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SOME NEW RESUTS CONCERING STRONG CONVERGENCE OF FEJÉR MEANS WITH RESPECT TO VILENKIN SYSTEMS*

ДЕЯКІ НОВІ РЕЗУЛЬТАТИ ЩОДО СТРОГОЇ ЗБІЖНОСТІ СЕРЕДНІХ ФЕЄРА ВІДНОСНО СИСТЕМ ВІЛЕНКІНА

We prove some new strong convergence theorems for partial sums and Fejér means with respect to the Vilenkin system.

Доведено деякі теореми про строгу збіжність часткових сум та середніх Феєра відносно системи Віленкіна.

1. Introduction. Concerning definitions and notations used in this introductions we refer to Section 2.

It is well-known (for details see, e.g., [1, 8, 10]) that Vilenkin system forms not basis in the space $L_1(G_m)$. Moreover, there is a function in the martingale Hardy space $H_1(G_m)$ such that the partial sums of f are not bounded in $L_1(G_m)$-norm. However, for all $p > 0$ and $f \in H_p$, there exists an absolute constant c_p such that

$$\|S_{M_k} f\|_p \leq c_p \|f\|_{H_p}. \tag{1}$$

In [5] (see also [11]) the following strong convergence result was obtained for all $f \in H_1(G_m)$:

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{\|S_k f - f\|_1}{k} = 0.$$

It follow that

$$\frac{1}{\log n} \sum_{k=1}^{n} \frac{\|S_k f\|_1}{k} \leq \|f\|_{H_1}, \quad n = 2, 3, \ldots.$$

In [19] was proved that for any $f \in H_1$ there exists an absolute constant c such that

$$\sup_{n\in\mathbb{N}} \frac{1}{n \log n} \sum_{k=1}^{n} \|S_k f\|_1 \leq \|f\|_{H_1}, \quad n = 1, 2, 3, \ldots.$$

Moreover, for every nondecreasing function $\varphi\colon \mathbb{N}_+ \to [1, \infty)$, satisfying the condition

$$\overline{\lim_{n\to\infty}} \frac{\log n}{\varphi_n} = +\infty,$$

there exists a function $f \in \mathbb{N}_1$ such that

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$$\sup_{n\in\mathbb{N}} \frac{1}{n\varphi_n} \sum_{k=1}^{n} \|S_k f\|_1 = \infty.$$

For the Vilenkin system Simon [12] proved that there is an absolute constant c_p, depending only on p, such that

$$\sum_{k=1}^{\infty} \frac{\|S_k f\|_p^p}{k^{2-p}} \leq c_p \|f\|_{H_p}^p$$

for all $f \in H_p(G_m)$, where $0 < p < 1$. In [16] was proved that for any nondecreasing function $\Phi : \mathbb{N}_+ \to [1, \infty)$, satisfying the condition $\lim_{n\to\infty} \Phi(n) = +\infty$, there exists a martingale $f \in H_p(G_m)$ such that

$$\sum_{k=1}^{\infty} \frac{\|S_k f\|_{\text{weak-}L_p}^p \Phi(k)}{k^{2-p}} = \infty \quad \text{for} \quad 0 < p < 1.$$

Strong convergence theorems of two-dimensional partial sums was investigate by Weisz [23], Goginava [6], Gogoladze [7], Tephnadze [18] (see also [9]).

Weisz [24] considered the norm convergence of Fejér means of Walsh – Fourier series and proved the following theorem.

Theorem W1 (Weisz). *Let $p > 1/2$ and $f \in H_p$. Then*

$$\|\sigma_k f\|_p \leq c_p \|f\|_{H_p}.$$

Moreover, Weisz [24] also proved that, for all $p > 0$ and $f \in H_p$, there exists an absolute constant c_p such that

$$\|\sigma_{M_k} f\|_p \leq c_p \|f\|_{H_p}. \tag{2}$$

Theorem W1 implies that

$$\frac{1}{n^{2p-1}} \sum_{k=1}^{n} \frac{\|\sigma_k f\|_p^p}{k^{2-2p}} \leq c_p \|f\|_{H_p}^p, \quad 1/2 < p < \infty.$$

