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A NOTE ON S-NAKAYAMA'S LEMMA

ЗАУВАЖЕННЯ ЩОДО S-ЛЕМИ НАКАЯМИ

We propose an S-version of Nakayama's lemma. Let R be a commutative ring, S a multiplicative subset of R, and M be an S-finite R-module. Also let I be an ideal of R. We show that if there exists $t \in S$ such that $tM \subseteq IM$, then $(t' + a)M = 0$ for some $t' \in S$ and $a \in I$. We also give an analog of Nakayama's lemma for a w-ideal and an S-w-finite R-module, where R is an integral domain. Thus, we generalize the result obtained by Wang and McCasland [Commun. Algebra, **25**, 1285 – 1306 (1997)].

Запропоновано S-версію леми Накаями. Нехай R — комутативне кільце, S — мультиплікативна підмножина R, а M — S-скінченний R-модуль. Крім того, нехай I — ідеал в R. Доведено, що у випадку, коли існує $t \in S$ таке, що $tM \subseteq IM$, маємо $(t' + a)M = 0$ для деяких $t' \in S$ та $a \in I$. Також наведено аналог леми Накаями для w-ідеалу та S-w-скінченного R-модуля, де R є інтегральною множиною. Таким чином, узагальнено результат, що був отриманий Вангом та МакКасландом [Commun. Algebra, **25**, 1285 – 1306 (1997)].

1. Introduction. Let R be a commutative ring with identity and let M be a unitary R-module. Nakayama's lemma is a well-known result, which states that every finitely generated R-module M such that $M = IM$ for some ideal I of R, implies that there exists an $a \in I$ such that $(1+a)M = 0$ [6] (Theorem 2.2). Some generalizations of Nakayama's lemma has been given and studied, in the literatures [1, 3].

In [2], Anderson and Dumitrescu generalized the notion of finitely generated module by introducing the concept of S-finite modules. Let R be a commutative ring with identity, $S \subseteq R$ be a given multiplicative set and M be an R-module. We say that M is S-finite if $sM \subseteq F$ for some finitely generated submodule F of M and some $s \in S$. Note that if S consists of units of R, then M is an S-finite module if and only if M is a finitely generated module.

The first result of this paper is to try to relaxing the condition finitely generated for M with weaker condition (S-finite) and we will study the Nakayama's lemma. Let R be a commutative ring, S be a multiplicative subset of R and M be an S-finite R-module. Let I be an ideal of R. We show that if there exist $t \in S$ such that $tM \subseteq IM$, then $(t' + a)M = 0$ for some $t' \in S$ and $a \in I$. Also, if there exist $t \in S$ such that $tM \subseteq IM + N$, for some submodule N of M, then $(t' + a)M \subseteq N$ for some $t' \in S$ and $a \in I$. Note that if S consists of units of R we find the Nakayama's lemma.

On the other hand, in [4], the authors gived an analogue of the Nakayama's lemma for a w-ideal and a w-module of finite type. First, let us recall the following notions. Let D be an integral domain with quotient field K and let J be an ideal of D. We say that J is a Glaz – Vasconcelos ideal (GV-ideal) if J is finitely generated and $J^{-1} = D$. Let $\mathrm{GV}(D)$ be the set of GV-ideals of D. Following [4], a torsion-free D-module M is called a w-module if $xJ \subseteq M$ for $J \in \mathrm{GV}(D)$ and $x \in M \otimes K$ imply that $x \in M$. M is a w-ideal if M is an ideal of D and is also a w-module. For a torsion-free D-module M, Wang and McCasland defined the w-envelope of M in [4] as $M_w = \{x \in M \otimes K \mid xJ \subseteq M \text{ for some } J \in GV(D)\}$. In particular, if I is a nonzero fractional ideal of D, then $I_w = \{x \in K \,|\, xJ \subseteq I \text{ for some } J \in GV(D)\}$. We say that a torsion-free R-module M is w-finite type if $M = N_w$, for some finitely generated submodule N of M. For a w-finite type R-module M and a proper w-ideal I of D, Wang and McCasland showed that if

$M = (IM)_w$, then $M = 0$. In [5], the authors generalized the notion of w-finite type module by introducing the concept of S-w-finite modules. Let D be an integral domain, S be a multiplicative subset of D and M be a (torsion-free) w-module as a D-module. We say that M is S-w-finite if $sM \subseteq F_w$ for some $s \in S$ and some finitely generated submodule F of M. Note that if S consists of units of D, then M is an S-w-finite module if and only if M is a w-finite module.

