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ASYMPTOTIC SOLUTIONS OF THE DIRICHLET PROBLEM FOR THE HEAT EQUATION AT A CHARACTERISTIC POINT *

АСИМПТОТИЧНІ РОЗВ'ЯЗКИ ЗАДАЧІ ДІРІХЛЕ ДЛЯ РІВНЯННЯ ТЕПЛОПРОВІДНОСТІ В ХАРАКТЕРИСТИЧНІЙ ТОЧЦІ

The Dirichlet problem for the heat equation in a bounded domain $\mathcal{G} \subset \mathbb{R}^{n+1}$ is characteristic because there are boundary points at which the boundary touches a characteristic hyperplane $t = c$, where c is a constant. It was I.G. Petrovskii (1934) who first established necessary and sufficient conditions on the boundary guaranteeing that the solution is continuous up to the characteristic point provided that the Dirichlet data are continuous. The appearance of the paper was stimulated by the existing interest in studying general boundary-value problems for parabolic equations in bounded domains. We contribute to the study by constructing a formal solution of the Dirichlet problem for the heat equation in a neighborhood of a cuspidal characteristic boundary point and showing its asymptotic character.

Задача Діріхле для рівняння теплопровідності в обмеженій області $\mathcal{G} \subset \mathbb{R}^{n+1}$ є характеристичною, оскільки існують граничні точки, в яких границя є дотичною до характеристичної гіперплощини $t = c$, де c є сталою. І. Г. Петровський (1934) уперше встановив необхідні та достатні умови на границю, що гарантують неперервність розв'язку аж до характеристичної точки за умови, що дані Діріхле є неперервними. Поява даної роботи була викликана постійним інтересом до вивчення загальних граничних задач для рівнянь параболічного типу в обмежених областях. Наш внесок у вивчення цієї проблеми полягає в побудові формального розв'язку задачі Діріхле для рівняння теплопровідності в околі гострокінцевої характеристичної граничної точки та дослідженні його асимптотичного характеру.

Introduction. The problem we consider in this paper goes back at least as far as [8] who proved the existence of a classical solution to the first boundary-value problem for the heat equation in a noncylindrical plane domain. By classical is meant "continuous up to the boundary," and a boundary point is called regular if any weak solution of the problem is continuous up to the point, provided the boundary data are continuous. The domain is assumed to be bounded by an interval $[a, b]$ of the x-axis and two curves $x = X_1(t)$ and $x = X_2(t)$ in the upper half-plane through $(a, 0)$ and $(b, 0)$, respectively. The Dirichlet data are posed on the interval and both lateral curves. All points of the interval $[a, b]$ are characteristic. The interval $[a, b]$ may shrink up to a point (say $(0, 0)$) in which case the origin is the only characteristic point.

The theory of [8] applies in particular to the plane domains $\mathcal{G}$ consisting of all $(x, t) \in \mathbb{R}^2$, such that $|x| < 1$ and $\mathfrak{f}(|x|) < t < \mathfrak{f}(1)$, where $\mathfrak{f}(r)$ is a C^1 function on $(0, 1]$ satisfying $\mathfrak{f}(r) > 0$, $\mathfrak{f}'(r) \neq 0$

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for all $r \in (0,1]$ and $\mathfrak{f}(0+) = 0$. The boundary point $(0,0)$ proves to be regular if $\mathfrak{f}^{-1}(t)$ satisfies the Hölder condition of exponent larger than $1/2$. When applied to the function $\mathfrak{f}(r) = r^p$, this obviously implies $0 < p < 2$. Note that for $1 < p < 2$ the origin is a true (i.e., smooth) characteristic point at the boundary while for $0 < p < 1$ this is a cuspidal (i.e., singular) boundary point.

The paper [8] exploited the fundamental solution of the heat equation and integral equations of potential theory. A more careful analysis led Petrovskii in [16] to an explicit necessary and sufficient condition for a boundary point to be regular. This latter paper initiated an extensive literature devoted to general boundary-value problems for parabolic equations (see, for example, [10, 14]). Mention that the classical paper [19] was actually motivated by the first boundary-value problem for the heat equation in a bounded domain $\mathcal{G} \subset \mathbb{R}^n$. On the other hand, [10] made essential use of function spaces of Slobodetskii [19]. Unfortunately, [10] suffers several drawbacks which, however, do not affect the main result of this seminal paper.

The most cited paper of V. A. Kondrat'ev is [11] studying boundary-value problems for elliptic equations in domains with conical points on the boundary. Asymptotics of solutions of general boundary-value problems for elliptic equations in domains with cusps remains still a challenge for mathematicians (see, for example, [6, 7, 9, 12] and reference therein).

According to the MathSciNet of the AMS there has been merely 8 citations to the paper [10] while this latter already contains all of the techniques of [11], especially the asymptotics of solutions at conical points. At the end of the 90's Kondrat'ev called the last author's attention to the paper [10] saying "Here are cusps." In spite of the fact that [10] deals with C^∞ boundaries the analysis near characteristic boundary points reveals Fuchs-type operators typical for conical singularities, provided that the contact degree of the boundary and characteristic plane is at least the anisotropy quotient (2 for the heat equation). If the contact degree is less than the anisotropy quotient, the analysis close to the characteristic point requires pseudodifferential operators typical for cuspidal points on the boundary (cf. [8] discussed above).

The structure of asymptotics at a conical point is completely determined by the spectrum of the problem frozen at the singular point. To an eigenvalue λ_n of multiplicity μ_n there correspond eigenfunctions $|x|^{-\imath\lambda_n}(\log|x|)^j$ with $j = 0, 1, \ldots, \mu_n - 1$. Each horizontal strip of finite width in the complex plane contains finitely many values λ_n, and the set of all λ_n is infinite. The expansions of solutions over these basic functions fail usually to converge, and so the series should be thought of as asymptotic.

Moreover, in the absence of embedding theorems the concept of asymptotic in the sense of Poincaré does not apply. We are thus led to asymptotic expansions related to certain filtrations on function spaces, a purely algebraic concept, which is a true substitution for Poincaré's asymptotics, see for instance [13] and elsewhere.

In mathematics, a (descending) filtration is an indexed set $\mathcal{F}_n$ of subspaces of a given vector space $\mathcal{F}$, with the index n running over entire numbers, subject to the condition that $\mathcal{F}_{n+1} \subset \mathcal{F}_n$ for all n. Let $\mathcal{F}_{-\infty}$ be the union of the $\mathcal{F}_n$. Given any $f \in \mathcal{F}_{-\infty}$, by

$$f \sim \sum_{n=n_f}^{\infty} f_n \tag{0.1}$$

with $f_n \in \mathcal{F}_n$ is meant that

$$f - \sum_{n=n_f}^{N} f_n \in \mathcal{F}_{N+1}$$

holds for every $N \geq n_f$. We intend to develop this generalisation of Poincaré's asymptotics in a forthcoming publication.

As filtration Kondrat'ev used in [10] weighted Slobodetskii spaces, where the weight functions are powers of the distance to the characteristic point. Analysis on manifolds with point and more general singularities has since exploited weighted function spaces.

In [3, 4] the first boundary-value problem is studied for the heat equation in a bounded plane domain with cuspidal points at the boundary at which the tangent coincides with a characteristic $t = c$, where c is a constant. The paper [2] contributed to the study of the first boundary-value problem for the **1D** heat equation in a bounded plane domain by evaluating the first term of the asymptotic of a solution at the characteristic point. The goal of the present paper is to explicitly compute full asymptotic expansions.

