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**A. Boua** (Sidi Mohamed Ben Abdellah Univ., Fez, Morocco),
**M. Ashraf** (Aligarh Muslim Univ., India)

# SOME ALGEBRAIC IDENTITIES IN 3-PRIME NEAR-RINGS
# ДЕЯКІ АЛГЕБРАЇЧНІ ТОТОЖНОСТІ ДЛЯ 3-ПРОСТИХ МАЙЖЕ КІЛЕЦЬ

We extend the domain of applicability of the concept of $(1,\alpha)$-derivations in 3-prime near-rings by analyzing the structure and commutativity of near-rings admitting $(1,\alpha)$-derivations satisfying certain differential identities.

Розширено область застосовності поняття $(1,\alpha)$-похідних для 3-простих майже кілець, як результат вивчення структури та комутативності майже кілець, що допускають $(1,\alpha)$-похідні, які задовольняють деякі диференціальні тотожності.

**1. Introduction.** Throughout this paper, $\mathcal{N}$ will denote a zero-symmetric left near-ring. A near-ring $\mathcal{N}$ is called zero symmetric if $0x = 0$ for all $x \in \mathcal{N}$ (recall that in a left near ring $x0 = 0$ for all $x \in \mathcal{N}$). $\mathcal{N}$ is called 3-prime if $x\mathcal{N}y = \{0\}$ implies $x = 0$ or $y = 0$. The symbol $Z(\mathcal{N})$ will represent the multiplicative center of $\mathcal{N}$, that is, $Z(\mathcal{N}) = \{x \in \mathcal{N} \mid xy = yx$ for all $y \in \mathcal{N}\}$. For any $x,\ y \in \mathcal{N}$, as usual, $[x,y] = xy - yx$ and $x \circ y = xy + yx$ will denote the well-known Lie product and Jordan product, respectively. Recall that $\mathcal{N}$ is called 2-torsion free if $2x = 0$ implies $x = 0$ for all $x \in \mathcal{N}$. For terminologies concerning near-rings we refer to G. Pilz [6].

An additive mapping $d\colon \mathcal{N} \to \mathcal{N}$ is said to be a derivation if $d(xy) = xd(y) + d(x)y$ for all $x,\ y \in \mathcal{N}$, or, equivalently, as noted in [7], that $d(xy) = d(x)y + xd(y)$ for all $x,\ y \in \mathcal{N}$. An additive mapping $d\colon \mathcal{N} \to \mathcal{N}$ is called a semiderivation if there exists a function $g\colon \mathcal{N} \to \mathcal{N}$ such that $d(xy) = d(x)g(y) + xd(y) = d(x)y + g(x)d(y)$ and $d(g(x)) = g(d(x))$ hold for all $x, y \in \mathcal{N}$. An additive mapping $d\colon \mathcal{N} \to \mathcal{N}$ is called a two sided $\alpha$-derivation if there exists a function $\alpha\colon \mathcal{N} \to \mathcal{N}$ such that $d(xy) = d(x)y + \alpha(x)d(y)$ and $d(xy) = d(x)\alpha(y) + xd(y)$ hold for all $x,\ y \in \mathcal{N}$. An additive mapping $d\colon \mathcal{N} \to \mathcal{N}$ is called a $(1,\alpha)$-derivation if there exists a function $\alpha\colon \mathcal{N} \to \mathcal{N}$ such that $d(xy) = d(x)y + \alpha(x)d(y)$ holds for all $x,\ y \in \mathcal{N}$. An additive mapping $d\colon \mathcal{N} \to \mathcal{N}$ is called an $(\alpha,1)$-derivation if there exists a function $\alpha\colon \mathcal{N} \to \mathcal{N}$ such that $d(xy) = d(x)\alpha(y) + xd(y)$ holds for all $x,\ y \in \mathcal{N}$. Obviously, a two sided $\alpha$-derivation is both $(1,\alpha)$-derivation as well as $(\alpha,1)$-derivation. Also, any derivation on $\mathcal{N}$ is a $(1,\alpha)$-derivation, but the converse is not true in general (see [5]). There are several results asserting that 3-prime near-rings with certain constrained derivations have ringlike behavior. Recently many authors (see [1, 2, 4], where further references can be found) studied commutativity of 3-prime near-rings satisfying certain identities involving derivations, semiderivations and two sided $\alpha$-derivations. Now our aim is to study the commutativity behavior of a 3-prime near-ring which admits $(1,\alpha)$-derivations satisfying certain properties. In fact, our results generalize, extend and complement several results obtained earlier in [1, 5, 8] on derivations, semiderivations and two sided $\alpha$-derivations for 3-prime near-rings.

**2. Some preliminaries.** In this section, we include some well-known results which will be used for developing the proof of our main result.

