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Lei Qiao (School Math. and Inform. Sci., Henan Univ. Economics and Law, Zhengzhou, China)

DIRICHLET PROBLEMS FOR HARMONIC FUNCTIONS IN HALF SPACES*

ЗАДАЧІ ДІРІХЛЕ ДЛЯ ГАРМОНІЧНИХ ФУНКЦІЙ У НАПІВПРОСТОРАХ

In the paper, we prove that if the positive part $u^+(x)$ of a harmonic function $u(x)$ in a half space satisfies a condition of slow growth, then its negative part $u^-(x)$ can also be dominated by a similar condition of growth. Moreover, we give an integral representation of the function $u(x)$. Further, a solution of the Dirichlet problem in the half space for a rapidly growing continuous boundary function is constructed by using the generalized Poisson integral with this boundary function.

Доведено, що у випадку, коли додатна частина $u^+(x)$ гармонічної функції $u(x)$ у напівпросторі задовольняє умову повільного зростання, її від'ємна частина $u^-(x)$ також може бути домінована подібною умовою зростання. Крім того, наведено інтегральне зображення для функції $u(x)$. Більш того, розв'язок задачі Діріхле в напівпросторі для швидко зростаючої неперервної граничної функції побудовано за допомогою узагальненого інтеграла Пуассона з цією граничною функцією.

1. Introduction and results. Let $\mathbf{R}$ and $\mathbf{R}_+$ be the sets of all real numbers and of all positive real numbers, respectively. Let $\mathbf{R}^n$, $n \geq 3$, denote the n-dimensional Euclidean space with points $x = (x', x_n)$, where $x' = (x_1, x_2, \ldots, x_{n-1}) \in \mathbf{R}^{n-1}$ and $x_n \in \mathbf{R}$. The boundary and closure of an open set D of $\mathbf{R}^n$ are denoted by ∂D and $\overline{D}$ respectively. The upper half space is the set $H = \{(x', x_n) \in \mathbf{R}^n : x_n > 0\}$, whose boundary is ∂H.

For a set E, $E \subset \mathbf{R}_+ \cup \{0\}$, we denote $\{x \in H : |x| \in E\}$ and $\{x \in \partial H : |x| \in E\}$ by HE and ∂HE, respectively. We identify $\mathbf{R}^n$ with $\mathbf{R}^{n-1} \times \mathbf{R}$ and $\mathbf{R}^{n-1}$ with $\mathbf{R}^{n-1} \times \{0\}$, writing typical points $x, y \in \mathbf{R}^n$ as $x = (x', x_n)$, $y = (y', y_n)$, where $y' = (y_1, y_2, \ldots, y_{n-1}) \in \mathbf{R}^{n-1}$ and putting

$$x \cdot y = \sum_{j=1}^{n} x_j y_j = x' \cdot y' + x_n y_n, \qquad |x| = \sqrt{x \cdot x}, \qquad |x'| = \sqrt{x' \cdot x'}.$$

Let $B_n(r)$ denote the open ball with center at the origin and radius $r (> 0)$ in $\mathbf{R}^n$. We use the standard notations $u^+ = \max\{u, 0\}$, $u^- = -\min\{u, 0\}$ and $[d]$ is the integer part of the positive real number d. In the sense of Lebesgue measure $dy' = dy_1 \ldots dy_{n-1}$ and $dy = dy' dy_n$. Let σ denote $(n-1)$-dimensional surface area measure and $\partial/\partial n$ denote differentiation along the inward normal into H. For positive functions h_1 and h_2, we say that $h_1 \lesssim h_2$ if $h_1 \leq d h_2$ for some positive constant d.

Given a continuous function f on ∂H, we say that h is a solution of the (classical) Dirichlet problem on H with f, if $\Delta h = 0$ in H and $\lim_{x \in H,\, x \to z'} h(x) = f(z')$ for every $z' \in \partial H$.

The classical Poisson kernel for H is defined by

$$P(x, y') = \frac{2x_n}{\omega_n |x - y'|^n},$$

where $\omega_n = 2\pi^{n/2}/\Gamma(n/2)$ is the area of the unit sphere in $\mathbf{R}^n$.

