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STRONGLY P-CLEAN AND SEMI-BOOLEAN GROUP RINGS

СИЛЬНО P-ЧИСТІ ТА НАПІВБУЛЕВІ ГРУПОВІ КІЛЬЦЯ

A ring R is called clean (resp., uniquely clean) if every element is (uniquely represented as) the sum of an idempotent and a unit. A ring R is called strongly P-clean if every its element can be written as the sum of an idempotent and a strongly nilpotent element that commute. The class of strongly P-clean rings is a subclass of classes of semi-Boolean and strongly nil clean rings. A ring R is called semi-Boolean if $R/J(R)$ is Boolean and idempotents lift modulo $J(R)$, where $J(R)$ denotes the Jacobson radical of R. The class of semi-Boolean rings lies strictly between the classes of uniquely clean and clean rings. We obtain a complete characterization of strongly P-clean group rings. It is proved that the group ring RG is strongly P-clean if and only if R is strongly P-clean and G is a locally finite 2-group. Further, we also study semi-Boolean group rings. It is proved that if a group ring RG is semi-Boolean, then R is a semi-Boolean ring and G is a 2-group and that the converse assertion is true if G is locally finite and solvable, or an FC group.

Кільце R називається чистим (відповідно, однозначно чистим), якщо кожний його елемент допускає (однозначне) зображення у вигляді суми ідемпотента та одиниці. Кільце R називається сильно P-чистим, якщо кожний його елемент допускає зображення у вигляді суми ідемпотента та сильно нільпотентного елемента, що комутують. Клас сильно P-чистих кілець є підкласом класів напівбулевих та сильно нульових чистих кілець. Кільце R називається напівбулевим, якщо $R/J(R)$ є булевим, а ідемпотенти піднімають по модулю $J(R)$, де $J(R)$ — радикал Джекобсона для R. Клас напівбулевих кілець лежить точно між класами однозначно чистих та чистих кілець. Отримано повну характеризацію сильно P-чистих групових кілець. Доведено, що групове кільце RG є сильно P-чистим тоді і тільки тоді, коли R є сильно P-чистим, а G — локально скінченна 2-група. Крім того, вивчаються також напівбулеві групові кільця. Доведено, що у випадку, коли групове кільце RG є напівбулевим, R — напівбулевим кільцем, а G — 2-групою, обернене твердження є справедливим, якщо G є локально скінченною та розв'язною або ж FC-групою.

1. Introduction. Throughout this paper, R is an associative ring with identity $1 \neq 0$. Further, $J(R)$ and $P(R)$ denote the Jacobson and Prime radical of R, respectively. A ring R is said to be *clean* if every element is sum of an idempotent and a unit; *uniquely clean* if every element is uniquely the sum of an idempotent and a unit. A ring R is *strongly clean* if every element is sum of an idempotent and a unit that commute with each other; *nil clean* if every element is sum of an idempotent and a nilpotent element. A ring R is *strongly P-clean* if every element $x \in R$ can be written as $x = e + b$, $e^2 = e$, $b \in P(R)$ and $eb = be$. A ring R is called *semi-Boolean* if $R/J(R)$ is Boolean and idempotents lift modulo $J(R)$. Clean rings were introduced by Nicholson [9]. Noncommutative uniquely clean rings were studied by Nicholson and Zhou [10]. Nil clean rings were studied by Diesl [4]. Strongly P-clean rings were introduced by Chen, Köse and Kurtulmaz [1]. Semi-Boolean rings were introduced by Nicholson and Zhou [11]. They show that the class of semi-Boolean rings lies strictly between the class of uniquely clean rings and the class of clean rings:

$$\text{uniquely clean} \Rightarrow \text{semi-Boolean} \Rightarrow \text{clean}.$$

None of the above implications are reversible. The ring of upper triangular matrices $T_n(\mathbb{Z}_2)$ over the field of two elements $\mathbb{Z}_2$, is not uniquely clean though it is semi-Boolean [11]. Also, $\mathbb{Z}_9$, the ring of integers modulo 9, is clean but it is not semi-Boolean [11].