If Theorem W1 should hold for $0 < p \leq \dfrac{1}{2}$, then we have

$$\sum_{k=1}^{\infty} \frac{\|\sigma_k f\|_p^p}{k^{2-2p}} \leq c_p \|f\|_{H_p}^p, \quad 0 < p < 1/2, \tag{3}$$

$$\frac{1}{\log n} \sum_{k=1}^{n} \frac{\|\sigma_k f\|_{1/2}^{1/2}}{k} \leq c \|f\|_{H_{1/2}}^{1/2}, \tag{4}$$

and

$$\frac{1}{n} \sum_{k=1}^{n} \|\sigma_k f\|_{1/2}^{1/2} \leq c \|f\|_{H_{1/2}}^{1/2}. \tag{5}$$

However, in [14] (see also [2, 3]) it was proved that the assumption $p > 1/2$ in Theorem W1 is essential. In particular, there exists a martingale $f \in H_{1/2}$ such that

$$\sup_{n \in \mathbb{N}} \|\sigma_n f\|_{1/2} = +\infty.$$

In [4] (see also [17]) it was proved that (3) and (4) hold though Theorem W1 is not true for $0 < p \leq 1/2$.

Moreover, in [4] it was proved that if $0 < p < 1/2$ and $\Phi : \mathbb{N}_+ \to [1, \infty)$ be any nondecreasing function satisfying condition

$$\overline{\lim_{k \to \infty}} \frac{k^{2-2p}}{\Phi_k} = \infty,$$

then there exists a martingale $f \in H_p$ such that

$$\sum_{m=1}^{\infty} \frac{\|\sigma_m f\|^p_{\text{weak-}L_p}}{\Phi_m} = \infty.$$

On the other hand, for the Walsh system (5) does not hold (see [17]). In particular, it was proved that there exists a martingale $f \in H_{1/2}$ such that

$$\sup_{n \in \mathbb{N}} \frac{1}{n} \sum_{m=1}^{n} \|\sigma_m f\|^{1/2}_{1/2} = \infty. \tag{6}$$

In this paper, we prove more general result for bounded Vilenkin system. In special case we also obtain (6).

This paper is organized as follows. In order not to disturb our discussions later on some definitions and notations are presented in Section 2. For the proofs of the main results we need some auxiliary lemmas, some of them are new and of independent interest. These results are presented in Section 3. The main result with proof is given in Section 4.

2. Definitions and notations. Let $\mathbb{N}_+$ denote the set of the positive integers, $\mathbb{N} := \mathbb{N}_+ \cup \{0\}$. Let $m := (m_0, m_1, \ldots)$ denote a sequence of the positive integers not less than 2.

Denote by

$$Z_{m_k} := \{0, 1, \ldots, m_k - 1\}$$

the additive group of integers modulo m_k.

Define the group G_m as the complete direct product of the group Z_{m_j} with the product of the discrete topologies of Z_{m_j}.

The direct product μ of the measures

$$\mu_k(\{j\}) := 1/m_k, \quad j \in Z_{m_k},$$

is the Haar measure on G_m with $\mu(G_m) = 1$.

If $\sup_{n \in \mathbb{N}} m_n < \infty$, then we call G_m a bounded Vilenkin group. If the generating sequence m is not bounded, then G_m is said to be an unbounded Vilenkin group. In this paper, we discuss bounded Vilenkin groups only.

The elements of G_m are represented by sequences

$$x := (x_0, x_1, \ldots, x_k, \ldots), \quad x_k \in Z_{m_k}.$$

It is easy to give a base for the neighbourhood of G_m namely

$$I_0(x) := G_m,$$

and

$$I_n(x) := \{y \in G_m \mid y_0 = x_0, \ldots, y_{n-1} = x_{n-1}\}, \quad x \in G_m, \quad n \in \mathbb{N}.$$

Denote $I_n := I_n(0)$ for $n \in \mathbb{N}$ and $\overline{I_n} := G_m \backslash I_n$.