To solve the Dirichlet problem on H, as in [3, 4, 8, 11], we use the following modified Poisson kernel of order m defined by

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$$P_m(x,y') = \begin{cases} P(x,y') & \text{when} \quad |y'| \le 1, \\ P(x,y') - \sum_{k=0}^{m-1} \dfrac{2x_n|x|^k}{\omega_n |y'|^{n+k}} C_k^{n/2}\left(\dfrac{x \cdot y'}{|x||y'|}\right) & \text{when} \quad |y'| > 1, \end{cases}$$

where m is a nonnegative integer, $C_k^{n/2}(t)$ is the ultraspherical (Gegenbauer) polynomials [10]. The expression arises from the generating function for Gegenbauer polynomials

$$(1 - 2tr + r^2)^{-n/2} = \sum_{k=0}^{\infty} C_k^{n/2}(t) r^k,$$

where $|r| < 1$ and $|t| \le 1$. The coefficient $C_k^{n/2}(t)$ is called the ultraspherical (Gegenbauer) polynomial of degree k associated with $n/2$, the function $C_k^{n/2}(t)$ is a polynomial of degree k in t.

Put

$$U(f)(x) = \int\limits_{\partial H} P(x,y') f(y') dy' \qquad \text{and} \qquad U_m(f)(x) = \int\limits_{\partial H} P_m(x,y') f(y') dy',$$

where $f(y')$ is a continuous function on ∂H.

For any positive real number α, we denote by $\mathcal{A}_\alpha$ the space of all measurable functions $f(y)$ on H satisfying

$$\int\limits_H \frac{y_n |f(y)| dy}{1 + |y|^{\alpha+2}} < \infty \tag{1.1}$$

and $\mathcal{B}_\alpha$ the set of all measurable functions $g(y')$ on ∂H such that

$$\int\limits_{\partial H} \frac{|g(y')| dy'}{1 + |y'|^{\alpha}} < \infty. \tag{1.2}$$

We also denote by $\mathcal{C}_\alpha$ the set of all continuous functions $u(x)$ on $\overline{H}$, harmonic on H with $u^+(y) \in \mathcal{A}_\alpha$ and $u^+(y') \in \mathcal{B}_\alpha$.

Theorem A (see [1, 9]). *If $u(x) > 0$ and $u \in \mathcal{C}_n$, then there exists a positive constant d_1 such that $u(x) = d_1 x_n + U(u)(x)$ for all $x \in H$.*

Using the modified Poisson kernel $P_m(x,y')$, H. Yoshida (cf. [11], Theorem 1) and Siegel – Talvila (cf. [8], Corollary 2.1) obtained classical solutions of the Dirichlet problem on H respectively.

Theorem B. *If $f \in \mathcal{B}_{n+m}$, then $U_m(f)(x)$ is a solution of the Dirichlet problem on H with f satisfying*

$$\lim_{|x|\to\infty,\, x\in H} |x|^{-m-1} U_m(f)(x) = 0.$$

Motivated by Theorems A and B, we first prove the following theorem.

Theorem 1. *If $\alpha \ge n$ and $u \in \mathcal{C}_\alpha$, then $u \in \mathcal{B}_\alpha$.*

Then we concerned with the growth property of $U_m(f)(x)$ at infinity on H. In the half plane, this result for $\alpha = 2$ was obtained by Pan – Deng ([7], Theorem 1.1 and Remark 1.1).

Theorem 2. *If $\alpha - n \leq m < \alpha - n + 1$ and $f \in \mathcal{C}_\alpha$, then $U_m(f)(x)$ is a solution of the Dirichlet problem on H with f satisfying*

$$\lim_{|x|\to\infty,\, x\in H} x_n^{n-1}|x|^{-\alpha}U_m(f)(x) = 0. \tag{1.3}$$

Then we are concerned with Dirichlet problems for harmonic functions of infinite order on H (see [5] (Definition 4.1) for the definition of the order). For this purpose, we define a nondecreasing and continuously differentiable function $\rho(R) \geq 1$ on the interval $[0, +\infty)$. We assume further that

$$\varepsilon_0 = \limsup_{R\to\infty} \frac{\rho'(R)R\ln R}{\rho(R)} < 1. \tag{1.4}$$

Let $\mathcal{D}(\rho, \beta)$ be the set of continuous functions f on ∂H such that

$$\int\limits_{\partial H} \frac{|f(y')|dy'}{1 + |y'|^{\rho(|y'|)+n+\beta-1}} < \infty, \tag{1.5}$$

where β is a positive real number.

Now we have the following theorem.

Theorem 3. *If $f \in \mathcal{D}(\rho, \beta)$, then the integral $U_{[\rho(|y'|)+\beta]}(f)(x)$ is a solution of the Dirichlet problem on H with f.*

Theorem B follows from Theorem 3 (the case $[\rho(|y'|) + \beta] = m$), Theorems 1 and 2 (the case $\alpha = n + m$).