Group ring of a group G and a ring R is denoted by RG. The augmentation map $\omega\colon RG \to R$ is defined by $\omega\left(\sum_{g\in G} r_g g\right) = \sum_{g\in G} r_g$. If H is a subgroup of G, then ωH will denote the right ideal of RG generated by $\{1 - h | h \in H\}$. It is easy to see that ωH is a two sided ideal of RG if

H is a normal subgroup of G. In particular, $RG/\omega H \cong R(G/H)$ and hence $RG/\omega G \cong R$. If I is an ideal of R, then IG is an ideal of RG and $RG/IG \cong (R/I)G$. For related results, we refer to Connell [3] and Passman [12].

The *FC-center* $\Delta(G)$ of a group G is the set of all elements of G which have finitely many conjugates in G. It is easy to see that $\Delta(G) = \{x \in G \,|\, |G : C_G(x)| < \infty\}$, and $\Delta^+(G) = \{x \in G \,|\, |G : C_G(x)| < \infty$ and $o(x) < \infty\}$. If $G = \Delta^+(G)$, then G is *locally normal*, i.e., every finite subset of G is contained in a finite normal subgroup of G.

In Section 2, we obtain a complete characterization of strongly P-clean group rings. It is proved that RG is strongly P-clean if and only if R is strongly P-clean and G is a locally finite 2-group. In Section 3, we determine when is a group ring RG semi-Boolean? It is proved that if RG is semi-Boolean, then R is semi-Boolean and G is a 2-group. This result is a generalization of [2] (Theorem 5). The converse holds, if G is a locally finite, solvable or an FC group. In case, when RG is a commutative semi-Boolean ring, then we show that RG is semi-Boolean if and only if R is semi-Boolean and G is a 2-group. This result is a generalization of [7] (Theorem 2.6).

2. Strongly P-clean group rings. In this section we completely characterize strongly P-clean group rings. It is proved that the group ring RG is strongly P-clean if and only if R is strongly P-clean, and G is a locally finite 2-group.

Recall that an ideal I of a ring R is *locally nilpotent*, if every finitely generated subring of I is nilpotent. An element $a \in R$ is *strongly nilpotent*, if every sequence $a = a_0, a_1, a_2, \ldots$ such that $a_{i+1} \in a_i R a_i$ terminates to zero. The prime radical, $P(R)$ of a ring R consists of precisely the strongly nilpotent elements [6, p. 170] (Ex. 10.17). Thus $P(R)$ is a nil ideal and $P(R) \subseteq J(R)$. Before we prove the main result of this section, we list some of the preliminary results about strongly P-clean rings from Chen, Köse and Kurtulmaz [1] as in the following lemma.

Lemma 2.1. (1) *A ring R is strongly P-clean if and only if R is strongly clean, $R/J(R)$ is Boolean and $J(R)$ is locally nilpotent.*

(2) *A ring R is strongly P-clean if and only if $R/P(R)$ is Boolean.*

(3) *Every homomorphic image of a strongly P-clean ring is strongly P-clean.*

(4) *Let I be a nilpotent ideal of a ring R. Then R is strongly P-clean if and only if R/I is strongly P-clean.*

For a commutative ring, the property of being strongly P-clean and nil clean are equivalent.

Lemma 2.2. *Let R be a commutative ring. Then R is strongly P-clean ring if and only if R is nil clean*

Proof. Obviously if R is strongly P-clean, then R is nil clean.

Conversely, let R be a nil clean ring. Then $R/J(R)$ is Boolean and $J(R)$ is nil by [4] (Corollary 3.20). Since, R is commutative, an element $a \in R$ is nilpotent if and only if it is strongly nilpotent. Also as $J(R)$ is nil, so it follows that $J(R) = P(R)$. Thus $R/P(R) = R/J(R)$ is Boolean. Hence, by Lemma 2.1(2), R is strongly P-clean.

Now if R is strongly P-clean, then $R/J(R)$ is Boolean and $J(R)$ is locally nilpotent (Lemma 2.1(1)). Since idempotents can be lifted modulo any nil ideal I of R. Therefore, every strongly P-clean ring is semi-Boolean. The following example shows that the converse is not true.

