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S. Z. Jafarov (Pamukkale Univ., Denizli, Turkey; Inst. Math. and Mech. Nat. Acad. Sci. Azerbaijan, Baku)

LINEAR METHODS OF SUMMING FOURIER SERIES AND APPROXIMATION IN WEIGHTED VARIABLE EXPONENT LEBESGUE SPACES

ЛІНІЙНІ МЕТОДИ ПІДСУМОВУВАННЯ РЯДІВ ФУР'Є ТА НАБЛИЖЕННЯ У ЗВАЖЕНИХ ПРОСТОРАХ ЛЕБЕГА ЗІ ЗМІННИМ ПОКАЗНИКОМ

In the present work, we study estimates for the periodic functions from the linear operators constructed on the basis of their Fourier series in weighted variable exponent Lebesgue spaces with Muckenhoupt weights. In this case, the obtained estimates depend on the sequence of the best approximation in weighted Lebesgue spaces with variable exponent.

Вивчаються оцінки періодичних функцій лінійних операторів, що побудовані на основі рядів Фур'є у зважених просторах Лебега зі змінним показником та вагою Макенхаупта. В даному випадку отримані оцінки залежать від послідовності найкращого наближення у зважених просторах Лебега зі змінним показником.

1. Introduction and the main results. Let $\mathbf{T}$ denote the interval $[0, 2\pi]$, $\mathbb{C}$ complex plane, and $L^p(\mathbf{T})$, $1 \leq p \leq \infty$, the Lebesgue space of measurable complex valued functions on $\mathbf{T}$.

We consider the sequence of the functions $\{\lambda_k(r)\}$ defined in the set E of the number line, satisfying the conditions that

$$\lambda_0(r) = 1, \quad \lim_{r \to r_0} \lambda_\nu(r) = 1$$

for an arbitrary fixed $\nu = 0, 1, 2, \ldots$.

Let

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} A_k(x, f), \; A_k(x, f) := a_k(f) \cos kx + b_k(f) \sin kx \tag{1.1}$$

be the Fourier series of the function $f \in L_1(\mathbf{T})$, where $a_k(f)$ and $b_k(f)$ are Fourier coefficients of the function f. The *nth partial sums* of the series (1.1) is defined

$$S_n(x, f) = \frac{a_0}{2} + \sum_{k=1}^{n} A_k(x, f).$$

Let us denote by $\wp$ the class of Lebesgue measurable functions $p\colon \mathbf{T} \to (1, \infty)$ such that $1 < p_* := \operatorname*{ess\,inf}_{x \in T} p(x) \leq p^* := \operatorname*{ess\,sup}_{x \in T} p(x) < \infty$. The conjugate exponent of $p(x)$ is shown by $p'(x) := \dfrac{p(x)}{p(x) - 1}$. For $p \in \wp$, we define a class $L^{p(.)}(\mathbf{T})$ of 2π-periodic measurable functions $f\colon \mathbf{T} \to \mathbb{C}$ satisfying the condition

$$\int_{\mathbf{T}} |f(x)|^{p(x)} dx < \infty.$$

This class $L^{p(.)}(\mathbf{T})$ is a Banach space with respect to the norm

$$\|f\|_{L^{p(.)}(\mathbf{T})} := \inf \left\{ \lambda > 0 \colon \int_{\mathbf{T}} \left| \frac{f(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\}.$$

A function $\omega\colon \mathbf{T} \to [0,\infty]$ is called a *weight function if* ω is a measurable and almost everywhere (a.e.) positive.

Let ω be a 2π-periodic weight function. We denote by $L^p_\omega(\mathbf{T})$ the weighted Lebesgue space of 2π-periodic measurable functions $f\colon \mathbf{T} \to \mathbb{C}$ such that $f\omega^{1/p} \in L^p(\mathbf{T})$. For $f \in L^p_\omega(\mathbf{T})$ we set

$$\|f\|_{L^p_\omega(\mathbf{T})} := \left\| f\omega^{1/p} \right\|_{L^p(\mathbf{T})}.$$

$L^{p(.)}_\omega(\mathbf{T})$ stands for the class of Lebesgue measurable functions $f\colon \mathbf{T} \to \mathbb{C}$ such that $\omega f \in$ $\in L^{p(.)}(\mathbf{T})$. $L^{p(.)}_\omega(\mathbf{T})$ is called the weighted Lebesgue space with variable exponent. The space $L^{p(.)}_\omega(\mathbf{T})$ is a Banach space with respect to the norm

$$\|f\|_{L^{p(.)}_\omega(\mathbf{T})} := \|f\omega\|_{L^{p(.)}(\mathbf{T})}.$$

Its known [17] that the set of trigonometric polynomials is dense in $L^{p(.)}_\omega(\mathbf{T})$, if $[\omega(x)]^{p(x)}$ is integrable on $\mathbf{T}$.