**Lemma 2.1** ([4], Theorem 2.9)**.** *Let $\mathcal{N}$ be a 3-prime near-ring. If $I$ is a nonzero semigroup ideal of $\mathcal{N}$ and $d$ is a nonzero derivation of $\mathcal{N}$, then the following assertions are equivalent:*

(i) $[u,v] \in Z(\mathcal{N})$ *for all* $u,\ v \in I$,

(ii) $[d(u), v] \in Z(\mathcal{N})$ *for all* $u, v \in I$,

(iii) $\mathcal{N}$ *is a commutative ring.*

**Lemma 2.2** ([4], Theorem 2.10)**.** *Let* $\mathcal{N}$ *be a* 2*-torsion free* 3*-prime near-ring. If* $u \circ v \in Z(\mathcal{N})$ *for all* $u, v \in \mathcal{N}$, *then* $\mathcal{N}$ *is a commutative ring.*

**Lemma 2.3** ([3], Lemma 1.5)**.** *Let* $\mathcal{N}$ *be a* 3*-prime near-ring. If* $\mathcal{N} \subseteq Z(\mathcal{N})$, *then* $\mathcal{N}$ *is a commutative ring.*

**Lemma 2.4.** *A near-ring* $\mathcal{N}$ *admits a* $(1,\alpha)$*-derivation* $d$ *associated with an additive map* $\alpha$ *if and only if it is zero-symmetric.*

***Proof.*** Let $\mathcal{N}$ be a zero-symmetric near-ring. Then the zero map is a $(1,\alpha)$-derivation $d$ on $\mathcal{N}$. Conversely, assume that $\mathcal{N}$ has an $(1,\alpha)$-derivation $d$ associated with an additive map $\alpha$. Let $x, y$ be two arbitrary elements of $\mathcal{N}$. By definition of $d$, we have

$$d(x0y) = d(x(0y)) = d(x)(0y) + \alpha(x)d(0y) =$$

$$= (d(x)0)y + \alpha(x)(d(0)y) + \alpha(x)(\alpha(0)d(y)) = 0y + (\alpha(x)d(0))y + (\alpha(x)\alpha(0))d(y) =$$

$$= 0y + (\alpha(x)0)y + (\alpha(x)\alpha(0))d(y) = 0y + 0y + (\alpha(x)0)d(y) = 0y + 0y + 0d(y).$$

On the other hand,

$$d(x0y) = d((x0)y)) = d(0y) =$$

$$= d(0)y + \alpha(0)d(y) = 0y + 0d(y).$$

By comparing the last two expressions, we find that $0y = 0$ for all $y \in \mathcal{N}$, and hence $\mathcal{N}$ is a zero-symmetric left near-ring.

***Remark.*** The above lemma has its independent interest in the study of arbitrary left near-rings (not necessarily zero-symmetric). It can also be easily seen that it is also true in the case of right near-ring.

**Lemma 2.5.** *Let* $\mathcal{N}$ *be a near-ring and* $d$ *be a* $(1,\alpha)$*-derivation associated with a map* $\alpha$. *Then* $\mathcal{N}$ *satisfies the following property:*

$$(d(x)y + \alpha(x)d(y))\, z =$$

$$= d(x)yz + \alpha(x)d(y)z + \alpha(x)\alpha(y)d(z) - \alpha(xy)d(z) \quad \textit{for all} \quad x, y, z \in \mathcal{N}.$$

***Proof.*** From the associative law we have

$$d((xy)z) = d(xy)z + \alpha(xy)d(z) =$$

$$= (d(x)y + \alpha(x)d(y))z + \alpha(xy)d(z) \quad \text{for all} \quad x, y \in \mathcal{N}.$$

Also

$$d(x(yz)) = d(x)yz + \alpha(x)d(yz) =$$

$$= d(x)yz + \alpha(x)d(y)z + \alpha(x)\alpha(y)d(z) \quad \text{for all} \quad x, y, z \in \mathcal{N}.$$

Combining the above two equalities, we find

$$(d(x)y + \alpha(x)d(y))\, z + \alpha(xy)d(z) =$$

$$= d(x)yz + \alpha(x)d(y)z + \alpha(x)\alpha(y)d(z) \quad \text{for all} \quad x, y, z \in \mathcal{N},$$

which is the required result.

**Lemma 2.6.** *Let $\mathcal{N}$ be a 3-prime near-ring and $d$ be a nonzero $(1, \alpha)$-derivation associated with an onto map $\alpha$.*

(i) *If $ad(\mathcal{N}) = \{0\}$, $a \in \mathcal{N}$ and $\alpha$ is an onto map, then $a = 0$.*

(ii) *If $d(\mathcal{N})a = \{0\}$ and $\alpha(xy) = \alpha(x)\alpha(y)$ for all $x,\ y \in \mathcal{N}$, then $a = 0$.*

***Proof.*** (i) If $ad(\mathcal{N}) = \{0\}$ and $a \in \mathcal{N}$, then $ad(xy) = 0$ for all $x,\ y \in \mathcal{N}$. This implies that $ad(x)y + a\alpha(x)d(y) = 0$ for all $x,\ y \in \mathcal{N}$, and, hence, $a\alpha(x)d(y) = 0$ for all $x,\ y \in \mathcal{N}$. Since $\alpha$ is onto, $a\mathcal{N}d(y) = \{0\}$ for all $y \in \mathcal{N}$. By 3-primeness of $\mathcal{N}$ and $d \neq 0$, we obtain $a = 0$.