Let

$$e_n := (0, \ldots, 0, x_n = 1, 0, \ldots) \in G_m, \quad n \in \mathbb{N}.$$

If we define the so-called generalized number system based on m in the following way:

$$M_0 := 1, \quad M_{k+1} := m_k M_k, \quad k \in \mathbb{N},$$

then every $n \in \mathbb{N}$ can be uniquely expressed as $n = \sum_{k=0}^{\infty} n_j M_j$, where $n_j \in Z_{m_j}$, $j \in \mathbb{N}$, and only a finite number of n_j's differ from zero. Let $\|n\| := \max\{j \in \mathbb{N}, n_j \neq 0\}$.

For the natural number $n = \sum_{j=1}^{\infty} n_j M_j$, we define

$$\delta_j = \operatorname{sign} n_j = \operatorname{sign}(\ominus n_j), \quad \delta_j^* = \| \ominus n_j - 1 \| \delta_j,$$

where $\ominus$ is the inverse operation for $a_k \oplus b_k = (a_k + b_k) \operatorname{mod} m_k$.

We define functions v and v^* by

$$v(n) = \sum_{j=0}^{\infty} |\delta_{j+1} - \delta_j| + \delta_0, \quad v^*(n) = \sum_{j=0}^{\infty} \delta_j^*.$$

The nth Lebesgue constant is defined in the following way:

$$L_n = \|D_n\|_1.$$

The norm (or quasinorm) of the space $L_p(G_m)$ is defined by

$$\|f\|_p := \left(\int_{G_m} \|f(x)\|^p d\mu(x) \right)^{1/p}, \quad 0 < p < \infty.$$

The space weak-$L_p(G_m)$ consists of all measurable functions f for which

$$\|f\|_{\text{weak-}L_p(G_m)} := \sup_{\lambda > 0} \lambda^p \mu\{f > \lambda\} < +\infty.$$

Next, we introduce on G_m an orthonormal system which is called the Vilenkin system.

At first, define the complex valued function $r_k(x) \colon G_m \to \mathbb{C}$, the generalized Rademacher functions as

$$r_k(x) := \exp(2\pi \imath x_k / m_k), \quad \imath^2 = -1, \quad x \in G_m, \quad k \in \mathbb{N}.$$

Now define the Vilenkin system $\psi := (\psi_n : n \in \mathbb{N})$ on G_m as

$$\psi_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x), \quad n \in \mathbb{N}.$$

Specially, we call this system the Walsh – Paley one if $m \equiv 2$.

The Vilenkin system is orthonormal and complete in $L_2(G_m)$ (for details see, e.g., [1, 10, 20]).

If $f \in L_1(G_m)$, then we can define Fourier coefficients, partial sums of the Fourier series, Fejér means, Dirichlet and Fejér kernels with respect to the Vilenkin system in the usual manner:

$$\widehat{f}(k) := \int_{G_m} f\overline{\psi}_k d\mu, \quad k \in \mathbb{N},$$

$$S_n f := \sum_{k=0}^{n-1} \widehat{f}(k)\psi_k, \quad n \in \mathbb{N}_+, \quad S_0 f := 0,$$

$$\sigma_n f := \frac{1}{n}\sum_{k=0}^{n-1} S_k f, \quad n \in \mathbb{N}_+,$$

$$D_n := \sum_{k=0}^{n-1} \psi_k, \quad n \in \mathbb{N}_+,$$

$$K_n := \frac{1}{n}\sum_{k=0}^{n-1} D_k, \quad n \in \mathbb{N}_+.$$

Recall that (for details see, e.g., [1, 8])

$$D_{M_n}(x) = \begin{cases} M_n, & x \in I_n, \\ 0, & x \notin I_n. \end{cases} \tag{7}$$

and

$$D_{s_n M_n} = D_{M_n} \sum_{k=0}^{s_n-1} \psi_{kM_n} = D_{M_n} \sum_{k=0}^{s_n-1} r_n^k, \quad 1 \le s_n \le m_n - 1. \tag{8}$$

