UDC 517.5

M. Goyal (IK Gujral Punjab Techn. Univ., Jalandhar, India)

ON COMBINATORIAL EXTENSIONS OF SOME RAMANUJAN'S MOCK THETA FUNCTIONS

ПРО КОМБІНАТОРНІ ПРОДОВЖЕННЯ ДЕЯКИХ ФІКТИВНИХ ТЕТА-ФУНКЦІЙ РАМАНУДЖАНА

Five mock theta functions of S. Ramanujan are combinatorially interpreted by means of certain associated lattice path functions and antihook differences. These results provide new combinatorial interpretations of five mock theta functions of Ramanujan. Using a bijection between the associated lattice path functions and the $(n+t)$-color partitions and then between the associated lattice path functions and the weighted lattice path functions, we extend the works by Agarwal and Agarwal and Rana to five new 3-way combinatorial identities. These results are further extended to 4-way combinatorial identities by using bijection between the $(n+t)$-color partitions and the partitions with certain antihook differences. These interesting results present elegant combinatorial links between Ramanujan's mock theta functions, $(n+t)$-color partitions, weighted lattice paths, associated lattice paths, and antihook differences.

Наведено комбінаторну інтерпретацію п'яти фіктивних тета-функцій С. Рамануджана за допомогою деяких асоційованих ґратчастих функцій шляху та антигачкових різниць. Отримані результати дають нову комбінаторну інтерпретацію п'яти фіктивних тета-функцій Рамануджана. За допомогою взаємно однозначної відповідності між асоційованими ґратчастими функціями шляху та $(n+t)$-кольоровими розбиттями, а також між асоційованими ґратчастими функціями шляху та зваженими ґратчастими функціями шляху узагальнено роботи Агарвала та Агарвала і Рана на випадок п'яти нових 3-шляхових комбінаторних тотожностей. Ці результати потім розширено на випадок 4-шляхових комбінаторних тотожностей за допомогою взаємно однозначної відповідності між $(n+t)$-кольоровими розбиттями та розбиттями з певними антигачковими різницями. Ці цікаві результати встановлюють елегантні комбінаторні зв'язки між фіктивними тета-функціями Рамануджана, $(n+t)$-кольоровими розбиттями, зваженими ґратчастими шляхами, асоційованими ґратчастими шляхами та антигачковими різницями.

1. Introduction. The world of mathematics owe a lot to the pioneering insights of the great Indian mathematician S. Ramanujan. The list of interesting directions pioneered by him is huge. Ramanujan has offered many important discoveries in different fields of mathematics such as in additive number theory, probabilistic number theory, theta functions, exponential sum (now called Ramanujan sum), zeta values, tau functions, modular equations, elliptic functions, magic squares, generating functions, continued fraction, Eulerian series, partitions, combinatorics and many more. His work is still being pursued actively and fruitfully (see, for instance, [10, 17, 19, 22]). The last gift of Ramanujan to the world of mathematics is "The Mock Theta Functions". In particular, Ramanujan gave a list of 17 mock theta functions which he divided into three classes. He defined four 3rd order, ten 5th order and three 7th order mock theta functions. For the definitions and orders of the mock theta functions, the interested readers are referred to [13]. Agarwal in [4, 5] interpreted four mock theta functions of Ramanujan combinatorially using n-color partitions and weighted lattice paths. One more mock theta function of order five had been interpreted combinatorially by Agarwal and Rana [12] using $(n+2)$-color partitions and weighted lattice paths. The purpose of this paper is to further explore these five mock theta functions using associated lattice paths as defined and studied by Anand and Agarwal [14] and antihook differences as defined by Agarwal and Andrews [6].

Let us first recall some definitions:

Definition 1.1 [7]. *A partition with* "$(n+t)$ *copies of* n", $t \geq 0$, *is a partition in which a part of size* n, $n \geq 0$, *can come in* $(n+t)$ *different colors denoted by subscripts:* $n_1, n_2, \ldots, n_{n+t}$. *Note* *that zeros are permitted if and only if t is greater than or equal to one. Also, zeros are not permitted to repeat in any partition.*

***Remark* 1.1.** We note that if we take $t = 0$, then these are nothing but the n-color partitions.

Definition 1.2. *The weighted difference of two parts g_k, h_l $(g \geq h)$ is defined by $g - h - k - l$ and is denoted by $((g_k - h_l))$.*

In [8] the *weighted lattice paths* are described as:

Definition 1.3. *All paths will be of finite length lying in the first quadrant. They will begin on the Y-axis and terminate on the X-axis. Only three moves are allowed at each step*:

northeast: *from (x, y) to $(x + 1, y + 1)$;*

southeast: *from (x, y) to $(x + 1, y - 1)$, only allowed if $y > 0$;*

horizontal: *from $(x, 0)$ to $(x + 1, 0)$, only allowed along X-axis.*

All our lattice paths are either empty or terminate with a southeast step: from $(x, 1)$ to $(x+1, 0)$.

In describing lattice paths, the following terminology is used:

Peak: *Either a vertex on the Y-axis which is followed by a southeast step or a vertex preceded by a northeast step and followed by a southeast step.*

Valley: *A vertex preceded by a southeast step and followed by a northeast step. Note that a southeast step followed by a horizontal step followed by a northeast step does not constitute a valley.*

Mountain: *A section of the path which starts on either the X- or Y-axis, which ends on the X-axis and which does not touch the X-axis anywhere in between the end points. Every mountain has at least one peak and may have more than one.*

Plain: *A section of the path consisting of only horizontal steps which starts either on the Y-axis or at a vertex preceded by a southeast step and ends at a vertex followed by a northeast step.*

Height: *Height of a vertex is its y-coordinate.*

Weight: *Weight of a vertex is its x-coordinate.*

Weight of a Lattice Path: It is the sum of the weights of its peaks.