***Example* 2.1.** Let $R = \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_8 \ldots$. Then R is not nil clean since the element $r = (0, 2, 2, 2, \ldots) \in R$ is not nil clean. Since R is commutative, by Lemma 2.2, R can not be strongly P-clean. However, since $\mathbb{Z}_2, \mathbb{Z}_4, \mathbb{Z}_8, \ldots$ are local rings with $\mathbb{Z}_2/J(\mathbb{Z}_2) \cong \mathbb{Z}_2, \mathbb{Z}_4/J(\mathbb{Z}_4) \cong$

$\cong \mathbb{Z}_2, \mathbb{Z}_8/J(\mathbb{Z}_8) \cong \mathbb{Z}_2, \ldots,$ we get $\mathbb{Z}_2, \mathbb{Z}_4, \mathbb{Z}_8, \ldots$ to be semi-Boolean rings. Hence, R is semi-Boolean [11] (Example 25(3)).

Lemma 2.3. *If R is a boolean ring and G is a locally finite 2-group, then RG is strongly P-clean.*

Proof. Suppose R is boolean and G is a locally finite 2-group. By [2] (Lemma 9), RG is uniquely clean. Thus $RG/J(RG)$ is Boolean by [10] (Theorem 20) and RG is strongly clean by [10] (Lemma 4). Since R is Boolean, $J(R) = 0$ and $2 = 0$ in R. We get $J(RG) = \omega G$ by [3] (Proposition 16(iv)). Further, by [3, p. 682] (Corollary), ωG is locally nilpotent. Hence, $J(RG)$ is locally nilpotent. So it follows from Lemma 2.1(1) that RG is strongly P-clean.

We now prove the main theorem of this section.

Theorem 2.1. *The group ring RG is strongly P-clean if and only if R is strongly P-clean and G is a locally finite 2-group.*

Proof. Let RG be strongly P-clean. Since the augmentation map $\omega\colon RG \to R$ is an epimorphism, R is strongly P-clean (Lemma 2.1(3)). Thus $R/P(R)$ is Boolean (Lemma 2.1(2)). Hence $\mathbb{Z}_2$ is an image of R, whence $\mathbb{Z}_2G$ is strongly P-clean (image of RG). So $\mathbb{Z}_2G/P(\mathbb{Z}_2G)$ is Boolean. We have $\overline{1-g} \in \mathbb{Z}_2G/P(\mathbb{Z}_2G)$ for all $g \in G$. As $\mathbb{Z}_2G/P(\mathbb{Z}_2G)$ is Boolean; so $\overline{1-g} = \overline{1} - \overline{g} =$ $= \overline{1} - \overline{1} = \overline{0}$. Thus $1 - g \in P(\mathbb{Z}_2G)$ for all $g \in G$. And hence $\omega G \subseteq P(\mathbb{Z}_2G) \subseteq J(\mathbb{Z}_2G)$. Since $\mathbb{Z}_2G/\omega G \cong \mathbb{Z}_2$, we get ωG to be maximal and $\omega G = P(\mathbb{Z}_2G) = J(\mathbb{Z}_2G)$. Now by Lemma 2.1(1), ωG is locally nilpotent. It follows from [3, p. 682] (Corollary), that G is a locally finite 2-group.

Conversely, suppose R is a strongly P-clean ring and G is a locally finite 2-group. By [3] (Proposition 9), we get $P(R)G \subseteq P(RG)$. Further, $P(RG/P(R)G) = P(RG)/P(R)G$. Thus

$$\frac{RG}{P(RG)} \cong \frac{RG/P(R)G}{P(RG)/P(R)G} = \frac{RG/P(R)G}{P(RG/P(R)G)} \cong \frac{\overline{R}G}{P(\overline{R}G)}$$

where $\overline{R} = R/P(R)$. Next, $\overline{R}$ is Boolean, so by Lemma 2.3, $\overline{R}G$ is strongly P-clean. Thus $\overline{R}G/P(\overline{R}G)$ is Boolean, and hence $RG/P(RG)$ is Bhoolean. Now it follows from Lemma 2.1(2) that RG is strongly P-clean.

Theorem 2.1 is proved.

3. Semi-Boolean group rings. First of all we list few of the preliminary results about semi-Boolean rings from Nicholson and Zhou [11].

Lemma 3.1. (1) *A ring R is semi-Boolean if and only if $R/J(R)$ is Boolean and idempotents lift modulo $J(R)$.*

(2) *A ring R is local and semi-Boolean if and only if $R/J(R) \cong \mathbb{Z}_2$.*

(3) *Every homomorphic image of a semi-Boolean ring is again semi-Boolean.*

(4) *A direct product $\prod_i R_i$ or a direct sum $\bigoplus_i R_i$ of rings is semi-Boolean if and only if each R_i is semi-Boolean.*

(5) *If $n \geq 1$, then $T_n(R)$ is semi-Boolean if and only if R is semi-Boolean.*

First we make an observation that if the coefficient ring R is semi-Boolean, then RG may not be semi-Boolean even if G is finite.