The σ-algebra generated by the intervals $\{I_n(x) : x \in G_m\}$ will be denoted by F_n, $n \in \mathbb{N}$. Denote by $f = (f_n, n \in \mathbb{N})$ a martingale with respect to F_n, $n \in \mathbb{N}$ (for details see, e.g., [21]). The maximal function of a martingale f is defined by

$$f^* = \sup_{n\in\mathbb{N}} |f_n|.$$

In the case $f \in L_1(G_m)$, the maximal functions are also be given by

$$f^*(x) = \sup_{n\in\mathbb{N}} \frac{1}{|I_n(x)|} \left| \int_{I_n(x)} f(u)\mu(u) \right|.$$

For $0 < p < \infty$ the Hardy martingale spaces $H_p(G_m)$ consist of all martingales for which

$$\|f\|_{H_p} := \|f^*\|_p < \infty.$$

If $f \in L_1(G_m)$, then it is easy to show that the sequence $(S_{M_n}f : n \in \mathbb{N})$ is a martingale. If $f = (f_n, n \in \mathbb{N})$ is martingale, then the Vilenkin – Fourier coefficients must be defined in a slightly different manner:

$$\widehat{f}(i) := \lim_{k\to\infty} \int_{G_m} f_k(x)\overline{\psi}_i(x)\, d\mu(x).$$

The Vilenkin – Fourier coefficients of $f \in L_1(G_m)$ are the same as those of the martingale $(S_{M_n}f : n \in \mathbb{N})$ obtained from f.

A bounded measurable function a is p-atom, if there exist an interval I such that

$$\int_I a d\mu = 0, \quad \|a\|_\infty \le \mu(I)^{-1/p}, \quad \operatorname{supp}(a) \subset I.$$

3. Auxiliary lemmas.

Lemma 1 [21, 22]. *A martingale $f = (f_n, n \in \mathbb{N})$ is in H_p, $0 < p \le 1$, if and only if there exist a sequence $(a_k,\ k \in \mathbb{N})$ of p-atoms and a sequence $(\mu_k,\ k \in \mathbb{N})$ of real numbers such that, for every $n \in \mathbb{N}$,*

$$\sum_{k=0}^{\infty} \mu_k S_{M_n} a_k = f_n \quad a.e., \tag{9}$$

where

$$\sum_{k=0}^{\infty} |\mu_k|^p < \infty.$$

Moreover,

$$\|f\|_{H_p} \backsim \inf \left(\sum_{k=0}^{\infty} |\mu_k|^p\right)^{1/p},$$

where the infimum is taken over all decomposition of f of the form (9).

By using atomic decomposition of $f \in H_p$ martingales, we can derive a counterexample, which play a central role to prove sharpness of main results and it will be used several times in the paper.

Lemma 2 [13]. *Let $n \in \mathbb{N}$ and $1 \le s_n \le m_n - 1$. Then*

$$s_n M_n K_{s_n M_n} = \sum_{l=0}^{s_n-1} \left(\sum_{t=0}^{l-1} r_n^t\right) M_n D_{M_n} + \left(\sum_{l=0}^{s_n-1} r_n^l\right) M_n K_{M_n}$$

and

$$|s_n M_n K_{s_n M_n}(x)| \ge \frac{M_n^2}{2\pi} \quad \text{for} \quad x \in I_{n+1}(e_{n-1} + e_n).$$

Moreover, if $x \in I_t / I_{t+1}$, $x - x_t e_t \notin I_n$ and $n > t$, then

$$K_{s_n M_n}(x) = 0. \tag{10}$$

Lemma 3 [4]. *Let* $n=\sum_{i=1}^{r}s_{n_i}M_{n_i}$, *where* $n_{n_1}>n_{n_2}>\ldots>n_{n_r}\geq 0$ *and* $1\leq s_{n_i}<m_{n_i}$ *for all* $1\leq i\leq r$ *as well as* $n^{(k)}=n-\sum_{i=1}^{k}s_{n_i}M_{n_i}$, *where* $0<k\leq r$. *Then*

$$nK_n=\sum_{k=1}^{r}\left(\prod_{j=1}^{k-1}r_{n_j}^{s_{n_j}}\right)s_{n_k}M_{n_k}K_{s_{n_k}M_{n_k}}+\sum_{k=1}^{r-1}\left(\prod_{j=1}^{k-1}r_{n_j}^{s_{n_j}}\right)n^{(k)}D_{s_{n_k}M_{n_k}}.$$