Anand and Agarwal [14] gave the following description of *associated lattice paths*.

Definition 1.4. *All paths will be of finite length lying in the first quadrant. They will begin on the Y-axis and terminate on the X-axis. Only three moves are allowed at each step*:

northeast: *from (x, y) to $(x + 1, y + 1)$;*

southeast: *from (x, y) to $(x + 1, y - 1)$, only allowed if $y > 0$;*

horizontal: *from (x, y) to $(x+1, y)$, only allowed when the first step is preceded by a northeast step and the last is followed by a southeast step.*

The following terminology is used in describing associated lattice paths:

Truncated Isosceles Trapezoidal Section (*TITS*): *A section of the path which starts on the X-axis with northeast steps followed by horizontal steps and then followed by southeast steps ending on the X-axis forms a TITS.*

Since the lower base lies on X-axis and is not a part of the path, hence the term truncated.

Slant Section (*SS*): *A section of the path consisting of only southeast steps which starts on the Y-axis (origin not included) and ends on the X-axis.*

Height of a slant section: *It is "t" if it starts from $(0, t)$. Clearly, a path can have an SS only in the beginning of the path. An associated lattice path can have at most one SS.*

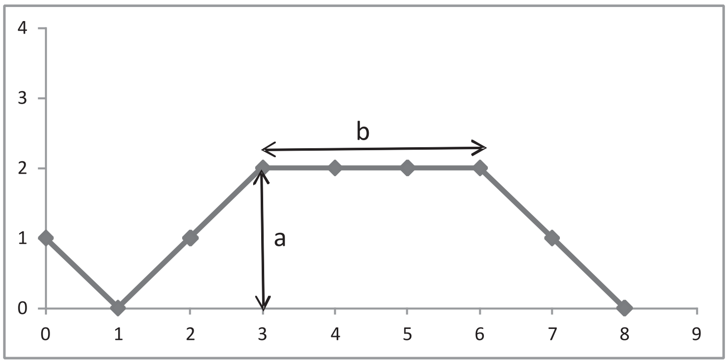


Fig. 1. One SS of height 1 and one TITS with ordered pair $\{2, 3\}$.

Weight of a TITS: *This is defined by representing every TITS by an ordered pair* $\{a, b\}$ *where* a *denotes its altitude and* b *the length of the upper base. Weight of a TITS with ordered pair* $\{a, b\}$ *is* a *units.*

Weight of an Associated Lattice Path: *It is the sum of weights of its TITSs.*

Note that Slant Section is assigned weight zero.

***Example* 1.1.** In this example, the associated lattice path has one SS of height 1 and one TITS with ordered pair $\{2, 3\}$ and its weight is 2 units (see Fig. 1).

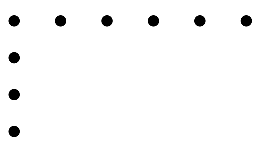
Agarwal and Andrews [6] gave the following definition of antihook differences.

Definition 1.5. *Let* Π *be a partition whose Ferrers graph is embedded in the fourth quadrant. Each node* (x, y) *of the fourth quadrant which is not in the Ferrers graph of* Π *is said to possess an antihook difference* $\xi_x - \zeta_y$ *relative to* Π, *where* ξ_x *is the number of nodes in the* x*th row of the fourth quadrant to the left of node* (x, y) *that are not in the Ferrers graph of* Π *and* ζ_y *is the number of nodes in the* y*th column of the fourth quadrant that lie above node* (x, y) *and are not in the Ferrers graph of* Π.

Definition 1.6. *The nodes* (x, y) *of* Π *for which* $x - y = d$ *are said to lie on diagonal* d.

Definition 1.7. *The rank of a partition is defined as the largest part minus the number of parts.*

Definition 1.8. *A right angle in the Ferrers graph of a partition is called a hook and will be denoted by* $[p, q]$ *if there are* p *nodes in the row and* q *nodes in the column. Thus, for instance,* [6, 4] *represents the hook*



Definition 1.9 [15]. *A two rowed array of nonnegative integers*

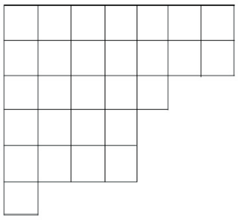
$$\begin{pmatrix} p_1 & p_2 & \dots & p_\nu \\ q_1 & q_2 & \dots & q_\nu \end{pmatrix},$$

where $p_1 \geq p_2 \geq \ldots \geq p_\nu \geq 0,\ q_1 \geq q_2 \geq \ldots \geq q_\nu \geq 0$ *is known as a generalized Frobenius partition or more simply an F-partition of* μ *if* $p_1 + p_2 + \ldots + p_\nu + q_1 + q_2 + \ldots + q_\nu + \nu = \mu$.

For example, $\mu = 28 = 4 + (6 + 5 + 2 + 0) + (5 + 3 + 2 + 1)$ *and the corresponding Frobenius symbol is*

$$\begin{pmatrix} 6 & 5 & 2 & 0 \\ 5 & 3 & 2 & 1 \end{pmatrix}.$$

The corresponding Ferrers graph is



and the associated antihook differences are given by

							0	1	2	3	4	5	
							-1	0	1	2	3	4	
					0	1	0	1	2	3	4	5	
				0	0	1	0	1	2	3	4	5	
				-1	-1	0	-1	0	1	2	3	4	
	0	1	2	1	1	2	1	2	3	4	5	6	
0	0	1	2	1	1	2	1	2	3	4	5	6	
-1	-1	0	1	0	0	1	0	1	2	3	4	5	
-2	-2	-1	0	-1	-1	0	-1	0	1	2	3	4	
-3	-3	-2	-1	-2	-2	-1	-2	-1	0	1	2	3	
-4	-4	-3	-2	-3	-3	-2	-3	-2	-1	0	1	2	
-5	-5	-4	-3	-4	-4	-3	-4	-3	-2	-1	0	1	
-6	-6	-5	-4	-5	-5	-4	-5	-4	-3	-2	-1	0	