(ii) If $d(\mathcal{N})a = \{0\}$, then $d(xy)a = 0$ for all $x, y \in \mathcal{N}$. By Lemma 2.5, we get $d(x)ya + \alpha(x)d(y)a + \alpha(x)\alpha(y)d(a) - \alpha(xy)d(a) = 0$ for all $x,\ y \in \mathcal{N}$. By the given hypothesis, we find that $d(x)ya = 0$ for all $x,\ y \in \mathcal{N}$, i.e., $d(x)\mathcal{N}a = \{0\}$ for all $x \in \mathcal{N}$. Since $d \neq 0$ and $\mathcal{N}$ is 3-prime, we arrive at $a = 0$.

**Lemma 2.7.** *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring. If $d$ is a nonzero $(1, \alpha)$-derivation associated with an onto map $\alpha$ such that $\alpha d = d\alpha$, then $d^2 \neq 0$.*

***Proof.*** Suppose that $d^2(\mathcal{N}) = \{0\}$. Then, for $x,\ y \in \mathcal{N}$, one can write

$$0 = d^2(xy) = d(d(xy)) = d(d(x)y + \alpha(x)d(y)) =$$

$$= d^2(x)y + \alpha(d(x))d(y) + d(\alpha(x))d(y) + \alpha^2(x)d^2(y) =$$

$$= \alpha(d(x))d(y) + d(\alpha(x))d(y) \quad \text{for all} \quad x, y \in \mathcal{N}.$$

Note that $\alpha(d(x)) = d(\alpha(x))$, we find that

$$2\alpha(d(x))d(y) = 0 \text{ for all } x, y \in \mathcal{N}.$$

Since $\mathcal{N}$ is 2-torsion free, we arrive at

$$d(\alpha(x))d(y) = 0 \text{ for all } x, y \in \mathcal{N}.$$

By using Lemma 2.6 and the fact that $\alpha$ is onto, we obtain that $d = 0$ a contradiction.

**3. Main results.** In [2], H. E. Bell and G. Mason proved that a 3-prime near-ring $\mathcal{N}$ must be commutative if it admits a derivation $d$ such that $d(\mathcal{N}) \subseteq Z(\mathcal{N})$. This result was generalized by the authors in [5, 8]. They replaced the derivation with a semiderivation or two sided $\alpha$-derivation. Our objective in the following theorems is to generalize these results by treating the case of $(1, \alpha)$-derivation where $\alpha$ is an onto map.

**Theorem 3.1.** *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring. If $\mathcal{N}$ admits a nonzero $(1, \alpha)$-derivation $d$ associated with an onto map $\alpha$ such that $\alpha d = d\alpha$ and $d(\mathcal{N}) \subseteq Z(\mathcal{N})$, then $\mathcal{N}$ is a commutative ring.*

***Proof.*** Suppose that $d(\mathcal{N}) \subseteq Z(\mathcal{N})$. By definition of $d$, we have

$$(d(x)y + \alpha(x)d(y))z = z(d(x)y + \alpha(x)d(y)) \quad \text{for all} \quad x, y, z \in \mathcal{N}.$$

This implies that

$$d(x)yz + \alpha(x)d(y)z + \alpha(x)\alpha(y)d(z) - \alpha(xy)d(z) =$$
$$= zd(x)y + z\alpha(x)d(y) \quad \text{for all} \quad x, y, z \in \mathcal{N}. \tag{3.1}$$

Replacing $z$ by $d(z)$ in (3.1), we get

$$\alpha(x)\alpha(y)d^2(z) = \alpha(xy)d^2(z) \quad \text{for all} \quad x, y, z \in \mathcal{N},$$

which reduces to

$$d^2(z)\mathcal{N}\big(\alpha(x)\alpha(y) - \alpha(xy)\big) = \{0\} \quad \text{for all} \quad x, y, z \in \mathcal{N}. \tag{3.2}$$

In view of 3-primeness of $\mathcal{N}$, (3.2) implies that

$$d^2 = 0 \text{ or } \alpha(xy) = \alpha(x)\alpha(y) \quad \text{for all} \quad x, y \in \mathcal{N}.$$

Since $d \neq 0$, we obtain $d^2 \neq 0$ by Lemma 2.7, and in this case the previous relation becomes only $\alpha(xy) = \alpha(x)\alpha(y)$ for all $x,\ y \in \mathcal{N}$ and by (3.1) we arrive at

$$d(x)yz + \alpha(x)d(y)z = zd(x)y + z\alpha(x)d(y) \quad \text{for all} \quad x, y, z \in \mathcal{N}. \tag{3.3}$$

