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REMARKS ON CERTAIN IDENTITIES WITH DERIVATIONS ON SEMIPRIME RINGS

ПРО ДЕЯКІ ТОТОЖНОСТІ ДЛЯ ПОХІДНИХ НА НАПІВПРОСТИХ КІЛЬЦЯХ

Let n be a fixed positive integer, R a $(2n)!$-torsion free semiprime ring, α an automorphism or an anti-automorphism of R, and $D_1, D_2 \colon R \to R$ derivations. We prove the following result: If $(D_1^2(x) + D_2(x)) \circ \alpha(x)^n = 0$ holds for all $x \in R$, then $D_1 = D_2 = 0$. The same is true if R is a 2-torsion free semiprime ring and $F(x) \circ \beta(x) = 0$ for all $x \in R$, where $F(x) = ({D_1}^2(x) + D_2(x)) \circ \alpha(x)$, $x \in R$, and β is any automorphism or anti-automorphism on R.

Припустимо, що n — фіксоване натуральне число, R — $(2n)!$ напівпросте кільце, вільне від кручення, α — автоморфізм або антиавтоморфізм на R, а $D_1, D_2 \colon R \to R$ — похідні. Доведено наступний результат: якщо $(D_1^2(x) + +D_2(x)) \circ \alpha(x)^n = 0$ виконується для всіх $x \in R$, то $D_1 = D_2 = 0$. Аналогічне твердження справджується, якщо R — 2-напівпросте кільце, вільне від кручення, і $F(x) \circ \beta(x) = 0$ для всіх $x \in R$, де $F(x) = ({D_1}^2(x) + D_2(x)) \circ \alpha(x)$, $x \in R$, і β — довільний автоморфізм або антиавтоморфізм на R.

1. Introduction. The aim of this paper is to generalize the results obtained in [9]. Let us first fix some notation. Throughout the paper, R will represent an associative ring with a center $Z(R)$. Let $n > 1$ be an integer. We say that a ring R is n-torsion free if $nx = 0$, $x \in R$, implies $x = 0$. As usual, the Lie product of elements $x, y \in R$ will be denoted by $[x, y]$ (i. e., $[x, y] = xy - yx$) and the Jordan product of elements $x, y \in R$ will be denoted by $x \circ y$ (i.e., $x \circ y = xy + yx$). Recall that a ring R is prime if $aRb = \{0\}$, $a, b \in R$, implies that either $a = 0$ or $b = 0$, and it is semiprime if $aRa = \{0\}$, $a \in R$, implies $a = 0$.

An additive mapping $f \colon R \to R$ is called centralizing on R if $[f(x), x] \in Z(R)$ holds for all $x \in R$. In a special case, when $[f(x), x] = 0$ for all $x \in R$, the mapping f is said to be commuting on R. Furthermore, an additive mapping $f \colon R \to R$ is skew-centralizing on R if $f(x) \circ x \in Z(R)$ for all $x \in R$, and it is called skew-commuting on R if $f(x) \circ x = 0$ is fulfilled for all $x \in R$. We say that an additive mapping $D \colon R \to R$ is a derivation on R if $D(xy) = D(x)y + xD(y)$ holds for all $x, y \in R$. A classical result of Posner [12] (Posner's second theorem) states that the existence of a nonzero centralizing derivation on a prime ring forces the ring to be commutative. On the other hand, Posner's second theorem in general cannot be proved for semiprime rings as shows the following example. Let R_1 and R_2 be prime rings with R_1 commutative and set $R = R_1 \oplus R_2$. Further, let $D_1 \colon R_1 \to R_1$ be a nonzero derivation. Then a mapping $D \colon R \to R$ given by $D((r_1, r_2)) = (D_1(r_1), 0)$ is a nonzero commuting derivation. It is also easy to show that every commuting derivation on a semiprime ring R maps R into $Z(R)$ (see, for example, the end of the proof of Theorem 2.1 in [13]).

In the present paper we continue the series of papers concerning arbitrary additive maps of prime and semiprime rings satisfying certain identities (see [1–5, 9] and the references therein). In particular, we generalize the main results obtained in [9].