The following are the five mock theta functions of S. Ramanujan:

$$\Psi(q) = \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q;q^2)_n}, \tag{1.1}$$

$$F_0(q) = \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q;q^2)_n}, \tag{1.2}$$

$$\Phi_0(q) = \sum_{n=0}^{\infty} q^{n^2}(-q;q^2)_n, \tag{1.3}$$

$$\Phi_1(q) = \sum_{n=1}^{\infty} q^{n^2}(-q;q^2)_{n-1}, \tag{1.4}$$

$$F_1(q) = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q;q^2)_{n+1}}, \tag{1.5}$$

where

$$(a;q)_n = \prod_{i=0}^{\infty} \frac{(1-aq^i)}{(1-aq^{n+i})}.$$

Mock theta functions (1.1)–(1.4) were interpreted by Agarwal [4, 5] in the form of following theorems.

Theorem 1.1. *For $\mu \geq 1$, let $A_1(\mu)$ denote the number of n-color partitions of μ such that* (i) *even parts appear with even subscripts and odd with odd*, (ii) *for some k, k_k is a part and* (iii) *the weighted difference between any two consecutive parts is 0. Let $B_1(\mu)$ denote the number of lattice paths of weight μ which start at $(0,0)$, such that* (iv) *they have no valley above height 0 and* (v) *there is no plain. Then $A_1(\mu) = B_1(\mu)$ for all μ and*

$$\sum_{\mu=1}^{\infty} A_1(\mu)q^{\mu} = \sum_{\mu=1}^{\infty} B_1(\mu)q^{\mu} = \Psi(q). \tag{1.6}$$

Theorem 1.2. *For $\mu \geq 0$, let $A_2(\mu)$ denote the number of n-color partitions of μ such that* (i) *even parts appear with even subscripts and odd with odd subscripts > 1*, (ii) *for some k, k_k is a part and* (iii) *the weighted difference between any two consecutive parts is 0. Let $B_2(\mu)$ denote the number of lattice paths of weight μ which start at $(0,0)$, such that* (iv) *they have no valley above height 0*, (v) *there is no plain and* (vi) *the height of each peak is ≥ 2. Then $A_2(\mu) = B_2(\mu)$ for all μ and*

$$\sum_{\mu=0}^{\infty} A_2(\mu)q^{\mu} = \sum_{\mu=0}^{\infty} B_2(\mu)q^{\mu} = F_0(q). \tag{1.7}$$

Theorem 1.3. *For $\mu \geq 0$, let $A_3(\mu)$ denote the number of n-color partitions of μ such that* (i) *the parts are of the form $(2j-1)_1$ or $(2j)_2$*, (ii) *the minimum part is 1_1 or 2_2 and* (iii) *the weighted difference between any two consecutive parts is 0. Let $B_3(\mu)$ denote the number of lattice paths of weight μ which start at $(0,0)$, such that* (iv) *they have no valley above height 0*, (v) *there is no plain and* (vi) *the height of each peak of odd weight is 1 while that of even weight is 2. Then $A_3(\mu) = B_3(\mu)$ for all μ and*

$$\sum_{\mu=0}^{\infty} A_3(\mu)q^{\mu} = \sum_{\mu=0}^{\infty} B_3(\mu)q^{\mu} = \Phi_0(q). \tag{1.8}$$

Theorem 1.4. *For $\mu \geq 1$, let $A_4(\mu)$ denote the number of n-color partitions of μ such that* (i) *the parts are of the form $(2j-1)_1$ or $(2j)_2$*, (ii) *the minimum part is 1_1 and* (iii) *the weighted difference between any two consecutive parts is 0. Let $B_4(\mu)$ denote the number of lattice paths of weight μ which start at $(0,0)$, such that* (iv) *they have no valley above height 0*, (v) *there is no* *plain,* (vi) *the height of each peak of odd weight is* 1 *while that of even weight is* 2 *and* (vii) *the weight of the first peak is* 1. *Then* $A_4(\mu) = B_4(\mu)$ *for all* μ *and*

$$\sum_{\mu=1}^{\infty} A_4(\mu)q^{\mu} = \sum_{\mu=1}^{\infty} B_4(\mu)q^{\mu} = \Phi_1(q). \tag{1.9}$$

Mock theta function (1.5) was interpreted by Agarwal and Rana [12] in the form of following theorem.

Theorem 1.5. *For* $\mu \geq 0$, *let* $A_5(\mu)$ *denote the number of* $(n+2)$*-color partitions of* μ *such that* (i) *even parts appear with even subscripts and odd with odd subscripts* > 1, (ii) *for some* i, i_{i+2} *is a part and* (iii) *the weighted difference between any two consecutive parts is* 0. *Let* $B_5(\mu)$ *denote the number of lattice paths of weight* μ *which start at* $(0, 2)$, *such that* (iv) *they have no valley above height* 0, (v) *there is no plain and* (vi) *the height of each peak is* ≥ 2. *Then* $A_5(\mu) = B_5(\mu)$ *for all* μ *and*

$$\sum_{\mu=0}^{\infty} A_5(\mu)q^{\mu} = \sum_{\mu=0}^{\infty} B_5(\mu)q^{\mu} = F_1(q). \tag{1.10}$$

In next section, we propose to further extend Theorems 1.1 – 1.5 using associated lattice paths. We will show that certain restricted associated lattice paths are also generated by the extreme right-hand sides of (1.6) – (1.10). This extends Theorems 1.1 – 1.5 to five new 3-way combinatorial identities. In Section 3, using antihook differences we further extend these results to 4-way combinatorial identities. In last section we conclude the results and pose some open problems.