We suppose that for an arbitrary fixed $r \in E$ and for every function $f \in L^{p(.)}_\omega(\mathbf{T})$, the series

$$U_r(x,f,\lambda) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \lambda_k(r)\, A_k(x,f) \tag{1.2}$$

converges in the space $L^{p(.)}_\omega(\mathbf{T})$.

For each linear operator $U_r(x,\ f,\lambda)$ we set

$$R_r(f,\lambda)_{L^{p(.)}_\omega(\mathbf{T})} := \|f - U_r(x,f,\lambda)\|_{L^{p(.)}_\omega(\mathbf{T})}. \tag{1.3}$$

Let $\mathcal{B}$ be the class of all intervals in $\mathbf{T}$. For $B \in \mathcal{B}$ we set

$$p_B := \left(\frac{1}{|B|} \int_B \frac{1}{p(x)} dx \right)^{-1}.$$

For given $p \in \wp$ the class of weights ω satisfying the condition [1]

$$\left\|\omega^{p(x)}\right\|_{A_{p(.)}} := \sup_{B\in\mathcal{B}} \frac{1}{|B|^{p_B}} \left\|\omega^{p(x)}\right\|_{L^1(B)} \left\| \frac{1}{\omega^{p(x)}} \right\|_{B_{(p'(.)/p(.))}} < \infty$$

will be denoted by $A_{p(.)}$.

We say that the variable exponent $p(x)$ satisfies *Local log-Hölder continuity condition,* if there is a positive constant c_1 such that

$$|p(x) - p(y)| \le \frac{c_1}{\log\left(e + \dfrac{1}{|x-y|}\right)}, \tag{1.4}$$

for all $x, y \in \mathbf{T}$.

A function $p \in \wp$ is said to belong to the class $\wp^{\log}$, if the condition (1.4) is satisfied.

We denote by $E_n(f)_{L^{p(.)}_\omega(\mathbf{T})}$ the best approximation of $f \in L^{p(.)}_\omega(\mathbf{T})$ by trigonometric polynomials of degree not exceeding n, i. e.,

$$E_n(f)_{L^{p(\cdot)}_\omega(\mathbf{T})} = \inf\{\|f - T_n\|_{L^{p(\cdot)}_\omega(\mathbf{T})} : T_n \in \Pi_n\},$$

where Π_n denotes the class of trigonometric polynomials of degree at most n.

In this study we obtain estimates of the deviation of the periodic function from the linear operators constructed on the basis of its Fourier series in variable exponent Lebesgue spaces with Muckenhoupt weights. In particular, we obtain the general estimate for the deviation $R_r(f,\lambda)_{L^{p(\cdot)}_\omega(\mathbf{T})}$ of the function f from its Abel–Poisson means $U_r(x,f,\lambda)$ $(0 \le r < 1,\ \lambda_\nu(r) = r^\nu,\ \nu = 0,1,2,3,\ldots)$ in variable exponent Lebesgue spaces with Muckenhoupt weights. Note that the estimate obtained in this study depends on the rate of decrease of sequences of the best approximation $E_n(f)_{L^{p(\cdot)}_\omega(\mathbf{T})}$.

Lebesgue spaces with variable exponents have been investigated intensively by many authors (see, for example, [21, 22, 24, 25, 27]).

The approximation problems in nonweighted and weighted Lebesgue spaces with variable exponents were studied in [1, 2, 11, 16–20, 28].

Similar problems of the approximation theory in the different spaces have been studied by several authors (see, for example, [4–10, 12–15, 23, 26, 29–32]).

Our main results are as follows.