The second result of this work gives an analogue of the Nakayama's lemma for a w-ideal and an S-w-finite module. Indeed, let D be an integral domain, S be a multiplicative subset of D and M be an S-w-finite D-module. Let I be a w-ideal of A disjoint with S, we show that, if $tM \subseteq (IM)_w$ for some $t \in S$, then $M = 0$. So we generalize the result of Wang and McCasland [4] (Corollary 2.10).

2. Main results. Let R be a commutative ring, S be a multiplicative subset of R and M be an R-module. Recall from [2] that M is called S-finite if $sM \subseteq F$ for some finitely generated submodule F of M and some $s \in S$. The next result give an S-version of Nakayama's lemma. Let M be an S-finite R-module, I be an ideal of R and N be a submodule of M. We show that if there exist $t \in S$ such that $tM \subseteq IM + N$, then $(t' + a)M \subseteq N$ for some $t' \in S$ and $a \in I$. The demonstration of this general statement reduces to that of the particular case $N = 0$.

Lemma 2.1. *Let R be a commutative ring, S be a multiplicative subset of R and M be an S-finite R-module. Let I be an ideal of R. If there exist $t \in S$ such that $tM \subseteq IM$, then $(t'+a)M = 0$ for some $t' \in S$ and $a \in I$.*

Proof. As M is S-finite, then there exist an $s \in S$ and a finitely generated submodule $F = a_1R + \ldots + a_nR$ of M such that $sM \subseteq F \subseteq M$. We have $stM \subseteq sIM \subseteq IF$. Then, for all $1 \leq i \leq n$, $sta_i = \sum_{j=1}^{n} y_{i,j}a_j$ with $y_{i,j} \in I$. Noting Y the matrix of $y_{i,j}$ and d the determinant of $stI_n - Y$. By Laplace formula we have, for all $1 \leq i \leq n$, $da_i = 0$, hence $d(sM) = 0$, because $sM \subseteq F$. Moreover, by developing the determinant it's easy to see that $d \in (st)^n + I$, then $sd \in s^{n+1}t^n + I$, thus $sd = t' + a$ where $t' = s^{n+1}t^n \in S$ and $a \in I$.

If S included in the set of units of D, we find the following corollary, which is a particular case of Nakayama's lemma $(N = 0)$ [6].

***Corollary* 2.1.** *Let R be a commutative ring, I be an ideal of R and M be a finitely generated module over R. If $M = IM$, then $(1 + a)M = 0$ for some $a \in I$.*

Our next result give an S-version of Nakayama's lemma.

Theorem 2.1. *Let R be a commutative ring, S be a multiplicative subset of R and M be an S-finite R-module. Let I be an ideal of R and N be a submodule of M. If there exist $t \in S$ such that $tM \subseteq IM + N$, then $(t' + a)M \subseteq N$ for some $t' \in S$ and $a \in I$.*

Proof. We put $N' = M/N$. Since M is an S-finite R-module, then N' is also an S-finite R-module. Moreover, $tN' = tM/N \subseteq (IM + N)/N \subseteq IM/N = IN'$. Thus by Lemma 2.1, there exist a $t' \in S$ and an $a \in I$ such that $(t' + a)N' = (t' + a)M/N = 0$, which implies that $(t' + a)M \subseteq N$.