Our scheme of construction of asymptotic series for a solution near a characteristic point consists in the following. In Section 1 we resolve singularities at the characteristic point by blowing-up this point to a segment of the x-axis containing the origin. The domain $\mathcal{G}$ close to the origin blows up to a half-strip. In Section 2 we construct a formal solution of the transformed problem in the half-strip. This is actually a formal Puiseux series in fractional powers of t unless $p = 2$. In Section 3 we construct a formal solution in the case $p = 2$, which reveals immediately asymptotic expansions on manifolds with conical points. To prove the asymptotic character of formal solution we need an existence theorem which is a part of Fredholm theory for the first boundary-value problem for the heat equation. To this end we describe in Section 4 a change of variables which transforms the characteristic point to the point at infinity along the t-axis. In Section 5 we discuss the Fredholm property of the first boundary-value problem for the heat equation. When the Fredholm property has been proved one obtains real solutions of the problem which expand as formal series. In this case one introduces the difference between the real solution and a partial sum of the formal series and substitutes this remainder to the equations. This yields a nonhomogeneous problem for the remainder, and the formal solvability might testify to the possibility of estimating the remainder. We follow this way to show in Section 6 the asymptotic character of formal solution in the sense (0.1).

Needless to say that our results go far beyond the first boundary-value problem for the heat equation and extend to general boundary-value problems for parabolic equations in bounded domains.

1. Blow-up techniques. Consider the first boundary-value problem for the heat equation in a bounded domain $\mathcal{G} \subset \mathbb{R}^2$. The boundary of $\mathcal{G}$ is assumed to be C^∞ except for a finite number of singular points. A boundary point is called characteristic if the boundary is smooth at this point and the tangent is orthogonal to the t-axis. Since $\mathcal{G}$ is bounded, there are at least two characteristic points on the boundary unless it has a singularity at a characteristic point. In this paper we restrict our discussion to characteristic points which may moreover bear boundary singularities. By the local principle of [18] it is sufficient to study the problem only in a small neighbourhood of any characteristic (singular) point. Thus, the domain $\mathcal{G}$ looks like that of Figure 1 with $n = 1$, i.e., it is bounded by a curve $t = |x|^p$, with $p > 0$ an arbitrary real number, from below and by a horizontal segment from above. This is a typical domain for problems of such a type. As usual, no conditions are posed on the upper segment see [20].

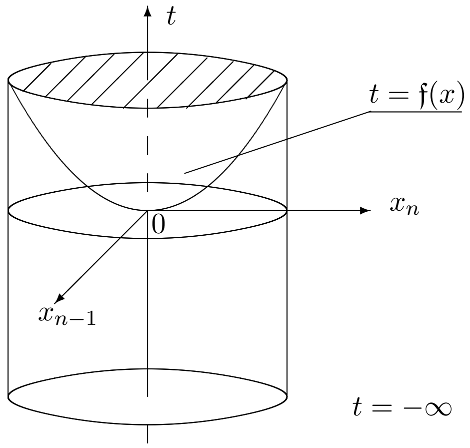

Fig. 1. Resolution of singularities at a characteristic point.

If $p > 1$, then the origin is a characteristic point of the boundary. If $0 < p \leq 1$, then the boundary has a singularity at the origin, which is a conical point for $p = 1$ and a cusp for $p < 1$. As mentioned, the case $p \geq 2$ was treated in [10] in the framework of analysis of Fuchs-type operators. The paper [2] demonstrates rather strikingly that, for $0 < p < 2$, the problem to be considered is specified in analysis on manifolds with cusps. A modern approach to studying boundary-value problems in domains with cuspidal boundary points is based on the so-called blow-up techniques, cf. [17]. While giving a complete characterisation of Fredholm problems, the approach falls short of providing asymptotics of solutions at singular points.

The first boundary-value problem for the heat equation in the domain $\mathcal{G}$ is formulated as follows: Write Σ for the set of all characteristic points $0, \ldots$ on the boundary of $\mathcal{G}$. Given functions f in $\mathcal{G}$ and u_0 at $\partial\mathcal{G} \setminus \Sigma$, find a function u on $\overline{\mathcal{G}} \setminus \Sigma$ which satisfies

$$u'_t - u''_{x,x} = f \qquad \text{in} \quad \mathcal{G}, \qquad u = u_0 \qquad \text{at} \quad \partial\mathcal{G} \setminus \Sigma. \tag{1.1}$$

By the local principle of Simonenko [18], the Fredholm property of problem (1.1) in suitable function spaces is equivalent to the local invertibility of this problem at each point of the closure of $\mathcal{G}$. Here we focus upon the characteristic points like the origin $P = (0, 0)$.

Suppose the domain $\mathcal{G}$ is described in a neighbourhood of the origin by the inequality

$$t > |x|^p, \tag{1.2}$$

where p is a positive real number. There is no loss of generality in assuming that $|x| \leq 1$.

We now blow up the domain $\mathcal{G}$ at P by introducing new coordinates (ω, r) with the aid of

$$x = r^{1/p}\,\omega, \quad t = r, \tag{1.3}$$

where $|\omega| < 1$ and $r \in (0, 1)$. It is clear that the new coordinates are singular at $r = 0$, for the entire segment $[-1, 1]$ on the ω-axis is blown down into the origin by (1.3). The rectangle $(-1, 1) \times (0, 1)$ transforms under the change of coordinates (1.3) into the part of the domain $\mathcal{G}$ nearby $(0, 0)$ lying below the line $t = 1$.

In the domain of coordinates (ω, r) problem (1.1) reduces to an ordinary differential equation with respect to the variable r with operator-valued coefficients. More precisely, under transformation

(1.3) the derivatives in t and x change by the formulas

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial r} - \frac{1}{r}\frac{\omega}{p}\frac{\partial u}{\partial \omega}, \qquad \frac{\partial u}{\partial x} = \frac{1}{r^{1/p}}\frac{\partial u}{\partial \omega},$$

and so (1.1) transforms into

$$r^Q U'_r - U''_{\omega,\omega} - r^{Q-1}\frac{\omega}{p}U'_\omega = r^Q F \quad \text{in} \quad (-1,1)\times(0,1), \qquad U = U_0 \quad \text{at} \quad \{\pm 1\}\times(0,1), \tag{1.4}$$

where $U(\omega, r)$ and $F(\omega, r)$ are pullbacks of $u(x,t)$ and $f(x,t)$ under transformation (1.3), respectively, and $Q = \dfrac{2}{p}$.

We are now interested in the local solvability of problem (1.4) near the edge $r = 0$ in the rectangle $(-1,1)\times(0,1)$. Note that the equation degenerates at $r = 0$, since the coefficient $r^{2/p}$ of the higher order derivative in r vanishes at $r = 0$. The exponent Q is of crucial importance for specifying the ordinary differential equation. If $p = 2$ then it is a Fuchs-type equation, these are also called regular singular equations. The Fuchs-type equations fit well into an algebra of pseudodifferential operators based on the Mellin transform. If $p > 2$, then the singularity of the equation at $r = 0$ is weak and so regular theory of finite smoothness applies. In the case $p < 2$ the degeneracy at $r = 0$ is strong and the equation can not be treated except by the theory of slowly varying coefficients [17].