Theorem 1.1. *Let* $\{\lambda_\nu(r)\}$ *be an arbitrary triangular matrix* $(r = 0,1,2,3,\ldots,\ \lambda_0(r) = 1,\ \lambda_\nu(r) = 0,\ \nu > r)$. *If* $p \in \wp^{\log}$, $\omega^{-p_0} \in A_{(p(.)/p_0)'}$ *for some* $p_0 \in (1,\ p_*)$, $\gamma := \min\{2,\ p_*\}$ *and* $f \in L^{p(\cdot)}_\omega(\mathbf{T})$, *there exists a positive constant* c_2 *depending on* p *such that*

$$R_r(f,\lambda)_{L^{p(\cdot)}_\omega(\mathbf{T})} \le c_2 \left\{ \left(\sum_{\mu=0}^{m} \delta^\gamma_{2^\mu}(r)\ E^\gamma_{2^\mu-1}(f)_{L^{p(\cdot)}_\omega(\mathbf{T})} \right)^{1/\gamma} + E_r(f)_{L^{p(\cdot)}_\omega(\mathbf{T})} \right\},$$

where $2^m \le r < 2^{m+1}$ *and*

$$\delta_{2^\mu}(r) = \sum_{\nu=2^\mu}^{2^{\mu+1}-1} |\lambda_\nu(r) - \lambda_{\nu+1}(r)| + |1 - \lambda_{2^{\mu+1}}(r)|.$$

***Corollary* 1.1.** 1. *Let* $r = 0,1,2,3,\ldots,$

$$\lambda_\nu(r) = \begin{cases} 1 - \dfrac{\nu}{r+1}, & 0 \le \nu \le r, \\ 0, & \nu > r. \end{cases}$$

Then for the Fejér means, the estimate

$$R_r(f,\lambda)_{L^{p(\cdot)}_\omega(\mathbf{T})} \le \frac{c_3}{r+1} \left\{ \sum_{\nu=0}^{r} (\nu+1)^{\gamma-1} E^\gamma_\nu(f)_{L^{p(\cdot)}_\omega(\mathbf{T})} \right\}^{1/\gamma}$$

holds with a constant $c_3 > 0$ *depending on* p.

2. *For* $r = 0,1,2,3,\ldots,$

$$\lambda_\nu(r) = \begin{cases} 1 - \dfrac{\nu^k}{(r+1)^k}, & 0 \le \nu \le r, \\ 0, & \nu > r. \end{cases}$$

Then for the Zygmund means, the estimate

$$R_r(f,\lambda)_{L^{p(\cdot)}_\omega(\mathbf{T})} \le \frac{c_4}{(\nu+1)^k}\left\{\sum_{\nu=0}^{r}(\nu+1)^{\gamma k-1}\ E^\gamma_\nu(f)_{L^{p(\cdot)}_\omega(\mathbf{T})}\right\}^{1/\gamma}$$

holds with a constant $c_4 > 0$ depending on p and k.

Theorem 1.2. *Let $0 \le r < 1$, $\lambda_\nu(r) = r^\nu$, $\nu = 0,1,2,3,\ldots$. If $p \in \wp^{\log}$, $\omega^{-p_0} \in A_{(p(\cdot)/p_0)'}$ for some $p_0 \in (1,p_*)$, $\gamma := \min\{2,\ p_*\}$ and $f \in L^{p(\cdot)}_\omega(\mathbf{T})$, then for the Abel – Poisson means the estimate*

$$R_r(f,\lambda)_{L^{p(\cdot)}_\omega(\mathbf{T})} \le c_5(1-r)\left\{\sum_{\nu=0}^{\infty}r^\nu(\nu+1)^{\gamma-1}E^\gamma_\nu(f)_{L^{p(\cdot)}_\omega(\mathbf{T})}\right\}^{1/\gamma}$$

holds with a constant $c_5 > 0$ depending on p.

In the proof main results we need the following theorems [18].

The following Littlewood – Paley type theorem holds:

Theorem 1.3. *Let $\sum_{\nu=1}^\infty A_\nu(x)$ be the Fourier series of $f \in L^{p(\cdot)}_\omega(\mathbf{T})$. Under the conditions of Theorem 1.1 there are constants $c_6,\ c_7 > 0$ such that*

$$c_6\left\|\left(\sum_{\mu=1}^{\infty}\Delta_\mu^2\right)^{1/2}\right\| \le \|f(x)\|_{L^{p(\cdot)}_\omega(\mathbf{T})} \le c_7\left\|\left(\sum_{\mu=1}^{\infty}\Delta_\mu^2\right)^{1/2}\right\|_{L^{p(\cdot)}_\omega(\mathbf{T})}, \tag{1.5}$$

where

$$\Delta_\mu := \Delta_\mu(f,x) = \sum_{\nu=2^{\mu-1}}^{2^\mu-1} A_\nu(x,f), \quad \mu = 1,2,\ldots.$$

The following theorem is true:

Theorem 1.4. *Let $\{\lambda_k\}_0^\infty$ be a sequence of numbers such that*

$$|\lambda_k| \le c_8 \quad \textit{and} \quad \sum_{k=2^{m-1}}^{2^m-1}|\lambda_k - \lambda_{k+1}| \le c_8, \tag{1.6}$$

where $c_8 > 0$ does not depend on k and m. Suppose that the conditions of Theorem 1.1 are satisfied. If $f \in L^{p(\cdot)}_\omega(\mathbf{T})$ has the Fourier series

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} A_k(x,f),$$

then there exists a function $F \in L^{p(\cdot)}_\omega(\mathbf{T})$ with the Fourier series

$$\frac{\lambda_0 a_0}{2} + \sum_{k=1}^{\infty}\lambda_k A_k(x,f),$$

and

$$\|F\|_{L^{p(\cdot)}_\omega(\mathbf{T})} \le c_9\|f\|_{L^{p(\cdot)}_\omega(\mathbf{T})}.$$

2. Proofs of theorems. ***Proof of Theorem* 1.1.** Let $f \in L^{p(\cdot)}_{.\omega}(\mathbf{T})$ and $2^m \le r < 2^{m+1}$. According to reference [11] the following inequality holds:

$$\|f(x) - S_r(x,f)\|_{L^{p(\cdot)}_\omega(\mathbf{T})} \le E_r(f)_{L^{p(\cdot)}_\omega(\mathbf{T})}. \qquad (2.1)$$

Using (2.1) we get

$$R_r(f,\lambda)_{L^{p(\cdot)}_\omega(\mathbf{T})} = \left\| f(x) - \sum_{\nu=0}^{r} \lambda_\nu(r) A_\nu(x,f) \right\|_{L^{p(\cdot)}_\omega(\mathbf{T})} \le$$

$$\le \left\| \sum_{\nu=1}^{r} (1-\lambda_\nu(r)) A_\nu(x,f) \right\|_{L^{p(\cdot)}_\omega(\mathbf{T})} + \left\| \sum_{\nu=r+1}^{\infty} A_\nu(x,f) \right\|_{L^{p(\cdot)}_\omega(\mathbf{T})} \le$$

$$\le \left\| \sum_{\nu=1}^{r} (1-\lambda_\nu(r)) A_\nu(x,f) \right\|_{L^{p(\cdot)}_\omega(\mathbf{T})} + E_r(f)_{L^{p(\cdot)}_\omega(\mathbf{T})}. \qquad (2.2)$$

According to inequality (2.2) and (1.5) we obtain

$$R_r(f,\ \lambda)_{L^{p(\cdot)}_\omega(\mathbf{T})} \le \left\| \left\{ \sum_{\mu=0}^{m} \left| \sum_{\nu=2^\mu}^{2^{\mu+1}-1} (1-\lambda_\nu(r)) A_\nu(x,\ f)) \right|^2 \right\}^{1/2} \right\|_{L^{p(\cdot)}_\omega(\mathbf{T})} +$$

$$+ E_r(f)_{L^{p(\cdot)}_\omega(\mathbf{T})}. \qquad (2.3)$$

Use of Abel's transformation leads to

$$\sigma_{r,\mu}(x) = \sum_{\nu=2^\mu}^{2^{\mu+1}-1} (1-\lambda_\nu(r)) A_\nu(x,f) =$$

$$= \sum_{\nu=2^\mu}^{2^{\mu+1}-1} \{S_\nu(f,x) - S_{2^\mu_{-1}}(f,x)\}\{\lambda_{\nu+1}(r) - \lambda_\nu(r)\} +$$

$$+ \{1 - \lambda_{2^{\mu+1}}(r)\}\{S_{2^{\mu+1}-1}(f,x) - S_{2^\mu-1}(f,x)\}. \qquad (2.4)$$

Using (2.1), (2.4), Minkowski's inequality and monotonicity of the best approximation sequence we reach

$$\|\sigma_{r,\mu}\|_{L^{p(\cdot)}_\omega(\mathbf{T})} \le \sum_{\nu=2^\mu}^{2^{\mu+1}-1} \|S_\nu(f,x) - S_{2^\mu-1}(f,x)\|_{L^{p(\cdot)}_\omega(\mathbf{T})} |\lambda_{\nu+1}(r) - \lambda_\nu(r)| +$$

$$+ |1 - \lambda_{2^{\mu+1}}(r)| \|S_{2^{\mu+1}-1}(f,x) - S_{2^\mu-1}(f,x)\| \le$$

$$\leq c_{10} E_{2^\mu-1}(f)_{L^{p(\cdot)}_\omega(\mathbf{T})} \left\{ \sum_{\nu=2^\mu}^{2^{\mu+1}-1} |\lambda_{\nu+1}(r)-\lambda_\nu(r)| + |1-\lambda_{2^{\mu+1}}(r)| \right\} = E_{2^\mu-1}(f)_{L^{p(\cdot)}_\omega(\mathbf{T})} \delta_{2^\mu}(r). \tag{2.5}$$