Replacing $z$ by $\alpha(x)$ in (3.3), we obtain

$$d(x)y\alpha(x) = \alpha(x)d(x)y \quad \text{for all} \quad x, y \in \mathcal{N}.$$

This yields that

$$d(x)\mathcal{N}[\alpha(x), y] = \{0\} \quad \text{for all} \quad x, y \in \mathcal{N}.$$

Since $\mathcal{N}$ is 3-prime, we find that

$$d(x) = 0 \quad \text{or} \quad \alpha(x) \in Z(\mathcal{N}) \quad \text{for all} \quad x \in \mathcal{N}. \tag{3.4}$$

Suppose there exists $x_0 \in \mathcal{N}$ such that $d(x_0) = 0$. Replacing $x$ by $x_0$ in (3.1), we get $\alpha(x_0)d(y)z = = z\alpha(x_0)d(y)$ for all $y, z \in \mathcal{N}$, which implies that

$$d(y)\mathcal{N}[\alpha(x_0), z] = \{0\} \quad \text{for all} \quad y, z \in \mathcal{N}.$$

Since $\mathcal{N}$ is 3-prime and $d \neq 0$, the last expression implies that $\alpha(x_0) \in Z(\mathcal{N})$, and the relation (3.4) yields $\alpha(x) \in Z(\mathcal{N})$ for all $x \in \mathcal{N}$. Now in this case (3.1) becomes

$$d(x)\mathcal{N}[y, z] = \{0\} \quad \text{for all} \quad y, z \in \mathcal{N}.$$

Since $\mathcal{N}$ is 3-prime and $d \neq 0$, we conclude that $\mathcal{N} \subseteq Z(\mathcal{N})$ and by Lemma 2.3, $\mathcal{N}$ is a commutative ring.

***Corollary* 3.1** ([2], Theorem 2)**.** *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring. If $\mathcal{N}$ admits a nonzero derivation $d$ such that $d(\mathcal{N}) \subseteq Z(\mathcal{N})$, then $\mathcal{N}$ is a commutative ring.*

***Corollary* 3.2.** *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring. If $\mathcal{N}$ admits a nonzero semiderivation $d$ associated with an onto map $\alpha$ such that $d(\mathcal{N}) \subseteq Z(\mathcal{N})$, then $\mathcal{N}$ is a commutative ring.*

**Theorem 3.2.** *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring which admits a nonzero $(1, \alpha)$-derivation $d$ associated with an onto map $\alpha$. Then the following assertions are equivalent:*

(i) $d([x, y]) = 0$ *for all* $x, y \in \mathcal{N}$,

(ii) $\mathcal{N}$ *is a commutative ring.*

***Proof.*** It is easy to verify that (ii) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii) Suppose that $d([x,y]) = 0$ for all $x, y \in \mathcal{N}$. Replacing $y$ by $xy$, we get

$$0 = d([x,xy]) = d(x[x,y]) =$$

$$= d(x)[x,y] + \alpha(x)d([x,y]) \quad \text{for all} \quad x, y \in \mathcal{N}.$$

This implies that

$$d(x)xy = d(x)yx \quad \text{for all} \quad x, y \in \mathcal{N}. \tag{3.5}$$

Replacing $y$ by $yt$ in (3.5) and using it again, we get

$$d(x)ytx = d(x)xyt =$$

$$= d(x)yxt \quad \text{for all} \quad x, y, t \in \mathcal{N},$$

which reduces to

$$d(x)\mathcal{N}[x,t] = \{0\} \quad \text{for all} \quad x, t \in \mathcal{N}.$$

By 3-primeness of $\mathcal{N}$, we obtain

$$d(x) = 0 \text{ or } x \in Z(\mathcal{N}) \quad \text{for all} \quad x \in \mathcal{N}. \tag{3.6}$$

Suppose there exists $x_0 \in \mathcal{N}$ such that $d(x_0) = 0$. Then by hypothesis, we have $d(x_0 y) = d(y x_0)$ for all $y \in \mathcal{N}$. Since $\mathcal{N}$ is zero-symmetric by Lemma 2.5, the last equation implies that

$$\alpha(x_0)d(y) = d(y)x_0 \quad \text{for all} \quad y \in \mathcal{N}. \tag{3.7}$$

Taking $yt$ instead of $y$ in (3.7) and using Lemma 2.5 together with the fact that $d(x_0) = 0$, we find that

$$\alpha(x_0)d(y)t + \alpha(x_0)\alpha(y)d(t) = d(y)tx_0 + \alpha(y)d(t)x_0 \quad \text{for all} \quad y, t \in \mathcal{N}.$$

By (3.7) the above expression implies that

$$d(y)x_0 t + \alpha(x_0)\alpha(y)d(t) = d(y)tx_0 + \alpha(y)\alpha(x_0)d(t) \quad \text{for all} \quad y, t \in \mathcal{N}.$$