About integral representations for harmonic functions of finite order on H, we have the following result.

***Corollary* 1.** *Let $u \in \mathcal{C}_\alpha$, $\alpha \geq n$, and let m be an integer such that $n + m < \alpha \leq n + m + 1$.*

(I) *If $\alpha = n$, then $U(u)(x)$ is a harmonic function on H and can be continuously extended to $\overline{H}$ such that $u(x') = U(u)(x')$ for $x' \in \partial H$. There exists a constant d_2 such that $u(x) = d_2 x_n + U(u)(x)$ for all $x \in H$.*

(II) *If $\alpha > n$, then $U_m(u)(x)$ is a harmonic function on H and can be continuously extended to $\overline{H}$ such that $u(x') = U_m(u)(x')$ for $x' \in \partial H$. There exists a harmonic polynomial $Q_m(u)(x)$ of degree at most $m - 1$ which vanishes on ∂H such that $u(x) = U_m(u)(x) + Q_m(u)(x)$ for all $x \in H$.*

Finally, we have the following theorem.

Theorem 4. *Let u be harmonic in H and continuous on $\overline{H}$. If $u \in \mathcal{D}(\rho, \beta)$, then we have*

$$u(x) = U_{[\rho(|y'|)+\beta]}(u)(x) + \Pi(x)$$

for all $x \in \overline{H}$, where $\Pi(x)$ is harmonic on H and vanishes continuously on ∂H.

2. Lemmas. The Carleman's formula refers to holomorphic functions in a half space, which is due to T. Carleman (see [2]). Recently, Y. H. Zhang, G. T. Deng and K. Kou (see [12], Lemma 1) generalized it to harmonic functions on H, which plays an important role in our discussions.

Lemma 1. *If $R > 1$ and $u(y)$ is a harmonic function on H with continuous boundary on ∂H, then we have*

$$m_-(R) + \int\limits_{\partial H(1,R)} u^-(y') \left(\frac{1}{|y'|^n} - \frac{1}{R^n} \right) dy' =$$

$$= m_+(R) + \int\limits_{\partial H(1,R)} u^+(y') \left(\frac{1}{|y'|^n} - \frac{1}{R^n} \right) dy' - d_3 - \frac{d_4}{R^n},$$

where

$$m_\pm(R) = \int\limits_{\{y \in H\colon |y|=R\}} u^\pm(y) \frac{ny_n}{R^{n+1}} d\sigma(y),$$

$$d_3 = \int\limits_{\{y \in H\colon |y|=1\}} \left((n-1) y_n u(y) + y_n \frac{\partial u(y)}{\partial n} \right) d\sigma(y),$$

and

$$d_4 = \int\limits_{\{y \in H\colon |y|=1\}} \left(y_n u(y) - y_n \frac{\partial u(y)}{\partial n} \right) d\sigma(y).$$

Lemma 2 (see [5], Lemma 4.2)**.**

$$|P_m(x, y')| \lesssim x_n |x|^{m-1} |y'|^{-n-m+1} \tag{2.1}$$

for any $y' \in \partial H[1, |x|/2)$,

$$|P_m(x, y')| \lesssim x_n^{1-n} \tag{2.2}$$

for any $y' \in \partial H[|x|/2, 2|x|)$,

$$|P_m(x, y')| \lesssim x_n |x|^m |y'|^{-n-m} \tag{2.3}$$

for any $y' \in \partial H[1, \infty) \cap \partial H[2|x|, \infty)$.

Lemma 3 (see [6], Theorem 10)**.** *Let $h(x)$ be a harmonic function on H such that $h(x)$ vanishes continuously on ∂H. If*

$$\lim_{|x| \to \infty,\, x \in H} |x|^{-m-1} h^+(x) = 0,$$

then $h(x) = Q_m(h)(x)$ on H, where $Q_m(h)$ is a polynomial of $(x', x_n) \in \mathbf{R}^n$ of degree less than m and odd with respect to the variable x_n.

3. Proof of Theorem 1. We distinguish the following two cases.

Case 1. $\alpha = n$.