2. Formal series expansion for homogeneous problem. To determine appropriate function spaces in which a solution of problem (1.4) is sought, one constructs formal series expansions for the solutions of the corresponding homogeneous problem. That is

$$r^Q U'_r - U''_{\omega,\omega} - r^{Q-1}\frac{\omega}{p}U'_\omega = 0 \quad \text{in} \quad (-1,1)\times(0,\infty), \qquad U(\pm 1, r) = 0 \quad \text{on} \quad (0,\infty). \tag{2.1}$$

We first consider the case $p \neq 2$. We look for a formal solution to (2.1) of the form

$$U(\omega, r) = e^{S(r)}\, V(\omega, r), \tag{2.2}$$

where S is a differentiable function of $r > 0$ and V expands as a formal Puiseux series with nontrivial principal part

$$V(\omega, r) = \frac{1}{r^{\mathfrak{e}N}}\sum_{j=0}^{\infty} V_j(\omega)\, r^{\mathfrak{e}j},$$

the (possibly) complex exponent N and real exponent $\mathfrak{e}$ have to be determined. In fact, the factor $r^{-\mathfrak{e}N}$ might be included into the definition of $\exp S$ as $\exp(-\mathfrak{e}N \ln r)$, however, we prefer to highlight the key role of Puiseux series.

Substituting (2.2) into (2.1) yields

$$r^Q\left(S'V + V'_r\right) - V''_{\omega,\omega} - r^{Q-1}\frac{\omega}{p}V'_\omega = 0 \quad \text{in} \quad (-1,1)\times(0,\infty), \qquad V(\pm 1, r) = 0 \quad \text{on} \quad (0,\infty).$$

In order to reduce this boundary-value problem to an eigenvalue problem we require the function S to satisfy the eikonal equation $r^Q S' = \lambda$ with a complex constant λ. When $Q \neq 1$, i.e., $p \neq 2$, this implies

$$S(r) = \lambda\,\frac{r^{1-Q}}{1-Q}$$

up to an inessential constant to be included into a factor of $\exp S$. In this manner the problem reduces to

$$r^Q V'_r - V''_{\omega,\omega} - r^{Q-1}\frac{\omega}{p} V'_\omega = -\lambda V \quad \text{in} \quad (-1,1)\times(0,\infty), \qquad V(\pm 1, r) = 0 \quad \text{on} \quad (0,\infty). \tag{2.3}$$

If $\mathfrak{e} = \dfrac{Q-1}{k}$ for some natural number k, then

$$r^Q V'_r = \sum_{j=k}^{\infty} \mathfrak{e}(j-N-k)\, V_{j-k} r^{\mathfrak{e}(j-N)},$$

$$V''_{\omega,\omega} = \sum_{j=0}^{\infty} V''_j r^{\mathfrak{e}(j-N)},$$

$$r^{Q-1} V'_\omega = \sum_{j=k}^{\infty} V'_{j-k} r^{\mathfrak{e}(j-N)},$$

as is easy to check. On substituting these equalities into (2.3) and equating the coefficients of the same powers of r we get two collections of Sturm – Liouville problems

$$-V''_j + \lambda\, V_j = 0 \quad \text{in} \quad (-1,1), \qquad V_j = 0 \quad \text{at} \quad \mp 1, \tag{2.4}$$

for $j = 0, 1, \ldots, k-1$, and

$$-V''_j + \lambda\, V_j = \frac{\omega}{p} V'_{j-k} - \mathfrak{e}(j-N-k)\, V_{j-k} \quad \text{in} \quad (-1,1), \qquad V_j = 0 \quad \text{at} \quad \mp 1, \tag{2.5}$$

for $j = mk, mk+1, \ldots, mk+(k-1)$, where m takes on all natural values.

Given any $j = 0, 1, \ldots, k-1$, the Sturm – Liouville problem (2.4) considered in space $L^2(-1,1)$ has obviously simple eigenvalues $\lambda_n = -\left(\dfrac{\pi}{2}n\right)^2$ for $n \geq 1$, a nonzero eigenfunction corresponding to λ_n being $\sin \dfrac{\pi}{2} n(\omega+1)$. It follows that

$$V_{n,j}(\omega) = c_{n,j} \sin \frac{\pi}{2} n(\omega + 1), \tag{2.6}$$

for $j = 0, 1, \ldots, k-1$, where $c_{n,j}$ are some constants. Without restriction of generality we can assume that the first coefficient $V_{n,0}$ in the Puiseux expansion of V is different from zero. Hence, $V_{n,j} = c_{n,j} V_{n,0}$ for $j = 1, \ldots, k-1$.

On having determined the functions $V_{n,0}, \ldots, V_{n,k-1}$ belonging to the standard Sobolev space $H^2(-1,1)$, we turn our attention to problems (2.5) with $j = k, \ldots, 2k-1$. Set

$$f_{n,j} = \frac{\omega}{p} V'_{n,j-k} - \mathfrak{e}(j-N-k)\, V_{n,j-k},$$

then for the inhomogeneous problem (2.5) to possess a nonzero solution $V_{n,j}$ it is necessary and sufficient that the right-hand side $f_{n,j}$ be orthogonal to all solutions of the corresponding homogeneous problem, to wit $V_{n,0}$. The orthogonality refers to the scalar product in $L^2(-1,1)$:

$$\big(f_{n,j}, V_{n,0}\big) = 0 \quad \text{for} \quad j = k, \ldots, 2k-1.$$

Let us evaluate the scalar product $(f_{n,j}, V_{n,0})$ for $j = k, \ldots, 2k-1$. We get

$$(f_{n,j}, V_{n,0}) = c_{n,j-k} \left(\frac{1}{p} \, (\omega V'_{n,0}, V_{n,0}) - \mathfrak{e}(j - N - k)(V_{n,0}, V_{n,0}) \right)$$

and

$$(\omega V'_{n,0}, V_{n,0}) = \omega \, |V_{n,0}|^2 \, \Big|_{-1}^{1} - (V_{n,0}, V_{n,0}) - (V_{n,0}, \omega V'_{n,0}) = -(V_{n,0}, V_{n,0}) - (\omega V'_{n,0}, V_{n,0}),$$

the latter equality being due to the fact that $V_{n,0}$ is real-valued and vanishes at ± 1. Hence,

$$(\omega V'_{n,0}, V_{n,0}) = -\frac{1}{2} \, (V_{n,0}, V_{n,0})$$

and

$$(f_{n,j}, V_{n,0}) = -c_{n,j-k} \left(\frac{1}{2p} + \mathfrak{e}(j - N - k) \right) (V_{n,0}, V_{n,0}) \tag{2.7}$$

for $j = k, \ldots, 2k-1$.

Since $V_{n,0} \neq 0$, the condition $(f_{n,j}, V_{n,0}) = 0$ fulfills for $j = k$ if and only if

$$\mathfrak{e} N = \frac{1}{2p}. \tag{2.8}$$

Under this condition, problem (2.5) with $j = k$ is solvable and its general solution has the form

$$V_{n,k} = W_{n,k} + c_{n,k} V_{n,0} \in H^2(-1, 1),$$

where $W_{n,k}$ is a particular solution of (2.5) and $c_{n,k}$ is an arbitrary constant. Moreover, for $(f_{n,j}, V_{n,0}) = 0$ to fulfill for $j = k+1, \ldots, 2k-1$ it is necessary and sufficient that $c_{n,1} = \ldots = c_{n,k-1} = 0$, i.e., all of $V_{n,1}, \ldots, V_{n,k-1}$ vanish. This in turn implies that $f_{n,k+1} = \ldots = f_{n,2k-1} = 0$, whence $V_{n,j} = c_{n,j} V_{n,0}$ for all $j = k+1, \ldots, 2k-1$, where $c_{n,j}$ are arbitrary constants. We choose the constants $c_{n,k+1}, \ldots, c_{n,2k-1}$ in such a way that the solvability conditions of the next k problems are fulfilled.