Putting $[u,v]$ instead of $t$ in the last expression, we get

$$d(y)x_0[u,v] = d(y)[u,v]x_0 \quad \text{for all} \quad u, v, y \in \mathcal{N}. \tag{3.8}$$

Replacing $y$ by $yt$ in (3.8) and using it again, we obtain

$$d(y)tx_0[u,v] + \alpha(y)d(t)x_0[u,v] =$$

$$= d(y)t[u,v]x_0 + \alpha(y)d(t)[u,v]x_0 \quad \text{for all} \quad y, t \in \mathcal{N},$$

which reduces to

$$d(y)\mathcal{N}\big(x_0[u,v] - [u,v]x_0\big) = \{0\} \quad \text{for all} \quad y, u, v \in \mathcal{N}.$$

Since $d \neq 0$, by 3-primeness of $\mathcal{N}$, we get $x_0[u,v] - [u,v]x_0 = 0$ for all $u, v \in \mathcal{N}$ in this case (3.6) becomes $[u,v] \in Z(\mathcal{N})$ for all $u, v \in \mathcal{N}$ and by Lemma 2.1, we conclude that $\mathcal{N}$ is a commutative ring.

***Corollary* 3.3** ([1], Theorem 4.1)**.** *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring. If $\mathcal{N}$ admits a nonzero derivation $d$ such that $d([x, y]) = 0$ for all $x,\ y \in \mathcal{N}$, then $\mathcal{N}$ is a commutative ring.*

***Corollary* 3.4.** *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring. If $\mathcal{N}$ admits a nonzero semiderivation $d$ associated with an onto map $\alpha$ such that $d([x, y]) = 0$ for all $x, y \in \mathcal{N}$, then $\mathcal{N}$ is a commutative ring.*

**Theorem 3.3.** *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring. Then there exists no nonzero $(1, \alpha)$-derivation $d$ associated with an onto map $\alpha$ such that $d(x \circ y) = 0$ for all $x,\ y \in \mathcal{N}$.*

***Proof.*** Assume that $d(x \circ y) = 0$ for all $x, y \in \mathcal{N}$. Replacing $y$ by $xy$, we get

$$0 = d(x \circ xy) = d(x(x \circ y)) =$$

$$= d(x)(x \circ y) + \alpha(x)d(x \circ y) \quad \text{for all} \quad x, y \in \mathcal{N}.$$

This implies that

$$d(x)xy = -d(x)yx \quad \text{for all} \quad x, y \in \mathcal{N}. \tag{3.9}$$

Replacing $y$ by $yt$ in (3.9) and using it again, we get

$$d(x)ytx = -d(x)xyt = d(x)xy(-t) =$$

$$= (-d(x)yx)(-t) = d(x)y(-x)(-t) \quad \text{for all} \quad x, y, t \in \mathcal{N}.$$

This can be rewritten as

$$d(x)\mathcal{N}(-t(-x) + (-x)t) = \{0\} \quad \text{for all} \quad x, y, t \in \mathcal{N}.$$

By 3-primeness of $\mathcal{N}$, the latter equation becomes

$$d(x) = 0 \quad \text{or} \quad -x \in Z(\mathcal{N}) \quad \text{for all} \quad x \in \mathcal{N}. \tag{3.10}$$

Suppose there exists $x_0 \in \mathcal{N}$ such that $d(x_0) = 0$. Then by hypothesis, we have $d(x_0y) = -d(yx_0)$ for all $y \in \mathcal{N}$. Since $\mathcal{N}$ is zero-symmetric, by definition of $d$ the last equation implies that

$$\alpha(x_0)d(y) = -d(y)x_0 \quad \text{for all} \quad y \in \mathcal{N}. \tag{3.11}$$

Taking $yt$ instead of $y$ in (3.11) and using Lemma 2.5 together with the fact that $d(x_0) = 0$, we find that

$$\alpha(x_0)d(y)t + \alpha(x_0)\alpha(y)d(t) = -\alpha(y)d(t)x_0 - d(y)tx_0 \quad \text{for all} \quad y, t \in \mathcal{N}.$$

By (3.11) the above expression implies that

$$(-d(y)x_0)t + \alpha(x_0)\alpha(y)d(t) = -\alpha(y)d(t)x_0 - d(y)tx_0 \quad \text{for all} \quad y, t \in \mathcal{N}.$$

Putting $u \circ v$ instead of $t$ in the last expression, we get

$$d(y)(-x_0)(u \circ v) = d(y)(u \circ v)(-x_0) \quad \text{for all} \quad u, v, y \in \mathcal{N}. \tag{3.12}$$