If $R > 2$, Lemma 1 gives

$$m_-(R) + \left(1 - \frac{1}{2^n} \right) \int\limits_{\partial H(1,R/2)} \frac{u^-(y')}{|y'|^n} dy' \le$$

$$\leq m_-(R) + \int\limits_{\partial H(1,R)} u^-(y') \left(\frac{1}{|y'|^n} - \frac{1}{R^n} \right) dy' \leq$$

$$\leq m_+(R) + \int\limits_{\partial H(1,R)} \frac{u^+(y')}{|y'|^n} dy' + |d_3| + |d_4|. \tag{3.1}$$

Since $u \in \mathcal{C}_n$, we obtain

$$\frac{1}{n} \int\limits_1^\infty \frac{m_+(R)}{R} dR = \int\limits_{H(1,\infty)} \frac{y_n u^+(y)}{|y|^{n+2}} dy \lesssim \int\limits_H \frac{y_n u^+(y)}{1 + |y|^{n+2}} dy < \infty$$

from (1.1) and hence

$$\liminf_{R\to\infty} m_+(R) = 0, \tag{3.2}$$

where $m_+(R)$ is defined in Lemma 1.

Then from (1.2), (3.1) and (3.2) we have

$$\liminf_{R\to\infty} \int\limits_{\partial H(1,R/2)} \frac{u^-(y')}{|y'|^n} dy' < \infty,$$

which gives

$$\int\limits_{\partial H} \frac{u^-(y')}{1 + |y'|^n} dy' < \infty.$$

Thus $u \in \mathcal{B}_n$ from $|u| = u^+ + u^-$.

Case 2. $\alpha > n$.

Since $u \in \mathcal{C}_\alpha$, we see from (1.1) that

$$\frac{1}{n} \int\limits_1^\infty \frac{m_+(R)}{R^{\alpha-n+1}} dR = \int\limits_{H(1,\infty)} \frac{y_n u^+(y)}{|y|^{\alpha+2}} dy \lesssim \int\limits_H \frac{y_n u^+(y)}{1 + |y|^{\alpha+2}} dy < \infty \tag{3.3}$$

and see from (1.2) that

$$\int\limits_1^\infty \frac{1}{R^{\alpha-n+1}} \int\limits_{\partial H(1,R)} u^+(y') \left(\frac{1}{|y'|^n} - \frac{1}{R^n} \right) dy' dR =$$

$$= \int\limits_{\partial H(1,\infty)} u^+(y') \int\limits_{|y'|}^\infty \frac{1}{R^{\alpha-n+1}} \left(\frac{1}{|y'|^n} - \frac{1}{R^n} \right) dR dy' \lesssim$$

$$\lesssim \frac{n}{(\alpha - n)\alpha} \int\limits_{\partial H(1,\infty)} \frac{u^+(y')}{|y'|^\alpha} dy' < \infty. \tag{3.4}$$

We have from (3.3), (3.4) and Lemma 1

$$\int\limits_{\partial H(1,\infty)} u^-(y') \int\limits_{|y'|}^{\infty} \frac{1}{R^{\alpha-n+1}} \left(\frac{1}{|y'|^n} - \frac{1}{R^n} \right) dR dy' \leq$$

$$\leq \int\limits_{1}^{\infty} \frac{m_+(R)}{R^{\alpha-n+1}} dR - \int\limits_{1}^{\infty} \frac{1}{R^{\alpha-n+1}} \left(d_3 + \frac{d_4}{R^n} \right) dR +$$

$$+ \int\limits_{1}^{\infty} \frac{1}{R^{\alpha-n+1}} \int\limits_{\partial H(1,R)} u^+(y') \left(\frac{1}{|y'|^n} - \frac{1}{R^n} \right) dy' dR < \infty.$$

Set

$$I(\alpha) = \lim_{|y'|\to\infty} |y'|^{\alpha} \int\limits_{|y'|}^{\infty} \frac{1}{R^{\alpha-n+1}} \left(\frac{1}{|y'|^n} - \frac{1}{R^n} \right) dR.$$

We get

$$I(\alpha) = \frac{n}{\alpha(\alpha - n)}$$

from the L'hospital's rule and hence we have

$$\int\limits_{|y'|}^{\infty} \frac{1}{R^{\alpha-n+1}} \left(\frac{1}{|y'|^n} - \frac{1}{R^n} \right) dR \gtrsim |y'|^{-\alpha}.$$

So

$$\int\limits_{\partial H[1,\infty)} \frac{u^-(y')}{|y'|^{\alpha}} dx' \lesssim \int\limits_{\partial H[1,\infty)} u^-(y') \int\limits_{|y'|}^{\infty} \frac{1}{R^{\alpha-n+1}} \left(\frac{1}{|y'|^n} - \frac{1}{R^n} \right) dR dy' < \infty.$$

Then $u \in \mathcal{B}_{\alpha}$ from $|u| = u^+ + u^-$.