2. The results. Before stating our main theorems, let us write some known facts which we will need in the sequel. So, let R be a 2-torsion free semiprime ring and $f \colon R \to R$ an additive mapping such that $[f(x), x^2] = 0$ holds for all $x \in R$. Then f must be commuting on R. This result was proved by Vukman and the second named author in [9]. Moreover, the same conclusion is true, if f satisfies $[f(x), x^n] = 0$, $x \in R$, where n is a fixed positive integer and R is a $n!$-torsion free semiprime ring (see [8], Theorem 2). Now, let α be an automorphism of R and suppose that an additive mapping $f\colon R \to R$ satisfies the relation

$$[f(x), \alpha(x)^n] = 0 \qquad (1)$$

for all $x \in R$. This means that

$$f(x)\alpha(x)^n - \alpha(x)^n f(x) = 0$$

for all $x \in R$. Since α is an automorphism of R, we have

$$\alpha^{-1}(f(x))x^n - x^n\alpha^{-1}(f(x)) = [\alpha^{-1}(f(x)), x^n] = 0.$$

Moreover, if α is an anti-automorphism of R such that (1) holds, then

$$x^n\alpha^{-1}(f(x)) - \alpha^{-1}(f(x))x^n = -[\alpha^{-1}(f(x)), x^n] = 0.$$

Thus, using Theorem 2 in [8], we have the next result.

Proposition 1. *Let n be a fixed positive integer, R a $n!$-torsion free semiprime ring, and α an automorphism or an anti-automorphism of R. Suppose that an additive mapping $f\colon R \to R$ satisfies the relation* (1) *for all $x \in R$. Then $[f(x), \alpha(x)] = 0$ holds for all $x \in R$.*

Next, let us take the Jordan product instead of the Lie product in (1) and observe the relation

$$f(x) \circ \alpha(x)^n \in Z(R), \qquad (2)$$

where f is an additive map on a $(2n)!$-torsion free semiprime ring R and α an automorphism or an anti-automorphism of R. Then we obtain

$$[f(x) \circ \alpha(x)^n, y] = 0$$

for all $y \in R$. Replacing y by $\alpha(x)^n$, we get

$$0 = [f(x) \circ \alpha(x)^n, \alpha(x)^n] = [f(x), \alpha(x)^{2n}].$$

Using Proposition 1, we have the next result which generalizes Theorem 3 in [8].

Proposition 2. *Let n be a fixed positive integer, R a $(2n)!$-torsion free semiprime ring, and α an automorphism or an anti-automorphism of R. Suppose that an additive mapping $f\colon R \to R$ satisfies the relation* (2) *for all $x \in R$. Then $[f(x), \alpha(x)] = 0$ holds for all $x \in R$.*

In particular, we will use the following corollary of Proposition 2.

***Corollary* 1.** *Let n be a fixed positive integer, R a $(2n)!$-torsion free semiprime ring, and α an automorphism or an anti-automorphism of R. Suppose that an additive mapping $f\colon R \to R$ satisfies*

$$f(x) \circ \alpha(x)^n = 0$$

for all $x \in R$. Then $[f(x), \alpha(x)] = 0$ holds for all $x \in R$.

Posner's first theorem [12] states that the composition of two nonzero derivations on a 2-torsion free prime ring cannot be a derivation. On the other hand, this conclusion is not true in the case of semiprime rings (see, for example, [6]). However, Herstein [10] (Lemma 1.1.9) showed that if R is a 2-torsion free semiprime ring and $D_1, D_2\colon R \to R$ derivations such that $D_1^2(x) = D_2(x)$ holds for all $x \in R$, then $D_1 = D_2 = 0$. The same is true if D_1 and D_2 satisfy the relation $(D_1^2(x) + D_2(x)) \circ x^2 = 0$ for all $x \in R$ (see [9]). These results motivated us to prove the following theorem which generalizes Theorem 8 in [9].