More precisely, we consider the problem (2.5) for $j = 2k$, the right-hand side being

$$f_{n,2k} = \left(\frac{\omega}{p} \, W'_{n,k} - \mathfrak{e}(k - N) \, W_{n,k} \right) + c_{n,k} \left(\frac{\omega}{p} \, V'_{n,0} - \mathfrak{e}(k - N) \, V_{n,0} \right) =$$

$$= \left(\frac{\omega}{p} \, W'_{n,k} - \mathfrak{e}(k - N) \, W_{n,k} \right) + c_{n,k} \left(f_{n,k} - \mathfrak{e} k \, V_{n,0} \right).$$

Combining (2.7) and (2.8) we conclude that

$$(f_{n,k} - \mathfrak{e} k \, V_{n,0}, V_{n,0}) = -\mathfrak{e} k \, (V_{n,0}, V_{n,0}) = (1 - Q) \, (V_{n,0}, V_{n,0})$$

is different from zero. Hence, the constant $c_{n,k}$ can be uniquely defined in such a way that $(f_{n,2k}, V_{n,0}) = 0$. Moreover, the functions $f_{n,2k+1}, \ldots, f_{n,3k-1}$ are orthogonal to $V_{n,0}$ if and only if $c_{n,k+1} = \ldots = c_{n,2k-1} = 0$. It follows that $V_{n,j}$ vanishes for each $j = k+1, \ldots, 2k-1$.

Continuing in this fashion we construct a sequence of functions $V_{n,j} \in H^2(-1,1)$, for $j = 0,1,\ldots$, satisfying equations (2.4) and (2.5). The functions $V_{n,j}(\omega)$ are defined uniquely up to a common constant factor $c_{n,0} = c_n$. Moreover, $V_{n,j}$ vanishes identically unless $j = mk$ with $m = 0,1,\ldots$. Therefore,

$$V_n(\omega,r) = \frac{1}{r^{\mathfrak{e}N}} \sum_{m=0}^{\infty} V_{n,mk}(\omega)\, r^{\mathfrak{e}mk} = \frac{1}{r^{Q/4}} \sum_{m=0}^{\infty} V_{n,mk}(\omega)\, r^{(Q-1)m} \tag{2.9}$$

is a unique (up to a constant factor) formal series expansions of the solution to the problem (2.3) corresponding to $\lambda = \lambda_n$.

From (2.9) it is seen that the powers in the formal series expansions don't depend on the parameter k, and the corresponding functions $V_{n,mk}(\omega)$ recovered from the system (2.4) with the right-hand side constructed from the functions $V_{n,m(k-1)}$. Thus, without loss of generality we may assume that $k = 1$ and write formal series expansions in the form

$$V_n(\omega,r) = \frac{1}{r^{Q/4}} \sum_{m=0}^{\infty} V_{n,m}(\omega)\, r^{(Q-1)m}.$$

Moreover, due to (2.5) we have the following reccurent relation for the functions $V_{n,m}$:

$$-V''_{n,0} + \lambda_n V_{n,0} = 0 \quad \text{in} \quad (-1,1), \qquad V_{n,0} = 0 \quad \text{at} \quad \mp 1, \tag{2.10}$$

and

$$\begin{gathered} -V''_{n,m} + \lambda_n V_{n,m} = \frac{\omega}{p} V'_{n,m-1} + \left(\frac{Q}{4} + (m-1)(1-Q)\right) V_{n,m-1} \quad \text{in} \quad (-1,1), \\ V_{n,m} = 0 \quad \text{at} \quad \mp 1, \end{gathered} \tag{2.11}$$

for $m \geq 1$.

Theorem 2.1. *Let $p \neq 2$. Then an arbitrary solution of homogeneous problem* (2.1) *has formal series expansions of the form*

$$U(\omega,r) = \sum_{n=1}^{\infty} \frac{c_n}{r^{Q/4}} \exp\left(\lambda_n \frac{r^{1-Q}}{1-Q}\right) \sum_{m=0}^{\infty} \frac{V_{n,m}(\omega)}{r^{(1-Q)m}},$$

where $\lambda_n = -\left(\frac{\pi}{2}n\right)^2$ are the eigenvalues of the Sturm–Liouville problem (2.4).

Proof. The theorem follows readily from (2.2).

In the original coordinates (x,t) close to the point $P = (0,0)$ in $\mathcal{G}$ the formal series expansions for the solution of problem (1.1) looks like

$$u(x,t) = \sum_{n=0}^{\infty} \frac{c_n}{t^{Q/4}} \exp\left(\lambda_n \frac{t^{1-Q}}{1-Q}\right) \sum_{m=0}^{\infty} V_{n,m}\left(\frac{x}{t^{1/p}}\right)\left(\frac{1}{t}\right)^{(1-Q)m}. \tag{2.12}$$

If $1 - Q > 0$, i.e., $p > 2$, expansion (2.12) behaves in much the same way as boundary layer expansion in singular perturbation problems, since the eigenvalues are all negative. The threshold value $p = 2$ is a turning contact order under which the boundary layer degenerates.

3. The exceptional case $p = 2$. In this section we consider the case $p = 2$ in detail. For $p = 2$, problem (2.1) takes the form

$$r\,U'_r - U''_{\omega,\omega} - \frac{\omega}{2}\,U'_\omega = 0 \quad \text{in} \quad (-1,1)\times(0,\infty), \qquad U(\pm1,r) = 0 \quad \text{on} \quad (0,\infty). \tag{3.1}$$

The problem is specified as Fuchs-type equation on the half-axis with coefficients in boundary-value problems on the interval $[-1,1]$. Such equations have been well understood, see [5] and elsewhere.

If one searches for a formal solution to (3.1) of the form $U(\omega,r) = e^{S(r)}\,V(\omega,r)$, then the eikonal equation $rS' = \lambda$ gives $S(r) = \lambda \ln r$, and so $e^{S(r)} = r^\lambda$, where λ is a complex number. It makes therefore no sense to looking for $V(\omega,r)$ being a formal Puiseux series in fractional powers of r. The choice $\mathfrak{e} = (Q-1)/k$ no longer works, and so a good substitute for a fractional power of r is the function $1/\ln r$. Thus,

$$V(\omega,r) = \sum_{j=0}^{\infty} V_j(\omega)\left(\frac{1}{\ln r}\right)^{j-N}$$

has to be a formal series expansions for the solution of

$$r\,V'_r - V''_{\omega,\omega} - \frac{\omega}{2}\,V'_\omega = -\lambda V \quad \text{in} \quad (-1,1)\times(0,\infty), \qquad V(\pm1,r) = 0 \quad \text{on} \quad (0,\infty),$$

N being a nonnegative integer. Substituting the series for $V(\omega,r)$ into these equations and equating the coefficients of the same powers of $\ln r$ yields two collections of Sturm–Liouville problems

$$-V_0'' - \frac{\omega}{2}\,V_0' + \lambda\,V_0 = 0 \quad \text{in} \quad (-1,1), \qquad V_0 = 0 \quad \text{at} \quad \mp 1, \tag{3.2}$$

for $j = 0$, and

$$-V_j'' - \frac{\omega}{2}\,V_j' + \lambda\,V_j = (j-N-1)V_{j-1} \quad \text{in} \quad (-1,1), \qquad V_j = 0 \quad \text{at} \quad \mp 1, \tag{3.3}$$

for $j \ge 1$.