Theorem 1 is proved.

4. Proof of Theorem 2. For any fixed $x \in H$, take a number R_1 satisfying $R_1 > \max\{1, 2|x|\}$, we have

$$\int\limits_{\partial H(R_1,\infty)} |P_m(x,y')||f(y')| dy' \lesssim$$

$$\lesssim x_n |x|^m \int\limits_{\partial H(R_1,\infty)} |y'|^{-n-m} |f(y')| dy' \lesssim$$

$$\lesssim x_n |x|^{\alpha-n} \int\limits_{\partial H(R_1,\infty)} |y'|^{-\alpha} |f(y')| dy' < \infty$$

from (2.3) and Theorem 1. Thus $U_m(f)(x)$ is finite for any $x \in H$. Since $P_m(x, y')$ is a harmonic function of $x \in H$ for any fixed $y' \in \partial H$. $U_m(f)(x)$ is also a harmonic function of $x \in H$.

To verify the boundary behavior of $U_m(f)(x)$. For any fixed boundary point $z' \in \partial H$, we can choose a number R_2 such that $R_2 > |z'| + 1$. Now we write

$$U_m(f)(x) = I_1(x) - I_2(x) + I_3(x),$$

where

$$I_1(x) = \int\limits_{\partial H[0,R_2]} P(x, y') f(y') dy',$$

$$I_2(x) = \sum_{k=0}^{m-1} \frac{2x_n |x|^k}{\omega_n} \int\limits_{\partial H(1,R_2]} \frac{1}{|y'|^{n+k}} C_k^{n/2} \left(\frac{x' \cdot y'}{|x||y'|} \right) f(y') dy',$$

and

$$I_3(x) = \int\limits_{\partial H(R_2,\infty)} P_m(x, y') f(y') dy'.$$

Notice that $I_1(x)$ is the Poisson integral of $f(y')\chi_{B_{n-1}(R_2)}(y')$, where $\chi_{B_{n-1}(R_2)}$ is the characteristic function of the ball $B_{n-1}(R_2)$. So it tends to $f(z')$ as $x \to z'$. Since $I_2(x)$ is a polynomial times x_n and $I_3(x) = O(x_n)$, both of them tend to zero as $x \to z'$. So the function $U_m(f)(x)$ can be continuously extended to $\overline{H}$ such that $U_m(f)(z') = f(z')$ for any $z' \in \partial H$. Then $U_m(f)(x)$ is a solution of the Dirichlet problem on H with f.

For any $\epsilon > 0$, there exists $R_\epsilon > 2$ such that

$$\int\limits_{\partial H[R_\epsilon,\infty)} \frac{|f(y')|}{1 + |y'|^\alpha} dy' < \epsilon \tag{4.1}$$

from Theorem 1. For any fixed $x \in H[2R_\epsilon, \infty)$, we write

$$U_m(f)(x) = \sum_{i=1}^{4} \int\limits_{G_i} P_m(x, y') f(y') dy' = \sum_{i=1}^{4} V_i(x),$$

where $G_1 = \partial H[0, 1)$, $G_2 = \partial H[1, |x|/2)$, $G_3 = \partial H[|x|/2, 2|x|)$, and $G_4 = \partial H[2|x|, \infty)$.

First note that

$$|V_1(x)| \lesssim x_n \left(\frac{|x|}{2} \right)^{-n} \int\limits_{G_1} |f(y')| dy' \lesssim x_n |x|^{-n}. \tag{4.2}$$

By (2.1) we have

$$|V_2(x)| \lesssim x_n |x|^{m-1} \int\limits_{G_2} |y'|^{-n-m+1} |f(y')| dy' \lesssim x_n |x|^{\alpha-n} \int\limits_{G_2} |y'|^{-\alpha} |f(y')| dy'. \tag{4.3}$$