Lemma 4. *Let*

$$n=\sum_{i=1}^{s}\sum_{k=l_i}^{m_i}n_kM_k,$$

where

$$0\leq l_1\leq m_1\leq l_2-2<l_2\leq m_2\leq\ldots\leq l_s-2<l_s\leq m_s.$$

Then

$$n|K_n(x)|\geq cM_{l_i}^2\quad \textit{for}\quad x\in I_{l_i+1}\left(e_{l_i-1}+e_{l_i}\right),$$

where $\lambda=\sup_{n\in\mathbb{N}}m_n$ *and* c *is an absolute constant.*

Proof. Let $x\in I_{l_i+1}(e_{l_i-1}+e_{l_i})$. By combining (10), (7) and (8), we obtain

$$D_{l_i}=0$$

and

$$D_{s_{n_k}M_{s_{n_k}}}=K_{s_{n_k}M_{s_{n_k}}}=0,\quad s_{n_k}>l_i.$$

Since $s_{n_1}>s_{n_2}>\ldots>s_{n_r}\geq 0$, we find

$$n^{(k)}=n-\sum_{i=1}^{k}s_{n_i}M_{n_i}=\sum_{i=k+1}^{s}s_{n_i}M_{n_i}\leq\sum_{i=0}^{n_{k+1}}(m_i-1)M_i=m_{n_{k+1}}M_{n_{k+1}}-1\leq M_{n_k}.$$

According to Lemma 3, we have

$$n|K_n|\geq|s_{l_i}M_{l_i}K_{s_{l_i}M_{l_i}}|-\sum_{r=1}^{i-1}\sum_{k=l_r}^{m_r}|s_kM_kK_{s_kM_k}|-\sum_{r=1}^{i-1}\sum_{k=l_r}^{m_r}|M_kD_{s_kM_k}|=I_1-I_2-I_3.$$

Let $x\in I_{l_i+1}(e_{l_i-1}+e_{l_i})$ and $1\leq s_{l_i}\leq m_{l_i}-1$. By using Lemma 2, we get

$$I_1=|s_{l_i}M_{l_i}K_{s_{l_i}M_{l_i}}|\geq\frac{M_{l_i}^2}{2\pi}\geq\frac{2M_{l_i}^2}{9}.$$

It is easy to see that

$$\sum_{s=0}^{k}n_s^2M_s^2\leq\sum_{s=0}^{k}(m_s-1)^2M_s^2\leq\sum_{s=0}^{k}m_s^2M_s^2-2\sum_{s=0}^{k}m_sM_s^2+\sum_{s=0}^{k}M_s^2=$$

$$=\sum_{s=0}^{k}M_{s+1}^2-2\sum_{s=0}^{k}M_{s+1}M_s+\sum_{s=0}^{k}M_s^2=$$

$$= M_{k+1}^2 + 2\sum_{s=0}^{k} M_s^2 - 2\sum_{s=0}^{k} M_{s+1}M_s - M_0^2 \leq M_{k+1}^2 - 1$$

and

$$\sum_{s=0}^{k} n_s M_s \leq \sum_{s=0}^{k}(m_s - 1)M_s = m_k M_k - m_0 M_0 \leq M_{k+1} - 2.$$

Since $m_{i-1} \leq l_i - 2$ if we use the estimates above, then we obtain

$$I_2 \leq \sum_{s=0}^{l_i-2} |n_s M_s K_{n_s M_s}(x)| \leq \sum_{s=0}^{l_i-2} n_s M_s \frac{n_s M_s + 1}{2} \leq$$

$$\leq \frac{(m_{l_i-2} - 1)M_{l_i-2}}{2} \sum_{s=0}^{l_i-2}(n_s M_s + 1) \leq$$

$$\leq \frac{(m_{l_i-2} - 1)M_{l_i-2}}{2} M_{l_i-1} + \frac{(m_{l_i-2} - 1)M_{l_i-2}}{2} l_i \leq$$

$$\leq \frac{M_{l_i-1}^2}{2} - \frac{M_{l_i-2}M_{l_i-1}}{2} + M_{l_i-1}l_i. \tag{11}$$