2. Combinatorial interpretations by using associated lattice paths. In this section the combinatorial interpretations of the mock theta functions (1.1) – (1.5) are given in terms of associated lattice paths. These results extend the work of Agarwal [4, 5] and Agarwal and Rana [12] to five new 3-way combinatorial identities.

2.1. Main results.

Theorem 2.1. *For* $\mu \geq 1$, *let* $C_1(\mu)$ *denote the number of associated lattice paths of weight* μ *such that* (i) *for any TITS with ordered pair* $\{a, b\}$, b *does not exceed* a, (ii) *the TITSs are arranged in order of nondecreasing altitudes and the TITSs with same altitude are ordered by the length of their upper base*, (iii) *there is always a TITS with ordered pair* $\{a, a\}$ *and* (iv) *for any two TITSs with respective ordered pairs* $\{a_1, b_1\}$ *and* $\{a_2, b_2\}$ $(a_1 \leq a_2)$, $a_2 - b_2 = a_1 + b_1$. *Then* $A_1(\mu) = B_1(\mu) = C_1(\mu)$ *for all* μ *and*

$$\sum_{\mu=1}^{\infty} A_1(\mu)q^{\mu} = \sum_{\mu=1}^{\infty} B_1(\mu)q^{\mu} = \sum_{\mu=1}^{\infty} C_1(\mu)q^{\mu} = \Psi(q).$$

Theorem 2.2. *For* $\mu \geq 0$, *let* $C_2(\mu)$ *denote the number of associated lattice paths of weight* μ *such that* (i) *for any TITS with ordered pair* $\{a, b\}$, b *does not exceed* a, (ii) *the TITSs are arranged in order of nondecreasing altitudes and the TITSs with same altitude are ordered by the length of their upper base*, (iii) *there is always a TITS with ordered pair* $\{a, a\}$, (iv) *the length of each upper base is greater than* 1 *and* (v) *for any two TITSs with respective ordered pairs* $\{a_1, b_1\}$ *and* $\{a_2, b_2\}$ $(a_1 \leq a_2)$, $a_2 - b_2 = a_1 + b_1$. *Then* $A_2(\mu) = B_2(\mu) = C_2(\mu)$ *for all* μ *and*

$$\sum_{\mu=0}^{\infty} A_2(\mu)q^{\mu} = \sum_{\mu=0}^{\infty} B_2(\mu)q^{\mu} = \sum_{\mu=0}^{\infty} C_2(\mu)q^{\mu} = F_0(q).$$

Theorem 2.3. *For $\mu \geq 0$, let $C_3(\mu)$ denote the number of associated lattice paths of weight μ such that* (i) *for any TITS with ordered pair $\{a, b\}$, b does not exceed a,* (ii) *the TITSs are arranged in order of nondecreasing altitudes and the TITSs with same altitude are ordered by the length of their upper base,* (iii) *if $\{a_1, b_1\}$ is the ordered pair of the first TITS of the path, then $a_1 = b_1 = 1$ or $a_1 = b_1 = 2$,* (iv) *the length of the upper base of a TITS with odd weight is* 1 *while that of even weight is* 2 *and* (v) *for any two TITSs with respective ordered pairs $\{a_1, b_1\}$ and $\{a_2, b_2\}$ $(a_1 \leq a_2)$, $a_2 - b_2 = a_1 + b_1$. Then $A_3(\mu) = B_3(\mu) = C_3(\mu)$ for all μ and*

$$\sum_{\mu=0}^{\infty} A_3(\mu)q^{\mu} = \sum_{\mu=0}^{\infty} B_3(\mu)q^{\mu} = \sum_{\mu=0}^{\infty} C_3(\mu)q^{\mu} = \Phi_0(q).$$

Theorem 2.4. *For $\mu \geq 1$, Let $C_4(\mu)$ denote the number of associated lattice paths of weight μ such that* (i) *for any TITS with ordered pair $\{a, b\}$, b does not exceed a,* (ii) *the TITSs are arranged in order of nondecreasing altitudes and the TITSs with same altitude are ordered by the length of their upper base*, (iii) *if $\{a_1, b_1\}$ is the ordered pair of the first TITS of the path, then $a_1 = b_1 = 1$,* (iv) *the length of the upper base of a TITS with odd weight is* 1 *while that of even weight is* 2 *and* (v) *for any two TITSs with respective ordered pairs $\{a_1, b_1\}$ and $\{a_2, b_2\}$ $(a_1 \leq a_2)$, $a_2 - b_2 = a_1 + b_1$. Then $A_4(\mu) = B_4(\mu) = C_4(\mu)$ for all μ and*

$$\sum_{\mu=1}^{\infty} A_4(\mu)q^{\mu} = \sum_{\mu=1}^{\infty} B_4(\mu)q^{\mu} = \sum_{\mu=1}^{\infty} C_4(\mu)q^{\mu} = \Phi_1(q).$$

Theorem 2.5. *For $\mu \geq 0$, let $C_5(\mu)$ denote the number of associated lattice paths of weight μ such that* (i) *for any TITS with ordered pair $\{a, b\}$, b does not exceed $(a + 2)$,* (ii) *the TITSs are arranged in order of nondecreasing altitudes and the TITSs with same altitude are ordered by the length of their upper base*, (iii) *there is one TITS with ordered pair $\{a, a + 2\}$ or an SS of height* 2, (iv) *the length of each upper base is greater than* 1 *and* (v) *for any two TITSs with respective ordered pairs $\{a_1, b_1\}$ and $\{a_2, b_2\}$ $(a_1 \leq a_2)$, $a_2 - b_2 = a_1 + b_1$. Then $A_5(\mu) = B_5(\mu) = C_5(\mu)$ for all μ and*

$$\sum_{\mu=0}^{\infty} A_5(\mu)q^{\mu} = \sum_{\mu=0}^{\infty} B_5(\mu)q^{\mu} = \sum_{\mu=0}^{\infty} C_5(\mu)q^{\mu} = F_1(q).$$

Detailed proof of Theorem 2.1 is discussed in the next subsection and the outline of the proofs of the remaining theorems are given in Subsection 2.3.