Replacing $y$ by $yt$ in (3.12) and using it again, we obtain

$$d(y)t(-x_0)(u \circ v) + \alpha(y)d(t)(-x_0)(u \circ v) = d(y)t(u \circ v)(-x_0) + \alpha(y)d(t)(u \circ v)(-x_0)$$

for all $y, t \in \mathcal{N}$, which reduces to

$$d(y)\mathcal{N}\big((-x_0)(u \circ v) - (u \circ v)(-x_0)\big) = \{0\} \quad \text{for all} \quad y, u, v \in \mathcal{N}.$$

Since $d \neq 0$, by 3-primeness of $\mathcal{N}$, we find that $(-x_0)(u \circ v) = (u \circ v)(-x_0)$ for all $u, v \in \mathcal{N}$, and in this case (3.10) implies $(-x)(u \circ v) = (u \circ v)(-x)$ for all $u, v, x \in \mathcal{N}$. Replacing $x$ by $-x$ in the last equation, we obtain $u \circ v \in Z(\mathcal{N})$ for all $u, v \in \mathcal{N}$. Now by Lemma 2.2, we conclude that $\mathcal{N}$ is a commutative ring. In this case, we obtain $2d(xy) = 0$ for all $x,\ y \in \mathcal{N}$ and by 2-torsion freeness, we have $d(xy) = 0$ for all $x, y \in \mathcal{N}$. By definition of $d$, we get $d(x)y + \alpha(x)d(y) = 0$ for all $x,\ y \in \mathcal{N}$. Replacing $y$ by $yz$ in the above expression we obtain that $d(x)yz = 0$ for all $x, y, z \in \mathcal{N}$, i.e., $d(x)\mathcal{N}z = \{0\}$ for all $x, z \in \mathcal{N}$ and by 3-primeness of $\mathcal{N}$ we conclude that $d = 0$, a contradiction.

***Corollary* 3.5.** *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring. Then there exists no nonzero derivation $d$ such that $d(x \circ y) = 0$ for all $x,\ y \in \mathcal{N}$.*

***Corollary* 3.6.** *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring. Then there exists no nonzero semiderivation $d$ associated with an onto map $\alpha$ such that $d(x \circ y) = 0$ for all $x,\ y \in \mathcal{N}$.*

**Theorem 3.4.** *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring which admits a nonzero $(1, \alpha)$-derivation $d$ associated with an onto homomophism $\alpha$. Then the following assertions are equivalent:*

(i) $d([x, y]) = [x, y]$ *for all* $x, y \in \mathcal{N}$,

(ii) $\mathcal{N}$ *is a commutative ring.*

***Proof.*** It is easy to verify that (ii) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii) Suppose that $d([x, y]) = [x, y]$ for all $x, y \in \mathcal{N}$. Replacing $y$ by $xy$, we get

$$[x, xy] = d([x, xy]) = d(x[x, y]) =$$

$$= d(x)[x, y] + \alpha(x)d([x, y]) \quad \text{for all} \quad x, y \in \mathcal{N}.$$

Since $[x, xy] = x[x, y]$, the above expression becomes

$$d(x)[x, y] + \alpha(x)[x, y] = x[x, y] \quad \text{for all} \quad x, y \in \mathcal{N}.$$

Taking $[u, v]$ instead of $x$ and using our hypothesis, we arrive at

$$\alpha([u, v])[[u, v], y] = 0 \quad \text{for all} \quad u, v, y \in \mathcal{N}.$$

This implies that

$$\alpha([u, v])y[u, v] = \alpha([u, v])[u, v]y \quad \text{for all} \quad u, v, y \in \mathcal{N}. \tag{3.13}$$

Replacing $y$ by $yt$ in (3.13) and using it again, we get

$$\alpha([u, v])yt[u, v] = \alpha([u, v])[u, v]yt =$$

$$= \alpha([u, v])y[u, v]t \quad \text{for all} \quad u, v, y, t \in \mathcal{N},$$

which forces that

$$\alpha([u, v])\mathcal{N}[[u, v], t] = \{0\} \quad \text{for all} \quad u, v, t \in \mathcal{N}.$$

Since $\mathcal{N}$ is 3-prime, we find that

$$\alpha([u, v]) = 0 \quad \text{or} \quad [u, v] \in Z(\mathcal{N}) \quad \text{for all} \quad u, v \in \mathcal{N}. \tag{3.14}$$

If there exist two elements $u_0, v_0 \in \mathcal{N}$ such that $[u_0, v_0] \in Z(\mathcal{N})$, then

$$d\big([u_0, v_0][x, y]\big) = d\big(\big[[u_0, v_0]x, y\big]\big) =$$
$$= [u_0, v_0][x, y] \quad \text{for all} \quad x, y \in \mathcal{N}.$$

By definition of $d$, we find that

$$[u_0, v_0][x, y] = d([u_0, v_0][x, y]) =$$
$$= d([u_0, v_0])[x, y] + \alpha([u_0, v_0])d([x, y]) =$$
$$= [u_0, v_0][x, y] + \alpha([u_0, v_0])[x, y] \quad \text{for all} \quad x, y \in \mathcal{N}.$$