Problem (3.2) has a nonzero solution V_0 if and only if λ is an eigenvalue of the operator $Lv = v'' + \dfrac{1}{2}\omega\,v'$ whose domain consists of all functions v from the Sobolev space $H^2(-1,1)$ vanishing at ∓1. Then, equalities (3.3) for $j = 1,\ldots,N$ mean that $V_1,\ldots,V_N$ are actually root functions of the operator corresponding to the eigenvalue λ. In other words, $V_{n,0},\ldots,V_{n,N}$ is a Jordan chain of length $N+1$ corresponding to the eigenvalue λ_n. Note that for $j = N+1$ the right-hand side of (3.3) vanishes, and so $V_{n,N+1}, V_{n,N+2},\ldots$ is also a Jordan chain corresponding to the eigenvalue λ_n. This suggests that the series breaks beginning at $j = N+1$. Furthermore, it follows from the Sturm–Liouville theory that problem (3.2) has a discrete sequence $\{\lambda_n\}_{n=1,2,\ldots}$ of real eigenvalues. If

$$-v'' - \frac{1}{2}\omega\,v' + \lambda v = 0$$

on $(-1,1)$ for some function $v \in H^2(-1,1)$ vanishing at ∓1, then

$$\|v'\|^2 + \lambda\|v\|^2 = \frac{1}{2}\,(\omega v', v), \tag{3.4}$$

where the scalar product and norm are those of $L^2(-1,1)$. By the Schwarz inequality, we get $|(\omega v', v)| \le \|v'\|\,\|v\|$. Since

$$\|v'\|^2 + \lambda\|v\|^2 = \frac{1}{2}\,\|v'\|\,\|v\| + \left(\|v'\| - \frac{1}{4}\,\|v\|\right)^2 + \left(\lambda - \frac{1}{16}\right)\|v\|^2 \ge$$

$$\ge \frac{1}{2}\,\|v'\|\,\|v\| + \left(\lambda - \frac{1}{16}\right)\|v\|^2,$$

we conclude that equality (3.4) fulfills only for the function $v = 0$ unless $\lambda \le \dfrac{1}{16}$. Hence, $\lambda_n \le \dfrac{1}{16}$ for all $n = 1, 2, \ldots$. Each eigenvalue λ_n is simple whence $N = 0$.

Theorem 3.1. *Suppose* $p = 2$. *Then an arbitrary formal series expansions for the solution of homogeneous problem* (2.1) *has the form* $U(\omega, r) = \sum_{n=1}^{\infty} r^{\lambda_n} V_{n,0}(\omega)$, *where* λ_n *is the eigenvalues of the problem* (3.2).

Proof. The theorem follows immediately from the above discussion.

In the original coordinates (x,t) near the point $P = (0,0)$ in $\mathcal{G}$ the formal series expansions for the solution proves to be

$$u(x,t) = \sum_{n=1}^{\infty} c_n\, t^{\lambda_n}\, V_{n,0}\left(\frac{x}{t^{1/2}}\right).$$

Of course, Theorem 3.1 can be proved immediately, for the homogeneous problem (2.1) admits a separation of variables. Namely, set $U(\omega, r) = R(r)\Omega(\omega)$. Substituting this into equation (3.1) yields

$$rR'\Omega - \Omega'' - \frac{\omega}{2}\Omega' R = 0,$$

which is equivalent to

$$rR' = \lambda R, \ \Omega'' - \frac{\omega}{2}\Omega' = \lambda\Omega.$$

Then $R(r) = r^{\lambda_n}$, where the parameter λ_n is determined from the boundary-value problem for Ω. The function Ω can be described in terms of parabolic cylinder functions, see [1]. To transform the equation for Ω to the equation of parabolic cylinder, set

$$\Omega(\omega) = \exp\left(\frac{\omega^2}{8}\right) y(\omega).$$

Then y satisfies

$$y'' + \left(\left(\frac{\omega}{4}\right)^2 + \lambda_n - \frac{1}{4}\right) y = 0.$$

Two linearly independent solutions of this equation are called functions of parabolic cylinder.

4. Resolution of singularities at infinity. Throughout this part we will assume that $0 < p < 2$, i.e., $Q = 2/p$ is greater than 1. As mentioned in the Introduction, this case is not included in the treatise [10] and it was first studied in [2]. For $1 < p < 2$, the origin is a characteristic boundary point of the domain $\mathcal{G}$. For $0 < p < 1$, the origin is a cuspidal point at the boundary.

We are actually interested in the local solvability of problem (1.4) near the edge $r = 0$ in the rectangle $(-1,1) \times (0,1)$. Note that the equation degenerates at $r = 0$, since the coefficient r^Q of the higher order derivative in r vanishes at $r = 0$. If $Q = 1$, the equation is of Fuchs type and is studied within the framework of Mellin calculus. In order to handle this degeneration in an orderly fashion for $Q > 1$, we find a change of coordinates $s = \delta(r)$ in the interval $(0, 1)$, such that

$$r^Q \frac{d}{dr} = \frac{d}{ds}.$$

Such a function δ is determined uniquely up to some constant from the equation $\delta'(r) = r^{-Q}$ and is given by

$$\delta(r) = \frac{r^{1-Q}}{1-Q} \tag{4.1}$$

for $r > 0$. Note that $\delta(0+) = -\infty$. Problem (1.4) becomes

$$\begin{gathered} U'_s - U''_{\omega,\omega} + \frac{1}{2-p}\frac{1}{s}\,\omega U'_\omega = \left(\frac{\delta(1)}{s}\right)^{\frac{2}{2-p}} F \quad \text{in} \quad (-1,1)\times(-\infty,\delta(1)), \\ U = U_0 \quad \text{at} \quad \{\pm 1\}\times(-\infty,\delta(1)), \end{gathered} \tag{4.2}$$

where we use the same letter to designate U and the push-forward of U under the transformation $s = \delta(r)$, and similarly for F. Above $\delta(1) = \dfrac{1}{1-Q} < 0$.

Thus we have transformed boundary-value problem (1.1) to the boundary-value problem (4.2) with the operator

$$\mathcal{A}(s)U = U'_s - U''_{\omega,\omega} + \frac{1}{2-p}\frac{\omega}{s}\,U'_\omega \tag{4.3}$$

considered in the space $\mathcal{H}^1_0(-\infty,\delta(1))$ of functions U such that $U = U_0$ for $\omega = \pm 1$, $s \in (-\infty, \delta(1))$ and

$$\|U\|^2_{\mathcal{H}^1_0(-\infty,\delta(1))} = \int\limits_{-\infty}^{\delta(1)} \Big(\|U'(s)\|^2_{L^2(-1,1)} + \|U(s)\|^2_{H^2(-1,1)}\Big)|s|^{\frac{3}{p-2}}\,d\omega\,ds < \infty. \tag{4.4}$$

Factor $s^{3/(p-2)}$ arise due to the chage of Lebesgue measure $dx\,dt$ under the change of coordinates (1.3) and (4.1).

We now rewrite formal series expansions for the solution to homogeneous problem (2.1) in the new coordinates (ω, s). On substituting (4.1) into Theorem 2.1 we get immediately

$$U(\omega,s) = \sum_{n=1}^{\infty} c_n\,((1-Q)s)^{\frac{1}{4}\frac{Q}{Q-1}} \exp(\lambda_n s) \sum_{m=0}^{\infty} \frac{V_{n,m}(\omega)}{((1-Q)s)^m} \tag{4.5}$$

for s in a neighbourhood of $-\infty$, where $\lambda_n = -\left(\dfrac{\pi}{2}n\right)^2$.