Write

$$V_2(x) = V_{21}(x) + V_{22}(x),$$

where

$$V_{21}(x) = \int\limits_{G_2 \bigcap B_{n-1}(R_\epsilon)} P_m(x, y') f(y') dy'$$

and

$$U_{22}(x) = \int\limits_{G_2 - B_{n-1}(R_\epsilon)} P_m(x, y') f(y') dy'.$$

If $|x| > 2R_{\epsilon_1}$, then

$$|V_{21}(x)| \lesssim R_\epsilon^{\alpha-n-m+1} x_n |x|^{m-1}.$$

Moreover, by (4.1) and (4.3) we obtain

$$|V_{22}(x)| \lesssim \epsilon x_n |x|^{\alpha-n}.$$

That is

$$|V_2(x)| \lesssim \epsilon x_n^{1-n} |x|^\alpha. \tag{4.4}$$

We also obtain the following estimates:

$$|V_3(x)| \lesssim \epsilon x_n^{1-n} |x|^\alpha, \tag{4.5}$$

$$|V_4(x)| \lesssim x_n |x|^m \int\limits_{G_4} |y'|^{-n-m} |f(y')| dy' \lesssim \epsilon x_n |x|^{\alpha-n} \tag{4.6}$$

from (2.2), (2.3), and (4.1).

Combining (4.2) and (4.4)–(4.6), (1.3) holds.

Theorem 2 is proved.

5. Proof of Theorem 3. Take a number r satisfying $r > R_3$, where R_3 is a sufficiently large positive number. For any ϵ $(0 < \epsilon < 1 - \epsilon_0)$, by (1.4) we have

$$\rho(r) < \rho(e)(\ln r)^{(\epsilon_0+\epsilon)}, \tag{5.1}$$

which yields that there exists a positive constant $M(r)$ dependentbonly on r such that

$$k^{-\frac{\beta}{2}} (2r)^{\rho(k+1)+\beta+1} \leq M(r) \tag{5.2}$$

for any $k \geq k_r = [2r] + 1$.

For any $x \in H$ and $|x| \leq r$, we obtain by (1.5), (2.3) and (5.2)

$$\sum_{k=k_r}^{\infty} \int\limits_{\partial H[k,k+1)} \frac{(2|x|)^{[\rho(|y'|)+\beta]+1}}{|y'|^{[\rho(|y'|)+\beta]+n}} |f(y')| dy' \lesssim$$

$$\lesssim \sum_{k=k_r}^{\infty} \int\limits_{\partial H[k,k+1)} \frac{(2r)^{\rho(k+1)+\beta+1}}{|y'|^{[\rho(|y'|)+\beta]-\rho(|y'|)-\frac{\beta}{2}+1}} \frac{|f(y')|}{|y'|^{\rho(|y'|)+n+\frac{\beta}{2}-1}} dy' \lesssim$$

$$\lesssim \sum_{k=k_r}^{\infty} k^{-\frac{\beta}{2}} (2r)^{\rho(k+1)+\beta+1} \int\limits_{\partial H[k,k+1)} \frac{2|f(y')|}{1+|y'|^{\rho(|y'|)+n+\frac{\beta}{2}-1}} dy' \lesssim$$

$$\lesssim \sum_{k=k_r}^{\infty} M(r) \int\limits_{\partial H[k,k+1)} \frac{|f(y')|}{1+|y'|^{\rho(|y'|)+n+\frac{\beta}{2}-1}} dy' \lesssim$$

$$\lesssim M(r) \int\limits_{\partial H[k_r,\infty)} \frac{|f(y')|}{1+|y'|^{\rho(|y'|)+n+\frac{\beta}{2}-1}} dy' < \infty. \tag{5.3}$$

Notice that

$$|U_{[\rho(|y'|)+\beta]}(f)(x)| \leq W_1(x) + W_2(x) + W_3(x),$$

where

$$W_1(x) = \int\limits_{\partial H[0,1]} \frac{2}{\omega_n x_n^{n-1}} |f(y')| dy',$$

$$W_2(x) = \int\limits_{\partial H(1,k_r]} \frac{2^{[\rho(|y'|)+\beta]}}{\omega_n} \left(2^n + [\rho(|y'|)+\beta] C^{n/2}_{[\rho(|y'|)+\beta]-1}(1)\right) \times$$

$$\times \frac{|x|^{[\rho(|y'|)+\beta]+n+1}}{x_n^{n-1}|y'|^{[\rho(|y'|)+\beta]+n-1}} |f(y')| dy',$$

and

$$W_3(x) = \int\limits_{\partial H(k_r,\infty)} \frac{2^{[\rho(|y'|)+\beta]+n+1} x_n x^{[\rho(|y'|)+\beta]}}{\omega_n |y'|^{[\rho(|y'|)+\beta]+n}} |f(y')| dy'.$$