***2.2. Proof of Theorem* 2.1.** The proof comprises of three steps. In first step we will show that the right-hand side of (1.1) generates the associated lattice paths enumerated by $C_1(\mu)$. Then we will show a bijection between the n-color partitions enumerated by $A_1(\mu)$ and the the associated lattice paths enumerated by $C_1(\mu)$. Finally, we will establish a bijection between the weighted lattice paths enumerated by $B_1(\mu)$ and the associated lattice paths enumerated by $C_1(\mu)$.

Step I. We shall prove that

$$\sum_{\mu=1}^{\infty} C_1(\mu)q^{\mu} = \sum_{m=1}^{\infty} \frac{q^{m^2}}{(q;q^2)_m} = \Psi(q). \tag{2.1}$$

In $\dfrac{q^{m^2}}{(q;q^2)_m}$ the factor q^{m^2} generates an associated lattice path having m TITSs such that ith TITS have the ordered pair $\{2i - 1, 1\}$.

For $m = 3$, the path begins as (see Fig. 2):

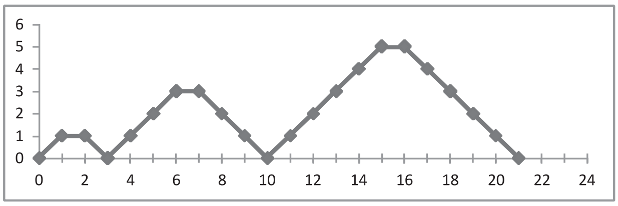


Fig. 2. TITSs for $m = 3$.

In the Fig. 3 we consider two successive TITSs, say, ith and $(i+1)$th. Their corresponding ordered pairs are $\{2i-1, 1\}$ and $\{2i+1, 1\}$, respectively.

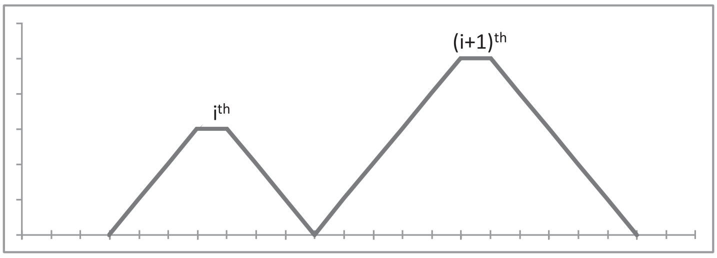


Fig. 3. ith and $(i+1)$th TITSs.

The factor $\dfrac{1}{(q;q^2)_m}$ generates m nonnegative multiples of $(2i-1)$, $1 \leq i \leq m$, say, $\beta_1 \times \times 1$, $\beta_2 \times 3, \ldots, \beta_m \times (2m-1)$. This is encoded by increasing the altitude of ith TITS by $2(\beta_m + \beta_{m-1} + \ldots + \beta_{m-i+2}) + \beta_{m-i+1}$ and the length of the upper base by β_{m-i+1}. So the associated ordered pair becomes $\{2i-1+2(\beta_m+\beta_{m-1}+\ldots+\beta_{m-i+2})+\beta_{m-i+1}, 1+\beta_{m-i+1}\}$.

Fig. 3 now changes to Fig. 4.

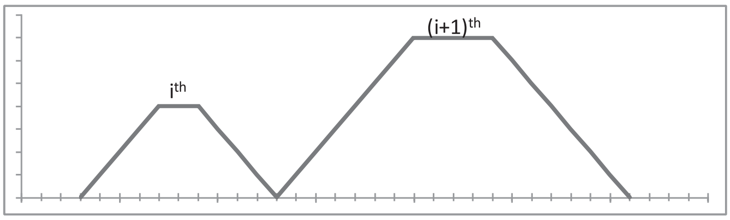


Fig. 4. ith and $(i+1)$th TITSs.

Every associated lattice path enumerated by $C_1(\mu)$ is uniquely generated in this manner. This proves (2.1).

Step II. We now establish a $1-1$ correspondence between the associated lattice paths enumerated by $C_1(\mu)$ and the n-color partitions enumerated by $A_1(\mu)$. We do this by encoding each associated lattice path as the sequence of weights of TITSs with each altitude of the TITS subscripted by the length of the respective upper base. Thus, if we denote the two TITS in Fig. 4 by P_r and Q_s, respectively, then

$$P = (2i-1) + 2(\beta_m + \beta_{m-1} + \ldots + \beta_{m-i+2}) + \beta_{m-i+1},$$

$$r = \beta_{m-i+1} + 1,$$

$$Q = (2i+1) + 2(\beta_m + \beta_{m-1} + \ldots + \beta_{m-i+1}) + \beta_{m-i},$$

$$s = \beta_{m-i} + 1.$$

Clearly, the parity of P and r depends upon β_{m-i+1}. If β_{m-i+1} is odd then both P and r are even and when β_{m-i+1} is even then both P and r are odd. This proves that even parts are appearing with even subscripts and odd with odd subscripts.

The weighted difference of these two parts is $((Q_s - P_r)) = Q - P - r - s = 0$.

The TITS with ordered pair $\{a, a\}$ corresponds to the part of the form a_a or we can say k_k in the corresponding colored partition.

To see the reverse implication, we consider two n-color parts of a partition enumerated by $A_1(\mu)$, say, P_r and Q_s with $Q \geq P$. Clearly $r \leq P$ and $s \leq Q$.