5. Fredholm property of the first boundary-value problem. In this section we state the solvability of the transformated boundary-value problem (4.2). For this we need to introduce the scale of spaces $\mathcal{H}^k_{\gamma,\mu}(-\infty,T)$ of functions with values in standard Sobolev spaces $H^{2k}(-1,1)$. In particular case, when $k = 1$, $\gamma = 0$, $\mu = \mu_0 \equiv \dfrac{3}{2(p-2)}$ and $T = \delta(1)$ these spaces coincide with the spaces $\mathcal{H}_0^1(-\infty, \delta(1))$ introduces in the previous part, so that $\mathcal{H}_0^1(-\infty, \delta(1)) = \mathcal{H}_{0,\mu_0}^1(-\infty, \delta(1))$. We say that function u with values in space $H^{2k}(-1,1)$ belongs to the space $\mathcal{H}_{\gamma,\mu}^k(-\infty, T)$, $T \leq \infty$ for $k \in \mathbb{N}$, $\gamma \leq 0$, and $\mu > \mu_0$ if the following norm is finite

$$\|U\|_{\mathcal{H}_{\gamma,\mu}^k(-\infty,T)} := \left(\int\limits_{-\infty}^{T} e^{-2\gamma s}\, s^{2\mu} \sum_{j=0}^{k} \|U^{(j)}(s)\|^2_{H^{2(k-j)}(-1,1)} ds \right)^{1/2}. \tag{5.1}$$

When $k = 0$ and $\mu = 0$ we denote the space $\mathcal{H}_{\gamma,0}^0(-\infty, T)$ by $\mathcal{L}_\gamma^2(-\infty, T)$ with the corresponding norm. Main statement of this section is given by the following theorem.

Theorem 5.1. *Let $\gamma < 0$, $\gamma \neq \lambda_n, n \geq 0$, where λ_n are the eigenvalues of operator Δ in space $L^2(-1,1)$. Then for any $\mu > -1$ there exists $T_0(\mu) \in \mathbb{R}$ such that for all $T < T_0$ operator* (4.3) *of problem* (4.2) *acting in the spaces*

$$\mathcal{A}(s)\colon \mathcal{H}_{\gamma,\mu}^1(-\infty, T) \mapsto \mathcal{L}_\gamma^2(-\infty, T) \tag{5.2}$$

is invertible and the following estimate holds:

$$\|U\|_{\mathcal{H}_{\gamma,\mu}^1(-\infty,T)} \leq C\, \|\mathcal{A}(s)U\|_{\mathcal{L}_\gamma^2(-\infty,T)}. \tag{5.3}$$

The proof of this theorem breaks into a sequence of lemmas.

Lemma 5.1. *Let $\gamma < 0$, $\gamma \neq \lambda_n$, $n \geq 0$, $\mu \in \mathbb{R}$ and $T < 0$. Then the operator*

$$(\partial_s - \Delta)^{-1}\colon \mathcal{L}_\gamma^2(-\infty, T) \mapsto \mathcal{H}_{\gamma,\mu}^1(-\infty, T)$$

is bounded and the following estimate holds:

$$\|U\|_{\mathcal{H}_{\gamma,\mu}^1(-\infty,T)} \leq C\, \|F\|_{\mathcal{L}_\gamma^2(-\infty,T)}, \tag{5.4}$$

$$U(s) = \frac{1}{2\pi} \int\limits_{\Im m\, \sigma = \gamma} e^{i\sigma s} (-i\sigma - \Delta)^{-1} \hat{F}(\sigma)\, d\sigma. \tag{5.5}$$

The statement of this lemma is also true for $\gamma = 0$. In this case μ should be less than $-\dfrac{1}{2}$.

Proof. Consider $F \in \mathcal{L}_\gamma^2(-\infty, T)$. For $|s| > |T|$ let us continue function F by zero, then $F \in \mathcal{L}_\gamma^2(\mathbb{R})$. Applying Fourier transform with respect to the variable s to the equation

$$\partial_s U(s) - \Delta U(s) = F(s) \tag{5.6}$$

we have

$$-i\sigma \hat{U} - \Delta \hat{U} = \hat{F}. \tag{5.7}$$

Since the line $\Im m\, \sigma = \gamma$ for $\gamma \neq \lambda_n$ consists of regular points of operator Δ and $\hat{F} \in L^2(-1,1)$ it follows that operator

$$(-i\sigma - \Delta)^{-1}\colon L^2(-1,1) \mapsto H^2(-1,1) \quad \text{is bounded} \tag{5.8}$$

and for all $v \in L^2(-1,1)$:

$$|\sigma| \cdot \|(i\sigma + \Delta)^{-1} v\|_{L^2(-1,1)} \le C' \|v\|_{L^2(-1,1)}, \tag{5.9}$$

where C' does not depend on σ.

Therefore solution $\hat{U}$ of equation (5.7) is given by

$$\hat{U} = \big(-i\sigma - \Delta\big)^{-1} \hat{F} \in H^2(-1,1). \tag{5.10}$$

Applying inverse Fourier transform, to the solution of equation (5.6) we get representation (5.5).

Denote $H_1 = H^2(-1,1)$ and $H_0 = L^2(-1,1)$. For $\gamma < 0$ we obtain

$$\|U\|^2_{\mathcal{H}^1_{\gamma,\mu}(-\infty,T)} = \frac{1}{2\pi} \int\limits_{-\infty}^{T} e^{-2\gamma s}\, s^{2\mu} \left\| \int\limits_{\Im m\, \sigma = \gamma} (i\sigma)\, e^{i\sigma s} (-i\sigma - \Delta)^{-1} \hat{F}(\sigma) d\sigma \right\|^2_{H_0} ds +$$

$$+ \frac{1}{2\pi} \int\limits_{-\infty}^{T} e^{-2\gamma s}\, s^{2\mu} \left\| \int\limits_{\Im m\, \sigma = \gamma} e^{i\sigma s} (-i\sigma - \Delta)^{-1} \hat{F}(\sigma) d\sigma \right\|^2_{H_1} ds \le$$

$$\le \frac{1}{2\pi} \int\limits_{-\infty}^{T} e^{-4\gamma s}\, s^{2\mu} \int\limits_{\Im m\, \sigma = \gamma} |\sigma|^2 \left\| (i\sigma + \Delta)^{-1} \hat{F}(\sigma) \right\|^2_{H_0} d\sigma\, ds +$$

$$+ \frac{1}{2\pi} \int\limits_{-\infty}^{T} e^{-4\gamma s}\, s^{2\mu} \int\limits_{\Im m\, \sigma = \gamma} \left\| (i\sigma + \Delta)^{-1} \hat{F}(\sigma) \right\|^2_{H_1} d\sigma\, ds. \tag{5.11}$$

Due to (5.8), (5.9) and the Parseval theorem, for

$$\|F\|^2_{\mathcal{L}^2_\gamma(\mathbb{R})} = 2\pi \int\limits_{\Im m\, \sigma = \gamma} \left\| \hat{F}(\sigma) \right\|^2_{L^2(-1,1)} d\sigma$$

we have

$$\|U\|^2_{\mathcal{H}^1_{\gamma,\mu}(-\infty,T)} \le C \int\limits_{-\infty}^{T} e^{-4\gamma s}\, s^{2\mu} \int\limits_{\Im m\, \sigma = \gamma} \left\| \hat{F}(\sigma) \right\|^2_{H_0} d\sigma\, ds =$$

$$= C' \int\limits_{-\infty}^{T} e^{-4\gamma s}\, s^{2\mu}\, \|F\|^2_{\mathcal{L}^2_\gamma(\mathbb{R})} ds = C''\, \|F\|^2_{\mathcal{L}^2_\gamma(-\infty,T)}. \tag{5.12}$$

To estimate expression in (5.12) in the case $\gamma = 0$ we only remark that operator Δ with zero Dirichlet boundary conditions has no eigenfunctions corresponding to zero eigenvalue, therefore it is invertible. Thus estimate (5.12) also holds for $\gamma = 0$. In this case μ should be less than $-\dfrac{1}{2}$ to make the integral above convergent.

Lemma 5.1 is proved.