For I_3 we have

$$I_3 \leq \sum_{k=0}^{l_i-2} |M_k D_{n_k M_k}(x)| \leq \sum_{k=0}^{l_i-2} n_k M_k^2 \leq M_{l_i-2} \sum_{k=0}^{l_i-2} n_k M_k \leq M_{l_i-1}M_{l_i-2} - 2M_{l_i-2}. \tag{12}$$

By combining (11), (12), we get

$$n|K_n(x)| \geq I_1 - I_2 - I_3 \geq \frac{M_{l_i}^2}{2\pi} + \frac{3}{2} + 2M_{l_i-2} - \frac{M_{l_i-1}M_{l_i-2}}{2} - \frac{M_{l_i-1}^2}{2} - M_{l_i-1}l_i \geq$$

$$\geq \frac{M_{l_i}^2}{2\pi} - \frac{M_{l_i}^2}{16} - \frac{M_{l_i}^2}{8} + \frac{7}{2} - M_{l_i-1}l_i \geq$$

$$\geq \frac{2M_{l_i}^2}{9} - \frac{3M_{l_i}^2}{16} + \frac{7}{2} - M_{l_i-1}l_i \geq \frac{M_{l_i}^2}{144} - M_{l_i-1}l_i.$$

Suppose that $l_i \geq 4$. Then

$$n|K_n(x)| \geq \frac{M_{l_i}^2}{36} - \frac{M_{l_i}}{4} \geq \frac{M_{l_i}^2}{36} - \frac{M_{l_i}^2}{64} \geq \frac{5M_{l_i}^2}{36 \cdot 16} \geq \frac{M_{l_i}^2}{144}.$$

Lemma 4 is proved.

4. Main result. The main result of this paper is the following theorem.

Theorem 1. 1. *Let $f \in H_{1/2}$. Then there exists an absolute constant c such that*

$$\sup_{n\in\mathbb{N}} \frac{1}{n\log n} \sum_{k=1}^{n} \|\sigma_k f\|_{H_{1/2}}^{1/2} \leq c\|f\|_{H_{1/2}}^{1/2}, \quad n = 1, 2, 3, \ldots.$$

2. *Let $\varphi\colon \mathbb{N}_+ \to [1, \infty)$ be a nondecreasing function satisfying the condition*

$$\overline{\lim_{n\to\infty}}\frac{\log n}{\varphi_n}=+\infty. \tag{13}$$

Then there exists a function $f\in H_{1/2}$ *such that*

$$\sup_{n\in\mathbb{N}_+}\frac{1}{n\varphi_n}\sum_{k=1}^{n}\|\sigma_k f\|_{H_{1/2}}^{1/2}=\infty.$$

***Corollary* 1.** *There exists a martingale* $f\in H_{1/2}$ *such that*

$$\sup_{n\in\mathbb{N}_+}\frac{1}{n}\sum_{k=1}^{n}\|\sigma_k f\|_{1/2}^{1/2}=\infty.$$

***Proof of Theorem* 1.** 1. In [15] was proved that there exists an absolute constant $c,$ such that

$$\|\sigma_k f\|_{H_{1/2}}^{1/2}\le c\log k\|f\|_{H_{1/2}}^{1/2},\quad k=1,2,\ldots.$$

Hence,

$$\frac{1}{n\log n}\sum_{k=1}^{n}\|\sigma_k f\|_{H_{1/2}}^{1/2}\le\frac{c\|f\|_{H_{1/2}}^{1/2}}{n\log n}\sum_{k=1}^{n}\log k\le c\|f\|_{H_{1/2}}^{1/2}.$$