By the last expression, we obtain

$$\alpha([u_0, v_0])xy = \alpha([u_0, v_0])yx \quad \text{for all} \quad x, y \in \mathcal{N}. \tag{3.15}$$

Replacing $x$ by $xt$ in (3.15) and using it again, we get

$$\alpha([u_0, v_0])xty = \alpha([u_0, v_0])yxt =$$
$$= \alpha([u_0, v_0])xyt \quad \text{for all} \quad x, y, t \in \mathcal{N}.$$

Using this expression, we arrive at

$$\alpha([u_0, v_0])\mathcal{N}[y, t] = \{0\} \quad \text{for all} \quad y, t \in \mathcal{N}.$$

Since $\mathcal{N}$ is 3-prime, by Lemma 2.3, we obtain $\alpha([u_0, v_0]) = 0$ or $\mathcal{N}$ is a commutative ring. In this case (3.14) becomes $\alpha([u, v]) = 0$ for all $u, v \in \mathcal{N}$ or $\mathcal{N}$ is a commutative ring. Since $\alpha$ is an onto homomorphism, we find that $\mathcal{N}$ is a commutative ring.

**Theorem 3.5.** *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring. Then $\mathcal{N}$ admits no nonzero $(1, \alpha)$-derivation $d$ associated with an onto homomorphism $\alpha$ satisfying any one of the following conditions:*

(i) $d(x \circ y) = x \circ y$ *for all* $x,\ y \in \mathcal{N}$,

(ii) $d([x, y]) = x \circ y$ *for all* $x,\ y \in \mathcal{N}$,

(iii) $d(x \circ y) = [x, y]$ *for all* $x,\ y \in \mathcal{N}$.

***Proof.*** (i) Suppose that $d(x \circ y) = x \circ y$ for all $x,\ y \in \mathcal{N}$. Replacing $x$ by $xy$, we have

$$x(x \circ y) = x \circ xy = d(x \circ xy) =$$
$$= d(x(x \circ y)) = d(x)(x \circ y) + \alpha(x)d(x \circ y) =$$
$$= d(x)(x \circ y) + \alpha(x)(x \circ y) \quad \text{for all} \quad x, y \in \mathcal{N}.$$

Putting $u \circ v$ instead of $x$ in the latter expression, we arrive at $\alpha(u \circ v)\big((u \circ v) \circ y\big) = 0$ for all $u, v \in \mathcal{N}$, which yields that

$$\alpha(u \circ v)y(u \circ v) = -\alpha(u \circ v)(u \circ v)y \quad \text{for all} \quad u, v, y \in \mathcal{N}. \tag{3.16}$$

Replacing $y$ by $yt$ in (3.16) and using it again, we get

$$\alpha(u \circ v)yt(u \circ v) = -\alpha(u \circ v)(u \circ v)yt = \alpha(u \circ v)(u \circ v)y(-t) =$$

$$= (-\alpha(u \circ v)y(u \circ v))(-t) = \alpha(u \circ v)y(-(u \circ v))(-t) \quad \text{for all} \quad u, v, t \in \mathcal{N}.$$

This reduces to

$$\alpha(u \circ v)\mathcal{N}\big(-t(-u \circ v) + (-u \circ v)t\big) = \{0\} \quad \text{for all} \quad u, v, t \in \mathcal{N}.$$

By 3-primeness of $\mathcal{N}$, we obtain

$$\alpha(u \circ v) = 0 \quad \text{or} \quad -u \circ v \in Z(\mathcal{N}) \quad \text{for all} \quad u, v \in \mathcal{N}. \tag{3.17}$$

Suppose there exist two elements $u_0, v_0 \in \mathcal{N}$ such that $-u_0 \circ v_0 \in Z(\mathcal{N})$, then

$$(-u_0 \circ v_0)(x \circ y) = (x(-u_0 \circ v_0) \circ y) = d((x(-u_0 \circ v_0) \circ y)) =$$

$$= d((-u_0 \circ v_0)(x \circ y)) = d((-u_0 \circ v_0))(x \circ y) + \alpha(-u_0 \circ v_0)d(x \circ y) =$$

$$= (-u_0 \circ v_0)(x \circ y) + \alpha(-u_0 \circ v_0)(x \circ y) \quad \text{for all} \quad x, y \in \mathcal{N}.$$

This implies that

$$\alpha(-u_0 \circ v_0)xy = -\alpha(-u_0 \circ v_0)yx \quad \text{for all} \quad x, y \in \mathcal{N}. \tag{3.18}$$

Replacing $y$ by $yt$ in (3.18) and using it again, we obtain

$$\alpha(-u_0 \circ v_0)y(tx - (-x)(-t)) = \{0\} \quad \text{for all} \quad x, y, t \in \mathcal{N}. \tag{3.19}$$

Taking $-x$ instead of $x$ in (3.19), we get

$$\alpha(-u_0 \circ v_0)\mathcal{N}(-tx + xt) = \{0\} \quad \text{for all} \quad x, t \in \mathcal{N}.$$

By 3-primeness of $\mathcal{N}$ and Lemma 2.3, we deduce that $\alpha(-u_0 \circ v_0) = 0$ or $\mathcal{N}$ is a commutative ring.