Lemma 5.2. *Let $T < 0$ be fixed. Then for all $\gamma \leq 0$, $\gamma \neq \lambda_n$, $n \geq 0$ and $\mu > -1$ operator*

$$B(s) = \frac{1}{2-p}\frac{\omega}{s}\frac{\partial}{\partial\omega} \colon \mathcal{H}^1_{\gamma,\mu}(-\infty, T) \mapsto \mathcal{L}^2_\gamma(-\infty, T)$$

is bounded and for all $\varphi \in \mathcal{H}^1_{\gamma,\mu}(-\infty, T)$

$$\|B(\cdot)\varphi\|_{\mathcal{L}^2_\gamma(-\infty,T)} \leq \frac{C}{|T|^{\mu+1}} \|\varphi\|_{\mathcal{H}^1_{\gamma,\mu}(-\infty,T)}. \tag{5.13}$$

Proof. As in Lemma 5.1 set $H_1 = H^2(-1,1)$ and $H_0 = L^2(-1,1)$. For $\varphi \in \mathcal{H}^1_{\gamma,\mu}(-\infty, T)$ we have

$$\|B(\cdot)\varphi\|^2_{\mathcal{L}^2_\gamma(-\infty,T)} = \int\limits_{-\infty}^{T} e^{-2\gamma s} \|B(s)\varphi(s)\|^2_{H_0}\, ds =$$

$$= C \int\limits_{-\infty}^{T} e^{-2\gamma s} s^{-2} \left\| \omega \frac{\partial\varphi(s)}{\partial\omega} \right\|^2_{H_0} ds \leq$$

$$\leq C \int\limits_{-\infty}^{T} e^{-2\gamma s} s^{-2} \left(\|\varphi'_s\|^2_{H_0} + \|\varphi\|^2_{H_1}\right) ds =$$

$$= C \int\limits_{-\infty}^{T} e^{-2\gamma s} \frac{s^{2\mu}}{s^{2(\mu+1)}} \left(\|\varphi'_s\|^2_{H_0} + \|\varphi\|^2_{H_1}\right) ds \leq$$

$$\leq \frac{C}{|T|^{2(\mu+1)}} \int\limits_{-\infty}^{T} e^{-2\gamma s} s^{2\mu} \left(\|\varphi'_s\|^2_{H_0} + \|\varphi\|^2_{H_1}\right) ds =$$

$$= \frac{C}{|T|^{2(\mu+1)}} \|\varphi\|^2_{\mathcal{H}^1_{\gamma,\mu}},$$

since $|s| > |T|$.

Lemma 5.2 is proved.

***Proof of Theorem* 5.1.** For $F \in \mathcal{L}^2_\gamma(-\infty, T)$ let us represent the problem

$$\mathcal{A}(s) \equiv \partial_s U - \Delta U + B(s)U = F$$

in the form

$$U - (\partial_s - \Delta)^{-1} B(s) U = (\partial_s - \Delta)^{-1} F. \tag{5.14}$$

Due to Lemmas 5.1 and 5.2 we may guarantee that for $\mu > -1$ there exists such T_0 that for any $T > T_0$ the norm of operator

$$(\partial_s - \Delta)^{-1} B(s)\colon \mathcal{H}^1_{\gamma,\mu}(-\infty, T) \mapsto \mathcal{H}^1_{\gamma,\mu}(-\infty, T)$$

is less than some $\delta < 1$:

$$\|(\partial_s - \Delta)^{-1} B(s)\|_{\mathcal{H}^1_{\gamma,\mu}(-\infty,T)\to\mathcal{H}^1_{\gamma,\mu}(-\infty,T)} < \delta. \tag{5.15}$$

This imply that equation (5.14) has unique solution for $F \in \mathcal{L}^2_\gamma(-\infty, T)$.

Theorem 5.1 is proved.

On changing the coordinates by

$$\omega = \frac{x}{t^{1/p}},$$

$$s = \frac{t^{1-Q}}{1-Q},$$

we pull back the function spaces $\mathcal{H}^1_{\gamma,\mu}(-\infty, \delta(1))$ to the original domain $\mathcal{G}$. Theorem 5.1 then yields a condition of local solvability of problem (1.1) at the characteristic point, see [2].

6. Asymptotic property of formal solution. We now turn to the proof of asymptotic property of formal series expansions of the solution of the first boundary-value problem for the heat equation at a characteristic point. To do this we denote by $\mathcal{H}^k_{\gamma,m}(-\infty, T)$ the spaces $\mathcal{H}^k_{\gamma,\mu}$ for $\mu = \mu_0 + m$, $m \in \mathbb{N}$ with $\mu_0 = \dfrac{3}{2(p-2)}$ and $\mathcal{H}^k_{\gamma,\mu_0}$ for $m = 0$. Let us remark that for any $k, m \geq 1$, $\gamma \in \mathbb{R}$, $T < \delta(1)$

$$\mathcal{H}^k_{\gamma,m+1}(-\infty, T) \subset \mathcal{H}^k_{\gamma,m}(-\infty, T) \subset \mathcal{H}^1_0(-\infty, \delta(1)) \equiv \mathcal{H}^1_{0,\mu_0}(-\infty, \delta(1)).$$

The main result of this paper reads as follows.

Theorem 6.1. *Suppose that $\lambda_{K+1} < \gamma < \lambda_K$. Then the formal series expansion* (4.5) *of the solution $U \in \mathcal{H}^1_0(-\infty, \delta(1))$ of problem* (4.2) *is actually asymptotic in the sense* (0.1).

Proof. Due to (4.5) the solution to the homogeneous boundary-value problem (4.2) in space $\mathcal{H}^1_0(-\infty, \delta(1))$ has the form

$$U(\omega, s) = \sum_{n=1}^{\infty} U_n(\omega, s), \tag{6.1}$$

where

$$U_n(\omega, s) = c_{n,Q}\, s^{\frac{1}{4}\frac{Q}{Q-1}} \exp(\lambda_n s) \sum_{m=0}^{\infty} \frac{V_{n,m}(\omega)}{((1-Q)s)^m}$$

and $\lambda_n = -\left(n\dfrac{\pi}{2}\right)$, $c_{n,Q} = c_n (1-Q)^{\frac{1}{4}\frac{Q}{Q-1}}$.

For each $M \geq 0$ and $K \geq 0$ we introduce the function

$$U_{K,M}(\omega, s) = \sum_{n=0}^{K} c_{n,Q}\, s^{\frac{1}{4}\frac{Q}{Q-1}} \exp(\lambda_n s) \sum_{m=0}^{M} \frac{V_{n,m}(\omega)}{(1-Q)^m s^m}$$

on $(-1, 1) \times (-\infty, S)$. Direct calculations show that for any finite K these functions belong to the space $\mathcal{H}^1_{\gamma,M}(-\infty, S)$ for $M > \mu_0$.

Given any nonnegative integers M and K, set

$$R_{K+1,M+1}(\omega,s)=U(\omega,s)-\sum_{n=0}^{K}c_{n,Q}\,s^{\frac{1}{4}\frac{Q}{Q-1}}e^{\lambda_n s}\sum_{m=0}^{M}\frac{V_{n,m}(\omega)}{(1-Q)^m s^m}$$

for $(\omega,s)\in[-1,1]\times(-\infty,S)$. Then we have

$$U(\omega,s)=U_{K,M}(\omega,s)+R_{K+1,M+1}(\omega,s).$$

Due to the Theorem 5.1 solution U of the problem (4.2) belongs to the space $\mathcal{H}^1_{\gamma,M}$, therefore $R_{K+1,M+1}\in\mathcal{H}^1_{\gamma,M}$, too. The theorem will be proved if we will establish that $R_{K+1,M+1}(\omega,s)\in$ $\in\mathcal{H}^1_{\gamma,M+1}$.