Since P_r and Q_s are the parts of n-color partition enumerated by $A_1(\mu)$, weighted difference equal $((Q_s - P_r)) = 0 \Rightarrow Q - P - r - s = 0 \Rightarrow Q - s = P + r$. Obviously, the part of the form k_k in the colored partition enumerated by $A_1(\mu)$ will correspond to the TITS with ordered pair $\{k, k\}$ or we can say $\{a, a\}$.

Step III. Finally we establish a bijection between the weighted lattice paths enumerated by $B_1(\mu)$ and the associated lattice paths enumerated by $C_1(\mu)$. We do this by mapping each peak of weight a and height b of a weighted lattice path enumerated by $B_1(\mu)$ to a TITS with ordered pair $\{a, b\}$ of an associated lattice path enumerated by $C_1(\mu)$ and conversely. Under this mapping, all the conditions on the weighted lattice paths enumerated by $B_1(\mu)$ are translated to the conditions on the associated lattice paths enumerated by $C_1(\mu)$ and vice-versa. Hence this completes the bijection between the weighted lattice paths enumerated by $B_1(\mu)$ and the associated lattice paths enumerated by $C_1(\mu)$.

***2.3. Outline of the proofs of Theorems* 2.2 – 2.5.** Here, the changes required to prove the remaining theorems are discussed briefly.

Theorem 2.2: This is treated in the same manner as Theorem 2.1. The only difference is now the path begins with m TITSs with ith TITS having the ordered pair $\{4i - 2, 2\}$.

Theorem 2.3: In this case we observe that $\beta_1, \ \beta_2, \ldots, \beta_m$ are 0 or 1 since the factor $(-q, q^2)_m$ generates m nonnegative distinct multiples of $(2i - 1), \ 1 \leq i \leq m$.

Theorem 2.4: This is treated in the same manner as Theorem 2.3. The only difference is the altitude of the first TITS is not increased since the factor $(-q, q^2)_{m-1}$ generates only $m-1$ nonnegative distinct multiples of $(2i - 1), \ 1 \leq i \leq m - 1$.

Theorem 2.5: An appeal to Theorem 2.2, the extra factor $\dfrac{q^{2m}}{1 - q^{2m+1}}$ puts an SS of height 2 in the beginning of the path or a TITS with ordered pair $\{a, a+2\}$. Clearly, it will correspond to a_{a+2} or we can say i_{i+2} part of the corresponding colored partition.

3. Combinatorial interpretations by using antihook differences. Here, we apply antihook differences to interpret mock theta functions (1.1) – (1.5) combinatorially. These results will further extend Theorems 2.1 – 2.5 to 4-way combinatorial identities.

3.1. Main results.

Theorem 3.1. *For $\mu \geq 1$, let $D_1(\mu)$ denote the number of partitions of μ such that* (i) *all antihook differences on diagonal* 0 *are equal to* 0 *or* 1, (ii) *if $[p, q]$ and $[r, s]$ are any two consecutive hooks such that $p > r$ and $q > s$ then $q = r + 1$ and* (iii) *if $[p, q]$ is the last hook, then $q = 0$. Then $A_1(\mu) = B_1(\mu) = C_1(\mu) = D_1(\mu)$ for all μ and*

$$\sum_{\mu=1}^{\infty} D_1(\mu)q^{\mu} = \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q;q^2)_n} = \Psi(q).$$

Theorem 3.2. *For $\mu \geq 0$, let $D_2(\mu)$ denote the number of partitions of μ such that* (i) *all antihook differences on diagonal 0 are equal to 0 or 1,* (ii) *there is no hook with rank ≤ 0,* (iii) *if $[p,q]$ and $[r,s]$ are any two consecutive hooks such that $p > r$ and $q > s$, then $q = r + 1$ and* (iv) *if $[p,q]$ is the last hook, then $p \neq 0,\ q = 0$. Then $A_2(\mu) = B_2(\mu) = C_2(\mu) = D_2(\mu)$ for all μ and*

$$\sum_{\mu=0}^{\infty} D_2(\mu)q^{\mu} = \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q;q^2)_n} = F_0(q).$$

Theorem 3.3. *For $\mu \geq 0$, let $D_3(\mu)$ denote the number of partitions of μ such that* (i) *all antihook differences on diagonal 0 are equal to 0 or 1,* (ii) *all hooks have rank 0 or 1,* (iii) *if $[p,q]$ and $[r,s]$ are any two hooks such that $p > r$ and $q > s$, then $q = r + 1$ and* (iv) *if $[p,q]$ is the last hook, then $q = 0$. Then $A_3(\mu) = B_3(\mu) = C_3(\mu) = D_3(\mu)$ for all μ and*

$$\sum_{\mu=0}^{\infty} D_3(\mu)q^{\mu} = \sum_{n=0}^{\infty} q^{n^2}(-q;q^2)_n = \Phi_0(q).$$

Theorem 3.4. *For $\mu \geq 1$, let $D_4(\mu)$ denote the number of partitions of μ such that* (i) *all antihook differences on diagonal 0 are equal to 0 or 1,* (ii) *all hooks have rank 0 or 1,* (iii) *if $[p,q]$ and $[r,s]$ are any two hooks such that $p > r$ and $q > s$, then $q = r + 1$ and* (iv) *if $[p,q]$ is the last hook, then $p = 0,\ q = 0$. Then $A_4(\mu) = B_4(\mu) = C_4(\mu) = D_4(\mu)$ for all μ and*

$$\sum_{\mu=1}^{\infty} D_4(\mu)q^{\mu} = \sum_{n=1}^{\infty} q^{n^2}(-q;q^2)_{n-1} = \Phi_1(q).$$

Theorem 3.5. *For $\mu \geq 0$, let $D_5(\mu)$ denote the number of partitions of μ such that* (i) *all antihook differences on diagonal 2 are equal to 1 or 2,* (ii) *there is no hook with rank ≥ 2,* (iii) *if $[p,q]$ and $[r,s]$ are any two consecutive hooks such that $p > r$ and $q > s$, then $q = r + 1$ and* (iv) *if $[p,q]$ is the last hook, then $p = 0$ or 2. Then $A_5(\mu) = B_5(\mu) = C_5(\mu) = D_5(\mu)$ for all μ and*

$$\sum_{\mu=0}^{\infty} D_5(\mu)q^{\mu} = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q;q^2)_{n+1}} = F_1(q).$$

Again the proofs of Theorems 3.1 – 3.5 are similar, we will provide a detailed proof of Theorem 3.1 and an outline of the remaining proofs.