By (1.5) we have

$$W_1(x) \lesssim x_n^{1-n} \int\limits_{\partial H[0,1]} \frac{|f(y')|}{1+|y'|^{\rho(|y'|)+n+\beta-1}} dy' < \infty \tag{5.4}$$

and

$$W_2(x) \leq \frac{2^{m_r}}{\omega_n} \left(2^n + m_r C^{n/2}_{m_r-1}(1)\right) \frac{r^{m_r+n-1}}{x_n^{n-1}} \int\limits_{\partial H(1,k_r]} \frac{2|y'|^{\rho(|y'|)+\beta-[\rho(|y'|)+\beta]}|f(y')|}{1+|y'|^{\rho(|y'|)+n+\beta-1}} dy' \leq$$

$$\leq \frac{2^{m_r}}{\omega_n} \left(2^n + m_r C^{n/2}_{m_r-1}(1)\right) \frac{r^{m_r+n-1}}{x_n^{n-1}} 2k_r \int\limits_{\partial H} \frac{|f(y')|}{1+|y'|^{\rho(|y'|)+n+\beta-1}} dy' < \infty, \tag{5.5}$$

where $m_r = [\rho(k_r)+\beta]$.

On the other hand we have, by (5.3), that

$$W_3(x) \le \sum_{k=k_r}^{\infty} \int\limits_{\partial H[k,k+1)} \frac{2^n}{\omega_n} \frac{(2|x|)^{[\rho(|y'|)+\beta]+1}}{|y'|^{n+[\rho(|y'|)+\beta]}} |f(y')| dy' \lesssim$$

$$\lesssim M(r) \int\limits_{\partial H} \frac{|f(y')|}{1+|y'|^{\rho(|y'|)+n+\frac{\beta}{2}-1}} dy' < \infty. \tag{5.6}$$

Thus $U_{[\rho(|y'|)+\beta]}(f)(x)$ is finite for any $x \in H$ from (5.4), (5.5) and (5.6). Since $P_{[\rho(|y'|)+\beta]}(x, y')$ is a harmonic function of $x \in H$ for any fixed $y' \in \partial H$. $U_{[\rho(|y'|)+\beta]}(f)(x)$ is also a harmonic function of $x \in H$.

To verify the boundary behavior of $U_{[\rho(|y'|)+\beta]}(f)(x)$. For any fixed boundary point $z' \in \partial H$, we can choose a number R_4 such that $R_4 > 2$. Let $D_{R_4} = H \cap B_{n-1}(R_4)$ and $\chi_{B_{n-1}(R_4)}$ be the characteristic function of $B_{n-1}(R_4)$.

Set

$$S_{[\rho(|y'|+\beta)]}(x, y') = \sum_{k=0}^{[\rho(|y'|+\beta)]-1} \frac{2x_n|x|^k}{\omega_n} \frac{1}{|y'|^{n+k}} C_k^{n/2}\left(\frac{x' \cdot y'}{|x||y'|}\right).$$

Write

$$U_{[\rho(|y'|)+\beta]}(f)(x) = X(x) - Y(x) + Z(x),$$

where

$$X(x) = \int\limits_{\partial H[0,2R_4]} P(x, y') f(y') dy',$$

$$Y(x) = \int\limits_{\partial H(1,2R_4]} S_{[\rho(|y'|+\beta)]}(x, y') f(y') dy', \quad \text{and} \quad Z(x) = \int\limits_{\partial H(2R_4,\infty)} P_{[\rho(|y'|+\beta)]}(x, y') f(y') dy'.$$

Note that $X(x) = \int_{\partial H} P(x, y') f(y') \chi_{B_{n-1}(R_4)}(y') dy'$, which tends to $f(z')$ as $x \to z'$. Further, $X(x)$ is harmonic on D_{R_4} and can be continuously extended to $\overline{D}_{R_4}$. Since $S_{[\rho(|y'|+\beta)]}(x, y')$ is a harmonic polynomial of x and tends to zero as $x \to z'$, $Y(x)$ is also a harmonic polynomial of x and tends to zero as $x \to z'$.

From (1.5), we have $Z(x)$ is a harmonic function on H. $P_{[\rho(|y'|+\beta)]}(x, y') = 0$ implies that $Z(z') = 0$, where $|z'| \le R_4$.

Moreover, (5.1) also implies that there exists a positive constant $M(R_4)$ dependent only on R_4 such that

$$\frac{2^{n+1}}{\omega_n} \frac{(2R_4)^{\rho(k+1)+\beta}}{k^{\frac{\beta}{2}}} \le M(R_4) \tag{5.7}$$

for any $k \ge [2R_4]$.