2. Under the condition (13) there exists an increasing sequence of the positive integers $\{\alpha_k: k\in\mathbb{N}\}$ such that

$$\overline{\lim_{k\to\infty}}\frac{\log M_{\alpha_k}}{\varphi_{2M_{\alpha_k}}}=+\infty$$

and

$$\sum_{k=0}^{\infty}\frac{\varphi_{2M_{\alpha_k}}^{1/2}}{\log^{1/2}M_{\alpha_k}}<c<\infty. \tag{14}$$

Let $f=(f_n,n\in\mathbb{N})$ be martingale, defined by

$$f_n:=\sum_{\{k;2\alpha_k<n\}}\lambda_k a_k,$$

where

$$a_k=M_{\alpha_k}r_{\alpha_k}D_{M_{\alpha_k}}=M_{\alpha_k}(D_{2M_{\alpha_k}}-D_{M_{\alpha_k}})$$

and

$$\lambda_k=\frac{\varphi_{2M_{\alpha_k}}}{\log M_{\alpha_k}}.$$

Since

$$S_{2^A}a_k=\begin{cases}a_k, & \alpha_k<A,\\ 0, & \alpha_k\ge A,\end{cases}$$

$$\operatorname{supp}(a_k)=I_{\alpha_k},\qquad\int\limits_{I_{\alpha_k}}a_k d\mu=0,\qquad\|a_k\|_\infty\le M_{\alpha_k}^2=\mu(\operatorname{supp}a_k)^{-2},$$

if we apply Lemma 1 and (14), we conclude that $f \in H_{1/2}$.

Moreover,

$$\widehat{f}(j) = \begin{cases} M_{\alpha_k}\lambda_k, & j \in \{M_{\alpha_k}, \ldots, 2M_{\alpha_k} - 1\}, \quad k \in \mathbb{N}, \\ 0, & j \notin \bigcup\limits_{k=1}^{\infty} \{M_{\alpha_k}, \ldots, 2M_{\alpha_k} - 1\}. \end{cases} \tag{15}$$

We have

$$\sigma_n f = \frac{1}{n} \sum_{j=0}^{M_{\alpha_k}-1} S_j f + \frac{1}{n} \sum_{j=M_{\alpha_k}}^{n-1} S_j f = I + II. \tag{16}$$

Let $M_{\alpha_k} \leq j < 2M_{\alpha_k}$. Since

$$D_{j+M_{\alpha_k}} = D_{M_{\alpha_k}} + \psi_{M_{\alpha_k}} D_j, \quad \text{when} \quad j \leq M_{\alpha_k},$$

if we apply (15), we obtain

$$S_j f = S_{M_{\alpha_k}} f + \sum_{v=M_{\alpha_k}}^{j-1} \widehat{f}(v)\psi_v = S_{M_{\alpha_k}} f + M_{\alpha_k}\lambda_k \sum_{v=M_{\alpha_k}}^{j-1} \psi_v =$$

$$= S_{M_{\alpha_k}} f + M_{\alpha_k}\lambda_k (D_j - D_{M_{\alpha_k}}) = S_{M_{\alpha_k}} f + \lambda_k \psi_{M_{\alpha_k}} D_{j-M_{\alpha_k}}. \tag{17}$$

According to (17) concerning II, we conclude that

$$II = \frac{n - M_{\alpha_k}}{n} S_{M_{\alpha_k}} f + \frac{\lambda_k M_{\alpha_k}}{n} \sum_{j=M_{2\alpha_k}}^{n-1} \psi_{M_{\alpha_k}} D_{j-M_{\alpha_k}} = II_1 + II_2.$$

We can estimate II_2 as follows:

$$|II_2| = \frac{\lambda_k M_{\alpha_k}}{n} \left| \psi_{M_{\alpha_k}} \sum_{j=0}^{n-M_{\alpha_k}-1} D_j \right| = \frac{\lambda_k M_{\alpha_k}}{n} (n - M_{\alpha_k}) |K_{n-M_{\alpha_k}}| \geq$$

$$\geq \lambda_k (n - M_{\alpha_k}) |K_{n-M_{\alpha_k}}|.$$

Let $n = \sum_{i=1}^{s} \sum_{k=l_i}^{m_i} M_k$, where

$$0 \leq l_1 \leq m_1 \leq l_2 - 2 < l_2 \leq m_2 \leq \ldots \leq l_s - 2 < l_s \leq m_s.$$