Theorem 1. *Let n be a fixed positive integer, R a $(2n)!$-torsion free semiprime ring, α an automorphism or an anti-automorphism of R, and $D_1, D_2 \colon R \to R$ derivations. Suppose that*

$$(D_1^2(x) + D_2(x)) \circ \alpha(x)^n = 0$$

holds for all $x \in R$. Then $D_1 = D_2 = 0$.

In the following, we shall use the fact that any semiprime ring R and its maximal right ring of quotients Q satisfy the same differential identities which is very useful since Q contains the identity element (see [11], Theorem 3). For the explanation of differential identities we refer the reader to [7].

***Proof of Theorem* 1.** By Theorem 3 in [11], we have

$$D(x) \circ \alpha(x)^n = 0 \tag{3}$$

for all $x \in Q$, where $D(x)$ stands for $D_1^2(x) + D_2(x)$. Since D is additive, by Corollary 1, we obtain $[D(x), \alpha(x)] = 0$ for all $x \in R$ and, again, using [11], this identity is true for all $x \in Q$.

Recall that $D(1) = 0$. Putting $x + 1$ instead of x in (3) we get

$$D(x) \sum_{k=0}^{n} \binom{n}{k} \alpha(x)^{n-k} + \sum_{k=0}^{n} \binom{n}{k} \alpha(x)^{n-k} D(x) = 0 \tag{4}$$

for all $x \in Q$. It follows from (3) and (4) that

$$D(x) \sum_{k=1}^{n} \binom{n}{k} \alpha(x)^{n-k} + \sum_{k=1}^{n} \binom{n}{k} \alpha(x)^{n-k} D(x) = 0 \tag{5}$$

holds for all $x \in Q$. Again, putting $x + 1$ instead of x and comparing the obtained equality with (5), we have

$$D(x) \sum_{k=2}^{n} t_k \alpha(x)^{n-k} + \sum_{k=2}^{n} t_k \alpha(x)^{n-k} D(x) = 0,$$

where $t_2, \ldots, t_n$ are the appropriate positive integers. Continuing with the same procedure for $(n-2)$-times, we get

$$n!(D(x)\alpha(x) + \alpha(x)D(x)) + (n-1)n!D(x) = 0$$

for every $x \in Q$. Since $[D(x), \alpha(x)] = 0$, we obtain

$$2D(x)\alpha(x) + (n-1)D(x) = 0$$

for all $x \in Q$. Again, putting $x + 1$ in the last identity, we get $2D(x) = 0$, $x \in Q$, and, therefore, $D = 0$. Recall that in the case $n = 1$ we do this procedure just for one time and if $n = 2$ we do this procedure for two times. In both cases we get the same conclusion, i.e., $D = 0$. At the end, using Lemma 1.1.9 in [10], we get $D_1 = 0$ and $D_2 = 0$, as asserted.

Theorem 1 is proved.

If we take $n = 2$ and $\alpha = id$, where id denotes the identity map on R, we have the next direct consequence of Theorem 1.

***Corollary* 2** ([9], Theorem 8)**.** *Let R be a 2-torsion free semiprime ring and let $D_1, D_2 \colon R \to R$ be derivations. Suppose that*

$$(D_1^2(x) + D_2(x)) \circ x^2 = 0$$

holds for all $x \in R$. Then $D_1 = D_2 = 0$.

***Remark* 1.** Let us point out that in Corollary 2 we do not have to restrict ourselves to 4!-torsion free semiprime rings, since the result holds true for 2-torsion free semiprime rings, as well. The main idea of the proof remains the same.

We proceed with the following result which generalizes Theorem 9 in [9].

Theorem 2. *Let R be a 2-torsion free semiprime ring, α an automorphism or an anti-automorphism of R, and $D_1, D_2 \colon R \to R$ derivations. Suppose that $F \colon R \to R$ is a mapping defined by*

$$F(x) = ({D_1}^2(x) + D_2(x)) \circ \alpha(x), \quad x \in R.$$

If $F(x) \circ \beta(x) = 0$ holds for all $x \in R$ and some automorphism or anti-automorphism β of R, then $D_1 = D_2 = 0$.