Hence it follows from (1.5) and (5.2) that

$$|Z(x) - Z(z')| = |Z(x)| \le$$

$$\leq \int\limits_{\partial H(2R_4,\infty)} |P_{[\rho(|y'|+\beta)]}(x,y')||f(y')|dy' \leq$$

$$\leq \int\limits_{\partial H[[2R_4],\infty)} \frac{2^{n+1}}{\omega_n} \frac{x_n(2R_4)^{[\rho(|y'|)+\beta]}}{|y'|^{[\rho(|y'|)+\beta]-\rho(|y'|)-\frac{\beta}{2}+1}} \frac{|f(y')|}{|y'|^{\rho(|y'|)+\frac{\beta}{2}+n-1}}dy' \leq$$

$$\leq \sum_{k=[2R_4]}^{\infty} \int\limits_{\partial H[k,k+1)} \frac{2^{n+1}}{\omega_n} \frac{x_n(2R_4)^{\rho(k+1)+\beta}}{k^{\frac{\beta}{2}}} \frac{2|f(y')|}{1+|y'|^{\rho(|y'|)+\frac{\beta}{2}+n-1}}dy' \lesssim$$

$$\lesssim x_n \sum_{k=[2R_4]}^{\infty} \int\limits_{\partial H[k,k+1)} M(R_4)\frac{|f(y')|}{1+|y'|^{\rho(|y'|)+\frac{\beta}{2}+n-1}}dy' \lesssim$$

$$\lesssim x_n M(R_4) \int\limits_{\partial H} \frac{|f(y')|}{1+|y'|^{\rho(|y'|)+\frac{\beta}{2}+n-1}}dy' \lesssim x_n M(R_4),$$

which tends to zero as $x \to z'$. Thus $Z(x)$ is harmonic on D_{R_4} and can be continuously extended to $\overline{D}_{R_4}$.

From the arbitrariness of R_4, we have that the function $U_{[\rho(|y'|)+\beta]}(f)(x)$ can be continuously extended to $\overline{H}$ such that $U_{[\rho(|y'|)+\beta]}(f)(z') = f(z')$ for any $z' \in \partial H$.

Theorem 3 is proved.

6. Proof of Corollary 1. To prove (II). Consider the function $u(x) - U_m(u)(x)$. Then it follows from Theorem 2 that this is harmonic in H and vanishes continuously on ∂H. Since

$$0 \leq (u(x) - U_m(u)(x))^+ \leq u^+(x) + U_m(u)^-(x) \tag{6.1}$$

for any $x \in H$ and

$$\liminf_{|x|\to\infty} |x|^{-m-1}u^+(x) = 0 \tag{6.2}$$

from (1.1), for every $x \in H$ we have

$$u(x) = U_m(u)(x) + Q_m(u)(x)$$

from (1.3), (6.1), (6.2) and Lemma 3, where $Q_m(u)$ is a polynomial in $\mathbf{R}^n$ of degree at most $m-1$ and odd with respect to the variable x_n. From these we evidently obtain Corollary (II).

If $u \in \mathcal{C}_n$, then $u \in \mathcal{C}_\alpha$ for $\alpha > n$. Corollary (II) gives that there exists a constant d_5 such that

$$u(x) = d_5 x_n + U_1(u)(x).$$

Put

$$d_2 = d_5 - \frac{1}{w_n} \int\limits_{\partial H[1,\infty)} \frac{f(y')}{|y'|^n}dy'.$$

It immediately follows that $u(x) = d_2 x_n + U(u)(x)$ for every $x \in H$, which is the conclusion of Corollary (I).

Corollary 1 is proved.

7. Proof of Theorem 4. Consider the function $u(x) - U_{[\rho(|y'|)+\beta]}(u)(x)$, which is harmonic in H, can be continuously extended to $\overline{H}$ and vanishes on ∂H.

The Schwarz Reflection Principle [1, p.68] applied to $u(x) - U_{[\rho(|y'|)+\beta]}(u)(x)$ shows that there exists a harmonic function $\Pi(x)$ on H such that $\Pi(x^*) = -\Pi(x) = U_{[\rho(|y'|)+\beta]}(u)(x) - u(x)$ for $x \in \overline{H}$, where $*$ denotes reflection on ∂H just as $x^* = (x', -x_n)$.

Thus $u(x) = U_{[\rho(|y'|)+\beta]}(u)(x) + \Pi(x)$ for all $x \in \overline{H}$, where $\Pi(x)$ is a harmonic function on H vanishing continuously on ∂H.

Theorem 4 is proved.

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