In addition, the following inequality holds:

$$\left\| \left\{ \sum_{\mu=0}^{m} |\sigma_{r,\mu}(x)|^2 \right\}^{1/2} \right\|_{L^{p(\cdot)}_\omega(\mathbf{T})} \leq c_{11} \left\{ \sum_{\mu=0}^{m} \|\sigma_{r,\mu}\|^\gamma_{L^{p(\cdot)}_\omega(\mathbf{T})} \right\}^{1/\gamma}. \tag{2.6}$$

Use of (2.3), (2.6) and (2.5) gives us

$$R_r(f,\lambda)_{L^{p(\cdot)}_\omega(\mathbf{T})} \leq c_{12} \left\{ \sum_{\mu=0}^{m} \|\sigma_{r,\mu}\|^\gamma_{L^{p(\cdot)}_\omega(\mathbf{T})} \right\}^{1/\gamma} + E_r(f)_{L^{p(\cdot)}_\omega(\mathbf{T})} \leq$$

$$\leq c_{13} \left\{ \sum_{\mu=0}^{m} E^\gamma_{2^\mu-1}(f)_{L^{p(\cdot)}_\omega(\mathbf{T})} \delta^\gamma_{2^\mu}(r) \right\}^{1/\gamma} + E_r(f)_{L^{p(\cdot)}_\omega(\mathbf{T})}.$$

Theorem 1.1 is proved.

***Proof of Theorem* 1.2.** Let $f \in L^{p(\cdot)}_\omega(\mathbf{T})$ and $\lambda_\nu(r) = r^\nu, 0 \leq r < 1,\ \nu = 0,1,2,\ldots$. We have

$$R_r(f,\lambda)_{L^{p(\cdot)}_\omega(\mathbf{T})} = \left\| f(x) - \sum_{\nu=0}^{\infty} r^\nu A_\nu(x,f) \right\|_{L^{p(\cdot)}_\omega(\mathbf{T})} =$$

$$= \left\| \sum_{\nu=0}^{\infty} (1-r^\nu) A_\nu(x,f) \right\|_{L^{p(\cdot)}_\omega(\mathbf{T})}. \tag{2.7}$$

Selecting m such that $2^m \leq n = \left[\frac{1}{1-r}\right] < 2^{m+1}$ (here, $[\beta]$ denotes the integer part of a real number β) from (2.7) we get

$$R_r(f,\lambda)_{L^{p(\cdot)}_\omega(\mathbf{T})} \leq \left\| \sum_{\nu=0}^{2^{m+1}-1} (1-r^\nu) A_\nu(x,f) \right\|_{L^{p(\cdot)}_\omega(\mathbf{T})} +$$

$$+ \left\| \sum_{\nu=2^{m+1}}^{\infty} (1-r^\nu) A_\nu(x,f) \right\|_{L^{p(\cdot)}_\omega(\mathbf{T})} = I_1 + I_2. \tag{2.8}$$

According to (1.5) we obtain

$$I_1 \leq c_{14} \left\| \left(\sum_{\nu=0}^{m} \left| \sum_{\mu=2^\nu}^{2^{\nu+1}-1} (1-r^\mu) A_\mu(x,f) \right|^2 \right)^{1/2} \right\|_{L^{p(\cdot)}_\omega(\mathbf{T})} \tag{2.9}$$

and therefore

$$I_1 \le c_{15}\left(\sum_{\nu=0}^{m}\left\|\sum_{\mu=2^\nu}^{2^{\nu+1}-1}(1-r^\mu)A_\mu(x,f)\right\|^\gamma_{L^{p(\cdot)}_\omega(\mathbf{T})}\right)^{1/\gamma}. \tag{2.10}$$