***Example* 3.1.** Let $\mathbb{Z}_{(2)}$ denotes the localization of $\mathbb{Z}$ at the prime ideal generated by 2 and C_7 be a cyclic group of order 7. The ring $\mathbb{Z}_{(2)}$ is local with $\mathbb{Z}_{(2)}/J(\mathbb{Z}_{(2)}) \cong \mathbb{Z}_2$. Thus $\mathbb{Z}_{(2)}$ is semi-Boolean. But $\mathbb{Z}_{(2)}C_7$ is not clean (by [13], Remark 18). Hence, $\mathbb{Z}_{(2)}C_7$ is not semi-Boolean.

We obtain a necessary condition for the group ring RG to be semi-Boolean.

Theorem 3.1. *If RG is semi-Boolean, then R is semi-Boolean and G is a 2-group.*

Proof. Clearly, the augmentation map $\omega\colon RG \to R$ is an epimorphism, and thus R is semi-Boolean by Lemma 3.1(3). Hence $R/J(R)$ is Boolean. Thus $\mathbb{Z}_2$ is an image of $R/J(R)$. Hence, $\mathbb{Z}_2$ is an image of R, and so $\mathbb{Z}_2G$ is an image of RG. Therefore, $\mathbb{Z}_2G$ is semi-Boolean and $\mathbb{Z}_2G/J(\mathbb{Z}_2G)$ is Boolean. Thus $\alpha - \alpha^2 \in J(\mathbb{Z}_2G)$ for all $\alpha \in \mathbb{Z}_2G$. In particular, $g \in \mathbb{Z}_2G$ for all $g \in G$. But then

$$1 - g = g^{-1}(g - g^2) \in J(\mathbb{Z}_2G)$$

for all $g \in G$. This proves that $\omega G \subseteq J(\mathbb{Z}_2G)$. By [3] (Proposition 15(i)), G is a 2-group.

Theorem 3.1 is proved.

The converse of the above theorem is true if G is locally finite. But, before that we prove a lemma which we will need.

Lemma 3.2. *A ring R is semi-Boolean if and only if R is clean and $R/J(R)$ is Boolean.*

Proof. ($\Rightarrow$) Every semi-Boolean ring is clean, and if R is semi-Boolean, then $R/J(R)$ is Boolean.

($\Leftarrow$) Suppose R is a clean ring and $R/J(R)$ is Boolean. Since every clean ring is an exchange ring. Thus idempotents lift modulo $J(R)$. Also since $R/J(R)$ is Boolean, R is semi-Boolean (by Lemma 3.1(1)).

Theorem 3.2. *If R is a ring and G is locally finite, then RG is semi-Boolean if and only if R is semi-Boolean and G is a 2-group.*

Proof. If RG is semi-Boolean, the result follows from Theorem 3.1.

Conversely, suppose R is a semi-Boolean ring and G is a 2-group. We have $R/J(R)$ is Boolean, and so $2 \in J(R)$. Thus it follows from [14] (Theorem 4) that RG is clean. Now by [2] (Lemma 9), $(R/J(R))G \cong RG/J(R)G$ is uniquely clean. Since G is locally finite, it follows from [3] (Proposition 9) that $J(R)G \subseteq J(RG)$. We now consider the map $\phi\colon RG/J(R)G \to RG/J(RG)$ defined as

$$\phi(\alpha + J(R)G) = \alpha + J(RG), \quad \alpha \in RG.$$

It is easy to see that ϕ is a ring epimorphism. So $RG/J(RG)$ is uniquely clean. Thus

$$\frac{RG/J(RG)}{J(RG/J(RG))} \cong RG/J(RG)$$

is Boolean. Hence, it follows from Lemma 3.2 that RG is semi-Boolean.

***Corollary* 3.1.** *If G is a solvable or an FC group, then RG is semi-Boolean if and only if R is semi-Boolean and G is a 2-group.*

Proof. It is easy to see that a torsion solvable group is locally finite. Also, if G is a torsion FC group, then $G = \Delta^+(G)$. It is known that $\Delta^+(G)$ is locally finite. The result follows from Theorem 3.2.