***3.2. Proof of Theorem* 3.1.** Let Π be a partition enumerated by $D_1(\mu)$. Let

$$\begin{pmatrix} p_1 & p_2 & \ldots & p_\nu \\ q_1 & q_2 & \ldots & q_\nu \end{pmatrix},$$

where $p_1 \geq p_2 \geq \ldots \geq p_\nu \geq 0,\ q_1 \geq q_2 \geq \ldots \geq q_\nu \geq 0$ and $p_1 + p_2 + \ldots + p_\nu + q_1 + q_2 + \ldots \ldots + q_\nu + \nu = \mu$, be the corresponding Frobenius symbol [15]. Then the antihook difference conditions of Theorem 3.1 are equivalent to

$$p_t \geq q_t, \tag{3.1}$$

$$q_t = p_{t+1} + 1, \tag{3.2}$$

and

$$q_\nu = 0. \tag{3.3}$$

We now establish a bijection between the ordinary partitions enumerated by $D_1(\mu)$ and the n-color partitions enumerated by $A_1(\mu)$. We do this by mapping each column $\begin{pmatrix} p \\ q \end{pmatrix}$ of the Frobenius symbol to a single part g_k of an n-color partition. The mapping is

$$\phi\colon \begin{pmatrix} p \\ q \end{pmatrix} \to \begin{cases} (p+q+1)_{q-p} & \text{if} \quad p < q, \\ (p+q+1)_{p-q+1} & \text{if} \quad p \geq q. \end{cases} \tag{3.4}$$

The inverse mapping ϕ^{-1} is given by

$$\phi^{-1}\colon g_k \to \begin{cases} \begin{pmatrix} (g-k-1)/2 \\ (g+k-1)/2 \end{pmatrix} & \text{if} \quad g \not\equiv k \pmod 2, \\ \begin{pmatrix} (g+k-2)/2 \\ (g-k)/2 \end{pmatrix} & \text{if} \quad g \equiv k \pmod 2. \end{cases} \tag{3.5}$$

Now for any two adjacent columns $\begin{pmatrix} p & r \\ q & s \end{pmatrix}$ in the Frobenius symbol with $\phi \begin{pmatrix} p \\ q \end{pmatrix} = g_k$ and $\phi \begin{pmatrix} r \\ s \end{pmatrix} = h_l$ as defined in (3.5), we have

$$((g_k - h_l)) = \begin{cases} 2q - 2r - 2 & \text{if} \quad p \geq q, r \geq s, \\ 2p - 2r - 1 & \text{if} \quad p < q, r \geq s, \\ 2q - 2s - 1 & \text{if} \quad p \geq q, r < s, \\ 2p - 2s & \text{if} \quad p < q, r < s. \end{cases} \tag{3.6}$$

Clearly (3.1) and (3.4) imply the condition (i) of Theorem 1.1. Also (3.1), (3.3), and (3.4) ensures the condition (ii) of Theorem 1.1 and then (3.2) and only the first line of (3.6) will imply the condition (iii) of Theorem 1.1. To see the reverse implication we note that by condition (i) of Theorem 1.1 $g \equiv k$, $h \equiv l \pmod 2$ and so under ϕ^{-1}

$$p - r = \frac{1}{2}((g_k - h_l)) + k, \tag{3.7}$$

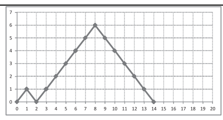
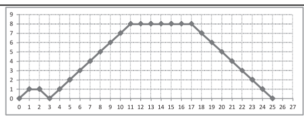
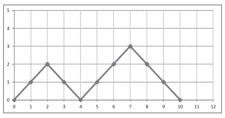
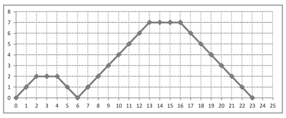
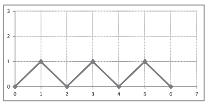
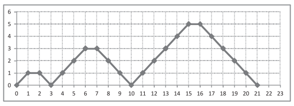
$$q - s = \frac{1}{2}((g_k - h_l)) + l, \tag{3.8}$$

$$p - q = k - 1, \tag{3.9}$$

$$q - r = \frac{1}{2}((g_k - h_l)) + 1. \tag{3.10}$$

Now (3.7) and (3.8) by condition (iii) of Theorem 1.1 guarantee that $p_t > p_{t+1}$ and $q_t > q_{t+1}$. (3.9) implies (3.1) and (3.10) by condition (iii) of Theorem 1.1 implies (3.2). Also, condition (ii) of Theorem 1.1 and only the second line of (3.5) will imply (3.3). This completes the proof of $A_1(\mu) = D_1(\mu)$.

To illustrate the constructed bijections we give an example for $\mu = 9$ shown in the following table.