Let us first calculate operator $\mathcal{A}(s)$ on $U_{K,M}$. We have

$$(U_{K,M}(\omega,s))'_s=\sum_{n=0}^{K}c_{n,Q}\,s^{\frac{1}{4}\frac{Q}{Q-1}}e^{\lambda_n s}\left(\sum_{m=0}^{M}\frac{\lambda_n V_{n,m}(\omega)}{(1-Q)^m s^m}-\right.$$

$$\left.-\sum_{m=1}^{M+1}\left(\frac{Q}{4}+(m-1)(Q-1)\right)\frac{V_{m-1,n}(\omega)}{(1-Q)^m s^m}\right),$$

$$(U_{K,M}(\omega,s))''_{\omega\omega}=\sum_{n=0}^{K}c_{n,Q}\,s^{\frac{1}{4}\frac{Q}{Q-1}}e^{\lambda_n s}\sum_{m=0}^{M}\frac{V''_{n,m}(\omega)}{(1-Q)^m s^m},$$

$$\frac{1}{2-p}\frac{\omega}{s}\,(U_{K,M}(\omega,s))'_\omega=\sum_{n=0}^{K}\frac{c_{n,Q}}{2-p}\,s^{\frac{1}{4}\frac{Q}{Q-1}}e^{\lambda_n s}\sum_{m=0}^{M}\frac{\omega V'_{n,m}(\omega)}{(1-Q)^m s^{m+1}}=$$

$$=-\sum_{n=0}^{K}c_{n,Q}s^{\frac{1}{4}\frac{Q}{Q-1}}e^{\lambda_n s}\sum_{m=1}^{M+1}\frac{\omega}{p}\frac{V'_{m-1,n}(\omega)}{(1-Q)^m s^m},$$

where we used that $\dfrac{1-Q}{2-p}=-\dfrac{1}{p}$.

Therefore for the operator $\mathcal{A}(s)$ (4.3) of boundary-value problem (4.2) we get

$$\mathcal{A}(s)U_{K,M}=\sum_{n=0}^{K}c_{n,Q}\,s^{\frac{1}{4}\frac{Q}{Q-1}}e^{\lambda_n s}\left(\sum_{m=0}^{M}\frac{-V''_{n,m}+\lambda_n V_{n,m}}{(1-Q)^m s^m}-\right.$$

$$\left.-\sum_{m=1}^{M+1}\frac{\dfrac{\omega V'_{n,m-1}}{p}+\left(\dfrac{Q}{4}+(m-1)(1-Q)\right)V_{n,m-1}}{(1-Q)^m s^m}\right)=$$

$$=-\sum_{n=0}^{K}c_{n,Q}\,s^{\frac{1}{4}\frac{Q}{Q-1}}e^{\lambda_n s}\,\frac{\dfrac{\omega}{p}V'_{n,M}+\left(\dfrac{Q}{4}+M(1-Q)\right)V_{n,M}}{(1-Q)^{M+1}s^{M+1}}. \tag{6.2}$$

Now we define a function $X_{K+1,M+1}(\omega,s)$ from the equality

$$R_{K+1,M+1}(\omega,s)=c_{K,Q}\,s^{\frac{1}{4}\frac{Q}{Q-1}}e^{\lambda_K s}\,\frac{X_{K+1,M+1}(\omega,s)}{(1-Q)^{M+1}s^{M+1}}. \tag{6.3}$$

Then we obtain

$$\mathcal{A}(s)R_{K+1,M+1}=\frac{c_{K,Q}\,s^{\frac{1}{4}\frac{Q}{Q-1}}e^{\lambda_K s}}{(1-Q)^{M+1}s^{M+1}}\,Y_{K+1,M+1}(\omega,s), \tag{6.4}$$

where

$$Y_{K+1,M+1}(\omega,s)=$$

$$=\big(X_{K+1,M+1}\big)''_{\omega\omega}-\frac{1}{2-p}\frac{\omega}{s}\big(X_{K+1,M+1}\big)'_{\omega}+$$

$$+\left(\lambda_K-\big(M+1-\frac{1}{4}\frac{Q}{Q-1}\big)\frac{1}{s}\right)X_{K+1,M+1}.$$

Thus, for homogeneous boundary-value problem (4.2), due to (6.2), we have

$$\mathcal{A}(s)U=\mathcal{A}(s)U_{K,M}+\mathcal{A}(s)R_{K+1,M+1}=$$

$$=-\sum_{n=0}^{K}\frac{c_{n,Q}\,s^{\frac{1}{4}\frac{Q}{Q-1}}e^{\lambda_n s}}{(1-Q)^{M+1}s^{M+1}}\left(\frac{\omega}{p}V'_{n,M}+\left(\frac{Q}{4}+M(1-Q)\right)V_{n,M}\right)+$$

$$+\frac{c_{K,Q}\,s^{\frac{1}{4}\frac{Q}{Q-1}}e^{\lambda_K s}}{(1-Q)^{M+1}s^{M+1}}Y_{K+1,M+1}(\omega,s)=0.$$

Therefore

$$Y_{K+1,M+1}(\omega,s)=$$

$$=\sum_{n=0}^{K}c'_{n,Q}e^{(\lambda_n-\lambda_K)s}\left(\frac{\omega}{p}V'_{n,M}+\left(\frac{Q}{4}+M(1-Q)\right)V_{n,M}\right)\overset{(2.11)}{=}$$

$$\overset{(2.11)}{=}\sum_{n=0}^{K}c'_{n,Q}e^{(\lambda_n-\lambda_K)s}\left(-V''_{n,M+1}+\lambda_n V_{n,M+1}\right).$$

Since $V_{n,M+1}\in H^2(-1,1)$ we get that $Y_{K+1,M+1}\in\mathcal{H}^0_{0,\mu_0}(-\infty,T)$. Thus, using representation (6.4), it is easy to see that

$$\mathcal{A}(s)R_{K+1,M+1}\in\mathcal{H}^0_{\gamma,M+1}(-\infty,T)\subset\mathcal{L}^2_{\gamma}(-\infty,T). \tag{6.5}$$

Indeed, due to the representation (6.4)

$$\mathcal{A}(s)R_{K+1,M+1} = C\, s^{\alpha} e^{\lambda_K s} Y_{K+1,M+1},$$

where $\alpha = \dfrac{1}{4}\dfrac{Q}{Q-1} - (M+1)$. By the definition of space $\mathcal{H}^0_{\gamma,M+1}(-\infty,T)$ we obtain

$$\|\mathcal{A}(s)R_{K+1,M+1}\|_{\mathcal{H}^0_{\gamma,M+1}(-\infty,T)} =$$

$$= \left(\int\limits_{-\infty}^{T} e^{-2\gamma s} s^{2(\mu_0+M+1)} \|\mathcal{A}(s)R_{K+1,M+1}\|^2_{L_2(-1,1)} ds\right)^{1/2} =$$

$$= C\left(\int\limits_{-\infty}^{T} e^{-2s(\gamma-\lambda_K)} s^{2(\mu_0+M+1)} s^{2\alpha} \|Y_{K+1,M+1}\|^2_{L_2(-1,1)} ds\right)^{1/2}$$

which is finite, for $\gamma < \lambda_K$ and integration runs in the negative half-axis. This implies (6.5).

Therefore, by Theorem 5.1, there exists $T_0 = T_0(M+1)$ such that $R_{K+1,M+1} \in \mathcal{H}^1_{\gamma,M+1}$, as desired.

Theorem 6.1 is proved.

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