By applying Lemma 4, we get

$$|II_2| \geq c\lambda_k \left| (n - M_{\alpha_k}) K_{n-M_{\alpha_k}}(x) \right| \geq c\lambda_k M_{l_i}^2 \quad \text{for} \quad x \in I_{l_i+1}(e_{l_i-1} + e_{l_i}).$$

Hence

$$\int_{G_m} |II_2|^{1/2} d\mu \geq \sum_{i=1}^{s-1} \int_{I_{l_i+1}(e_{l_i-1}+e_{l_i})} |II_2|^{1/2} d\mu \geq c \sum_{i=1}^{s-1} \int_{I_{l_i+1}(e_{l_i-1}+e_{l_i})} \lambda_k^{1/2} M_{l_i} d\mu \geq$$

$$\geq c\lambda_k^{1/2}(s-1) \geq c\lambda_k^{1/2} v(n - M_{\alpha_k}). \tag{18}$$

In view of (1), (2) and (16), we find

$$\|I\|^{1/2} = \left\| \frac{M_{\alpha_k}}{n} \sigma_{M_{\alpha_k}} f \right\|_{1/2}^{1/2} \leq \left\| \sigma_{M_{\alpha_k}} f \right\|_{1/2}^{1/2} \leq c \, \|f\|_{H_{1/2}}^{1/2} \tag{19}$$

and

$$\|II_1\|^{1/2} = \left\| \frac{n - M_{\alpha_k}}{n} S_{M_{\alpha_k}} f \right\|_{1/2}^{1/2} \leq \left\| S_{M_{\alpha_k}} f \right\|_{1/2}^{1/2} \leq c \, \|f\|_{H_{1/2}}^{1/2}. \tag{20}$$

By combining (18) – (20), we have

$$\|\sigma_n f\|_{1/2}^{1/2} \geq \|II_2\|_{1/2}^{1/2} - \|II_1\|_{1/2}^{1/2} - \|I\|_{1/2}^{1/2} \geq c\lambda_k^{1/2} v(n - M_{\alpha_k}) - c\|f\|_{H_{1/2}}^{1/2}.$$

By using estimates with the above, we conclude that

$$\sup_{n \in \mathbb{N}_+} \frac{1}{n\varphi_n} \sum_{k=1}^{n} \|\sigma_k f\|_{1/2}^{1/2} \geq \frac{1}{M_{\alpha_k+1}\varphi_{2M_{\alpha_k}}} \sum_{\{M_{\alpha_k} \leq l \leq 2M_{\alpha_k}\}} \|\sigma_l f\|_{1/2}^{1/2} \geq$$

$$\geq \frac{c}{M_{\alpha_k+1}\varphi_{2M_{\alpha_k}}} \sum_{\{M_{\alpha_k} \leq l \leq 2M_{\alpha_k}\}} \left(\lambda_k^{1/2} v\,(l - M_{\alpha_k}) - c\|f\|_{H_{1/2}}^{1/2} \right) \geq$$

$$\geq \frac{c\lambda_k^{1/2}}{M_{\alpha_k}\varphi_{2M_{\alpha_k}}} \sum_{l=1}^{M_{\alpha_k}} v(l) - \frac{c\|f\|_{H_{1/2}}^{1/2}}{M_{\alpha_k}\varphi_{2M_{\alpha_k}}} \sum_{\{M_{\alpha_k} \leq l \leq 2M_{\alpha_k}\}} 1 \geq$$

$$\geq \frac{c\lambda_k^{1/2}}{M_{\alpha_k}\varphi_{2M_{\alpha_k}}} \sum_{l=1}^{M_{\alpha_k}-1} v(l) - c \geq c\, \frac{\log^{1/2} M_{\alpha_k}}{\varphi_{2M_{\alpha_k}}^{1/2}} \to \infty \quad \text{as} \quad k \to \infty.$$

Theorem 1 is proved.

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