Partitions enumerated by $A_1(9)$	Lattice paths enumerated by $B_1(9)$	Assosiated lattice paths enumerated by $C_1(9)$	Partitions enumerated by $D_1(9)$	Frobenius symbols for partitions enumerated by $D_1(9)$
$8_6 + 1_1$			7+2	$\begin{pmatrix} 6 & 0 \\ 1 & 0 \end{pmatrix}$
$7_3 + 2_2$			5+3+1	$\begin{pmatrix} 4 & 1 \\ 2 & 0 \end{pmatrix}$
$5_1 + 3_1 + 1_1$			3+3+3	$\begin{pmatrix} 2 & 1 & 0 \\ 2 & 1 & 0 \end{pmatrix}$

***3.3. Outline of the proofs of Theorems* 3.2 – 3.5.** Now let us discuss the essential steps to treat the proofs of Theorems 3.2 – 3.5.

Theorem 3.2: In this case, the antihook difference conditions are equivalent to

$$p_t \geq q_t + 1,$$

$$q_t = p_{t+1} + 1,$$

$$p_\nu \neq 0,$$

and

$$q_\nu = 0.$$

The map ϕ is

$$\phi\colon \begin{pmatrix} p \\ q \end{pmatrix} \to \begin{cases} (p+q+1)_{q-p} & \text{if} \quad p < q+1, \\ (p+q+1)_{p-q+1} & \text{if} \quad p \geq q+1 \end{cases}$$

and ϕ^{-1} is given by

$$\phi^{-1}: g_k \to \begin{cases} \begin{pmatrix} (g-k-1)/2 \\ (g+k-1)/2 \end{pmatrix} & \text{if} \quad g \not\equiv k \,(\operatorname{mod} 2), \\ \begin{pmatrix} (g+k-2)/2 \\ (g-k)/2 \end{pmatrix} & \text{if} \quad g \equiv k \,(\operatorname{mod} 2), \quad k \neq 1. \end{cases}$$

Theorem 3.3: In this case, the antihook difference conditions are equivalent to

$$p_t = q_t \quad \text{or} \quad p_t = q_t + 1,$$
$$q_t = p_{t+1} + 1,$$

and

$$q_\nu = 0.$$

The map ϕ is

$$\phi\colon \begin{pmatrix} p \\ q \end{pmatrix} \to \begin{cases} (p+q+1)_{q-p} & \text{if} \quad p \neq q \quad \text{and} \quad p \neq q+1, \\ (p+q+1)_{p-q+1} & \text{if} \quad p = q \quad \text{or} \quad p = q+1 \end{cases}$$

and ϕ^{-1} is given by

$$\phi^{-1}: g_k \to \begin{cases} \begin{pmatrix} (g-k-1)/2 \\ (g+k-1)/2 \end{pmatrix} & \text{if} \quad g \not\equiv k \,(\operatorname{mod} 2), \quad \text{here} \quad k = 1 \quad \text{or} \quad 2, \\ \begin{pmatrix} (g+k-2)/2 \\ (g-k)/2 \end{pmatrix} & \text{if} \quad g \equiv k \,(\operatorname{mod} 2), \quad \text{here} \quad k = 1 \quad \text{or} \quad 2. \end{cases}$$

Theorem 3.4: In this case, the antihook difference conditions are equivalent to

$$p_t = q_t \quad \text{or} \quad p_t = q_t + 1,$$
$$q_t = p_{t+1} + 1,$$

and

$$p_\nu = q_\nu = 0.$$

The map ϕ is

$$\phi\colon \begin{pmatrix} p \\ q \end{pmatrix} \to \begin{cases} (p+q+1)_{q-p} & \text{if} \quad p \neq q \quad \text{and} \quad p \neq q+1, \\ (p+q+1)_{p-q+1} & \text{if} \quad p = q \quad \text{or} \quad p = q+1 \end{cases}$$

and ϕ^{-1} is given by

$$\phi^{-1} : g_k \to \begin{cases} \begin{pmatrix} (g-k-1)/2 \\ (g+k-1)/2 \end{pmatrix} & \text{if} \quad g \not\equiv k \,(\bmod\, 2), \quad \text{here} \quad k=1 \quad \text{or} \quad 2, \\ \begin{pmatrix} (g+k-2)/2 \\ (g-k)/2 \end{pmatrix} & \text{if} \quad g \equiv k \,(\bmod\, 2), \quad \text{here} \quad k=1 \quad \text{or} \quad 2. \end{cases}$$

Theorem 3.5: Lastly, in this case, we observe that the antihook difference conditions are equivalent to

$$p_t \le q_t + 1,$$

$$p_t = q_{t+1} + 3,$$

and

$$p_\nu = 0 \quad \text{or} \quad 2.$$

The map ϕ is

$$\phi \begin{pmatrix} p \\ q \end{pmatrix} \to \begin{cases} (p+q+1)_{q-p+3} & \text{if} \quad p \le q+1, \quad p \ne 1, \\ (p+q+1)_{p-q-1} & \text{if} \quad p > q+1 \end{cases}$$

and ϕ^{-1} is given by

$$\phi^{-1} : g_k \to \begin{cases} \begin{pmatrix} (g+k+1)/2 \\ (g-k-3)/2 \end{pmatrix} & \text{if} \quad g \not\equiv k+2 \,(\bmod\, 2), \\ \begin{pmatrix} (g-k+2)/2 \\ (g+k-4)/2 \end{pmatrix} & \text{if} \quad g \equiv k+2 \,(\bmod\, 2), \quad g \ne k. \end{cases}$$

4. Conclusion. In literature we find that a single basic series may have many combinatorial interpretations in terms of different combinatorial objects (see, for instance, [1–3, 9, 11, 22–25]). In this paper five mock theta functions of S. Ramanujan have been interpreted combinatorially using associated lattice paths and antihook differences. After Ramanujan many more mock theta functions of different orders have been found by several authors such as in [16, 18, 20, 21, 26, 27]. Gordon and McIntosh [20] found some basic functions which they used to establish modular transformation formulas for eighth order mock theta function. Now it would be of interest if these mock theta functions could also be interpreted combinatorially using associated lattice paths and antihook differences.

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