***Corollary* 2.2.** *Let M be a finitely generated module over R, I be an ideal of R and N be a submodule of M such that $M \subseteq IM + N$, then $(1 + a)M \subseteq N$ for some $a \in I$.*

***Remark* 2.1.** Let R be a commutative ring, S be a multiplicative subset of R and M be an S-finite R-module. Let N be a submodule of M and I be an ideal of R contained in the Jacobson radical of R. If there exist $t \in S$ such that $tM \subseteq IM+N$, then by the previous theorem, $(t'+a)M \subseteq N$ for some $t' \in S$ and $a \in I$.

Remark that if S consists of units of R, then $t' + a$ is an unit of R, thus $M = N$.

The next two results are analogues of the S-version of Nakayama's lemma for a w-ideal and an S-w-finite D-module.

Theorem 2.2. *Let D be an integral domain, S be a multiplicative subset of D and M be an S-finite torsion-free D-module. Let I be a w-ideal of D disjoint with S. If $tM_w \subseteq (IM)_w$ for some $t \in S$, then $M = 0$.*

Proof. As M is S-finite, then there exists an $s \in S$ and a finitely generated submodule $F = a_1D + \ldots + a_nD$ of M such that $sM \subseteq F \subseteq M$. Since F is a finitely generated and $tF \subseteq tM \subseteq tM_w \subseteq (IM)_w$, then $tJF \subseteq IM$ for some $J \in \mathrm{GV}(D)$, which implies that $stJM \subseteq IM$. Let $r \in J$. For all $1 \leq i \leq n$, we have $s^2tra_i \in s^2tJM \subseteq sIM \subseteq IF$, then, for all $1 \leq i \leq n$, $s^2tra_i = \sum_{j=1}^{n} y_{i,j}a_j$ with $y_{i,j} \in I$. Noting Y the matrix of $y_{i,j}$ and d the determinant of $s^2trI_n - Y$. By Laplace formula, we have, for all $1 \leq i \leq n$, $da_i = 0$, hence $d(sM) = 0$, because $sM \subseteq F$. By developing the determinant it's easy to see that $d \in (s^2tr)^n + I$, then $sd \in s^{2n+1}t^nr^n + I$. If $M \neq 0$, then $sd = 0$ (since $sdM = 0$) which implies that $t'r^n \in I$ where $t' = s^{2n+1}t^n \in S$. Thus, $t'J \subseteq \sqrt{I}$, because for each $r \in J$, we have $(t'r)^n = (t')^{n-1}t'r^n \in I$. Therefore, by [4] (Proposition 1.6), $t' \in (\sqrt{I})_w = \sqrt{I}$, then there exists a $q \in \mathbb{N}$, such that $(t')^q \in I$ hence $(t')^q \in S \cap I$, a contradiction.

Let D be an integral domain, S be a multiplicative subset of D and M be a (torsion-free) w-module as D-module. We say that M is S-w-finite if $sM \subseteq F_w$ for some $s \in S$ and some finitely generated submodule F of M ([5]).

***Corollary* 2.3.** *Let D be an integral domain, S be a multiplicative subset of D and M be an S-w-finite D-module. Let I be a w-ideal of A disjoint with S. If $tM \subseteq (IM)_w$ for some $t \in S$, then $M = 0$.*

Proof. Since M is S-w-finite, then there exist an $s \in S$ and a finitely generated submodule F of M such that $sM \subseteq F_w \subseteq M$. We have $stF_w \subseteq stM \subseteq s(IM)_w = (sIM)_w \subseteq (IF_w)_w = (IF)_w$ [4] (Proposition 2.8). Then $stF_w \subseteq (IF)_w$. Since F is a finitely generated submodule of M, then F is an S-finite D-module. Moreover, F is a torsion-free D-module. Thus by the previous theorem $F = 0$, which implies that $sM = 0$, hence $M = 0$.

In the particular case when S consists of units of D, we find the result of Wang and McCasland [4] (Corollary 2.10).

***Corollary* 2.4.** *Let D be an integral domain, M be a w-finite type D-module and I be a proper w-ideal of D. If $M = (IM)_w$, then $M = 0$.*

References

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