Proof. By the assumption, we have

$$\big(D(x)\alpha(x) + \alpha(x)D(x)\big) \circ \beta(x) = 0$$

for all $x \in R$, where $D(x) = {D_1}^2(x) + D_2(x)$. This means that

$$\big(D(x)\alpha(x) + \alpha(x)D(x)\big)\beta(x) + \beta(x)\big(D(x)\alpha(x) + \alpha(x)D(x)\big) = 0$$

for all $x \in R$. According to Theorem 3 in [11], the above identity holds for all $x \in Q$. Replacing x by $x + 1$, we obtain

$$0 = \big(D(x)\alpha(x) + \alpha(x)D(x)\big)\beta(x) + \beta(x)\big(D(x)\alpha(x) + \alpha(x)D(x)\big) +$$

$$+2\big(D(x)\alpha(x) + \alpha(x)D(x)\big) + 2\big(D(x)\beta(x) + \beta(x)D(x)\big) + 4D(x)$$

for all $x \in Q$. Combining the last two relations, it follows that

$$D(x)\alpha(x) + \alpha(x)D(x) + D(x)\beta(x) + \beta(x)D(x) + 2D(x) = 0 \tag{6}$$

for all $x \in Q$. Again, putting $x + 1$ instead of x in the above identity and comparing so obtained equality with the relation (6), we get $4D(x) = 0$ for all $x \in Q$. This yields that $D(x) = 0$ for all $x \in R$ and, by Lemma 1.1.9 in [10], $D_1 = D_2 = 0$.

Theorem 2 is proved.

Taking $\alpha = \beta = id$, we have the next direct consequence of Theorem 2.

***Corollary* 3** ([9], Theorem 9)**.** *Let R be a 2-torsion free semiprime ring and let $D_1, D_2 \colon R \to R$ be derivations. Suppose that $F \colon R \to R$ is a mapping defined by*

$$F(x) = ({D_1}^2(x) + D_2(x)) \circ x, \quad x \in R.$$

If F is skew-commuting on R, then $D_1 = D_2 = 0$.

***Remark* 2.** At the end, let us point out that (with the same main idea) we can prove the conclusion of Theorem 2 even if we replace the identity $F(x) \circ \beta(x) = 0$ with the identity $F(x) \circ \beta(x)^n = 0$, where n is any fixed positive integer. We only have to restrict ourselves to suitable torsion free semiprime rings. In the following, we will write just a sketch of the proof since the poof is rather technical but the main idea remains the same.

Firstly, we know that

$$(D(x)\alpha(x) + \alpha(x)D(x)) \circ \beta(x)^n = 0$$

for all $x \in R$. This means that

$$\big(D(x)\alpha(x) + \alpha(x)D(x)\big)\beta(x)^n + \beta(x)^n\big(D(x)\alpha(x) + \alpha(x)D(x)\big) = 0$$

for all $x \in Q$, as well. Replacing x by $x+1$, we obtain

$$0 = \big(D(x)\alpha(x) + \alpha(x)D(x) + 2D(x)\big)\sum_{k=0}^{n}\binom{n}{k}\beta(x)^{n-k} +$$

$$+\sum_{k=0}^{n}\binom{n}{k}\beta(x)^{n-k}\big(D(x)\alpha(x) + \alpha(x)D(x) + 2D(x)\big)$$

for all $x \in Q$. Combining the last two relations, it follows that

$$0 = \big(D(x)\alpha(x) + \alpha(x)D(x) + 2D(x)\big)\sum_{k=1}^{n}\binom{n}{k}\beta(x)^{n-k} +$$

$$+2\big(D(x)\beta(x)^n + \beta(x)^n D(x)\big) + \sum_{k=1}^{n}\binom{n}{k}\beta(x)^{n-k}\big(D(x)\alpha(x) + \alpha(x)D(x) + 2D(x)\big).$$

Again, putting $x+1$ instead of x in the above identity and continuing with the same procedure for n-times, we get $D(x) = 0$ for all $x \in Q$. This yields that $D(x) = 0$ for all $x \in R$ and, by Lemma 1.1.9 in [10], $D_1 = D_2 = 0$.

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