps all $a \in \mathbb{R}_+$ to 0 and $+\infty$ to $+\infty$ and any of this functional does not satisfy in the relation $\mu \circ T = f$. We note that condition (2.2) does not hold for this T, for example, $T(0,1) = T(1,0)$, but $(0,1) + N \neq (1,0) + N$.

***Remark* 2.1.** For topological vector spaces, an operator T is one-to-one if and only if $\ker(T) = \{0\}$, but in locally convex cones, this is not true. In locally convex cones if T is one-to-one, then $\ker(T) = \{0\}$, but the converse is not true. For instance, in the Example 2.1 we have $\ker(T) = \{(0,0)\}$, however, T is not one-to-one.

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