Since $\alpha$ is an additive map, (3.17) becomes

$$\alpha(u \circ v) = 0 \quad \text{for all} \quad u, v \in \mathcal{N} \quad \text{or} \quad \mathcal{N} \quad \text{is a commutative ring.}$$

Using the fact that $\alpha$ is onto homomorphism, we deduce that

$$u \circ v = 0 \quad \text{for all} \quad u, v \in \mathcal{N} \text{ or } \mathcal{N} \text{ is a commutative ring.}$$

By using Lemma 2.2, we conclude that $\mathcal{N}$ is a commutative ring. Returning to our assumptions and using 2-torsion freeness of $\mathcal{N}$, we obtain $d(xy) = xy$ for all $x,\ y \in \mathcal{N}$. By definition of $d$, we get $d(x)y + \alpha(x)d(y) = xy$ for all $x,\ y \in \mathcal{N}$. Replacing $x$ by $xz$, we obtain $d(xz)y + \alpha(xz)d(y) = xzy$ for all $x, y, z \in \mathcal{N}$, which means that $\alpha(xz)d(y) = 0$ for all $x, y, z \in \mathcal{N}$. Since $\alpha$ is an onto homomorphism, the last expression becomes $xzd(y) = 0$ for all $x, y, z \in \mathcal{N}$. Hence $x\mathcal{N}d(y) = \{0\}$ for all $x, y \in \mathcal{N}$. By 3-primeness of $\mathcal{N}$, we obtain that $d = 0$; a contradiction.

(ii) Assume that $d([x, y]) = x \circ y$ for all $x,\ y \in \mathcal{N}$. Since $\mathcal{N}$ is 2-torsion free, in particular, for $x = y$, we find that $x^2 = 0$ for all $x \in \mathcal{N}$. This implies that $x(x + y)^2 = 0$ for all $x,\ y \in \mathcal{N}$, hence by a simple calculation, we obtain $xyx = 0$ for all $x, y \in \mathcal{N}$. By 3-primeness of $\mathcal{N}$, we conclude that $\mathcal{N} = \{0\}$; a contradiction.

(iii) Using the same techniques as used in (i) and (ii), we obtain the required result. The following example shows that the existence of ”3-primeness” in the hypotheses of Theorems 3.2 and 3.3 is not superfluous.

***Example.*** Let $S$ be a zero symmetric left near-ring and

$$\mathcal{N} = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \middle| a, b, c \in S \right\}.$$

Then it can be easily seen that $\mathcal{N}$ is a zero-symmetric left near-ring which is not 3-prime. Define maps $d,\ \alpha : \mathcal{N} \to \mathcal{N}$ such that

$$d\begin{pmatrix} 0 & a & b \\ 0 & 0 & 0 \\ 0 & c & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & c & 0 \end{pmatrix}, \qquad \alpha\begin{pmatrix} 0 & a & b \\ 0 & 0 & 0 \\ 0 & c & 0 \end{pmatrix} = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $d$ is a $(1, \alpha)$-derivation satisfying $d([x, y]) = 0$ and $d(x \circ y) = 0$. However, $\mathcal{N}$ is not a commutative ring.

## References


1. M. Ashraf, A. Shakir, *On $(\sigma, \tau)$-derivations of prime near-rings-II*, Sarajevo J. Math., **4**, № 16, 23 – 30 (2008).
2. H. E. Bell, G. Mason, *On derivations in near-rings. Near-rings and near-fields,* North-Holland Math. Stud., **137** (1987).
3. H. E. Bell, *On derivations in near-rings. II. Nearrings, nearfields and K-loops (Hamburg, 1995),* Math. Appl., **426**, 191 – 197 (1997).
4. H. E. Bell, A. Boua, L. Oukhtite, *Semigroup ideals and commutativity in* 3*-prime near rings*, Commun. Algebra. **43**, 1757 – 1770 (2015).
5. A. Boua, L. Oukhtite, *Semiderivations satisfying certain algebraic identities on prime near-rings,* Asian-Eur. J. Math., **6**, № 3 (2013), 8 p.
6. G. Pilz, *Near-rings, 2nd ed.,* **23**, North Holland, Amsterdam (1983).
7. X. K. Wang, *Derivations in prime near-rings,* Proc. Amer. Math. Soc., **121**, № 2, 361 – 366 (1994).
8. M. S.Samman, L. Oukhtite, A. Boua, A. Raji, *Two sided $\alpha$-derivations in* 3*-prime near-rings,* Rocky Mountain J. Math., **46**, № 4, 1379 – 1393 (2016).


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