Using Abel's transformation and (2.1), we find that

$$\left\|\sum_{\mu=2^\nu}^{2^{\nu+1}-1}(1-r^\mu)A_\mu(x,f)\right\|_{L^{p(\cdot)}_\omega(\mathbf{T})} =$$

$$= \Bigg\|\sum_{\mu=2^\nu}^{2^{\nu+1}-1}\{S_\mu(f,x)-S_{2^\nu-1}(f,x)\}(r^{\mu+1}-r^\mu)+$$

$$+\{S_{2^{\nu+1}-1}(f,x)-S_{2^\nu-1}(f,x)\}(1-r^{2^{\nu+1}})\Bigg\|_{L^{p(\cdot)}_\omega(\mathbf{T})} \le$$

$$\le c_{16}2^{\nu+1}(1-r)E_{2^\nu-1}(f)_{L^{p(\cdot)}_\omega(\mathbf{T})}. \tag{2.11}$$

Using monotonicity of the best approximation sequence $E_n(f)_{L^{p(\cdot)}_\omega(T)}$ from (2.11) we have

$$\left\|\sum_{\mu=2^\nu}^{2^{\nu+1}-1}(1-r^\mu)A_\nu(x,f)\right\|^\gamma_{L^{p(\cdot)}_\omega(\mathbf{T})} \le c_{17}2^{(\nu+1)\gamma}(1-r)^\gamma E^\gamma_{2^\nu-1}(f)_{L^{p(\cdot)}_\omega(\mathbf{T})} \le$$

$$\le c_{18}(1-r)^\gamma\sum_{\mu=2^{\nu-1}}^{2^\nu-1}E^\gamma_{2^\nu-1}(f)_{L^{p(\cdot)}_\omega(\mathbf{T})}. \tag{2.12}$$

Consideration of (2.10) and (2.12) gives us we

$$I_1 \le c_{19}\Bigg\{(1-r)^\gamma\|A_1(x,\ f)\|^\gamma_{L^{p(\cdot)}_\omega(\mathbf{T})}+$$

$$+\sum_{\nu=1}^{m}(1-r)^\gamma\sum_{\mu=2^{\nu-1}}^{2^\nu-1}\mu^{\gamma-1}E^\gamma_\mu(f)_{L^{p(\cdot)}_\omega(\mathbf{T})}\Bigg\}^{1/\gamma} \le$$

$$\le c_{20}(1-r)\left\{\sum_{\mu=0}^{n}(\mu+1)^{\gamma-1}E^\gamma_\mu(f)_{L^{p(\cdot)}_\omega(\mathbf{T})}\right\}^{1/\gamma}. \tag{2.13}$$

Applying Theorem 1.4 for the sequence $\{\lambda_k\}_0^\infty$ satisfying the condition (1.5) and (2.1), we obtain

$$I_2 \le c_{21}\left\|\sum_{\nu=2^{m+1}}^{\infty}A_\nu(x,\ f)\right\|_{L^{p(\cdot)}_\omega(\mathbf{T})} \le c_{22}E_n(f)_{L^{p(\cdot)}_\omega(\mathbf{T})}. \tag{2.14}$$

Therefore according to (2.8), (2.13) and (2.14), we have

$$R_r(f,\lambda)_{L^{p(\cdot)}_\omega(\mathbf{T})} \le \left\| \sum_{\nu=0}^{2^{m+1}-1} (1-r^\nu) A_\nu(x,f) \right\|_{L^{p(\cdot)}_\omega(\mathbf{T})} +$$

$$+ \left\| \sum_{\nu=2^{m+1}}^{\infty} (1-r^\nu) A_\nu(x,f) \right\|_{L^{p(\cdot)}_\omega(\mathbf{T})} \le$$

$$\le c_{20}(1-r) \left\{ \sum_{\mu=0}^{n} (\mu+1)^{\gamma-1} E_\mu^\gamma(f)_{L^{p(\cdot)}_\omega(\mathbf{T})} \right\}^{1/\gamma} + c_{22} E_n(f)_{L^{p(\cdot)}_\omega(\mathbf{T})} \le$$

$$\le c_{23}(1-r) \left\{ \sum_{\nu=0}^{n} (\nu+1)^{\gamma-1} E_\nu^\gamma(f)_{L^{p(\cdot)}_\omega(\mathbf{T})} \right\}^{1/\gamma} \le$$

$$\le c_{24}(1-r) \left\{ \sum_{\nu=0}^{\infty} r^\nu (\nu+1)^{\gamma-1} E_\nu^\gamma(f)_{L^{p(\cdot)}_\omega(\mathbf{T})} \right\}^{1/\gamma}.$$

Theorem 1.2 is proved.

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