Theorem 3.3. *Let R be an Artinian ring, then RG is semi-Boolean iff RG is clean and $(R/J(R))G$ is semi-Boolean.*

Proof. One way is straight forward as every semi-Boolean ring is clean and $(R/J(R))G$ is semi-Boolean (image of RG).

Conversely, suppose $(R/J(R))G$ is semi-Boolean. Since R is Artinian, $J(R)G \subseteq J(RG)$ (by [3], Proposition 9). Now consider the map $\phi\colon RG/J(R)G \to RG/J(RG)$ defined as

$$\phi(\alpha + J(R)G) = \alpha + J(RG), \quad \alpha \in RG.$$

Following the proof of Theorem 3.2, we get that $RG/J(RG)$ is Boolean. Also since RG is clean, RG is semi-Boolean (by Lemma 3.2).

Theorem 3.3 is proved.

The following result about strongly nil clean rings is proved by Koşan, Wang and Zhou [5].

Lemma 3.3 ([5], Theorem 2.7)**.** *A ring R is strongly nil clean if and only if $R/J(R)$ is Boolean and $J(R)$ is nil.*

It is well known that if I is any nil ideal of a ring R, then idempotents modulo I can be lifted to R. Thus every strongly nil clean ring is semi-Boolean. So the class of strongly nil clean rings is contained in the class of semi-Boolean rings.

Remark **3.1.** A semi-Boolean ring may not be strongly nil clean ring. The ring $R = \mathbb{Z}_2 \times \mathbb{Z}_4 \times \times \mathbb{Z}_8, \ldots,$ of Example 2.1, is semi-Boolean, but not strongly nil clean.

The following characterization for a commutative group ring to be nil clean is due to McGovern, Raja and Sharp [7].

Lemma 3.4 ([7], Theorem 2.6)**.** *Suppose R is a commutative ring and G is an Abelian group. The group ring RG is nil clean if and only if R is nil clean and G is a 2-group.*

Since every Abelian group is an FC group, Corollary 3.1 gives a generalization of the above result. We state this as the following corollary.

Corollary **3.2.** *Let R be a commutative ring and G be an Abelian group. Then the group ring RG is semi-Boolean if and only if R is semi-Boolean and G is a 2-group.*

Theorem 3.4. *Let R be a local ring and G be a locally finite group, then the following are equivalent:*

(1) *$T_n(RG)$ is semi-Boolean,*

(2) *$RG/J(RG) \cong \mathbb{Z}_2$,*

(3) *RG is uniquely clean,*

(4) *$R/J(R) \cong \mathbb{Z}_2$ and $\mathbb{Z}_2G$ is semi-Boolean.*

Proof. $1) \Rightarrow 2)$ Suppose $T_n(RG)$ is semi-Boolean, then by Lemma 3.1(5), RG is semi-Boolean. So by Theorem 3.1, R is semi-Boolean and G is a 2-group. Also since R is local, $R/J(R) \cong \mathbb{Z}_2$. Thus by [8] (Theorem), RG is local. Now RG is semi-Boolean and local. Hence, $RG/J(RG) \cong \mathbb{Z}_2$.

$2) \Rightarrow 3)$ Follows from [10] (Theorem 15).

$3) \Rightarrow 4)$ Let RG be uniquely clean, then R is uniquely clean and $R/J(R)$ is Boolean. Also since R is local, $R \cong \mathbb{Z}_2$. Thus $\mathbb{Z}_2$ is an image of R. So $\mathbb{Z}_2G$ is an image of RG. Hence, $\mathbb{Z}_2G$ is uniquely clean. Since a uniquely clean ring is semi-Boolean, $\mathbb{Z}_2G$ is semi-Boolean.

$4) \Rightarrow 1)$ If $R/J(R) \cong \mathbb{Z}_2$, then R is semi-Boolean (by [11], Proposition 31). Also if $\mathbb{Z}_2G$ is semi-Boolean, then, by Theorem 3.1, G is a 2-group. So, by Theorem 3.2, RG is semi-Boolean. Hence, by Lemma 3.1(5), $T_n(RG)$ is semi-Boolean.

Theorem 3.4 is proved.

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