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SOME SUBCLASSES OF UNIVALENT FUNCTIONS ASSOCIATED WITH k-RUSCHEWEYH DERIVATIVE OPERATOR

ДЕЯКІ ПІДКЛАСИ УНІВАЛЕНТНИХ ФУНКЦІЙ, АСОЦІЙОВАНИХ З ОПЕРАТОРОМ k-ПОХІДНОЇ РУШЕВЕЯ

The purpose of the present paper is to investigate some subordination, other properties and inclusion relations for functions in certain subclasses of univalent functions in the open unit disc which are defined by k-Ruscheweyh derivative operator.

Мета цієї роботи — дослідження деякого підпорядкування та інших властивостей, а також співвідношень включення для функцій деяких підкласів унівалентних функцій у відкритому одиничному диску, які визначаються оператором k-похідної Рушевея.

1. Introduction. Denote $\mathcal{A}$ by the class of analytic functions in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ of the form

$$f(z) = z + \sum_{n=1}^{\infty} a_{n+1} z^{n+1}, \quad z \in \mathbb{U}. \tag{1.1}$$

For function $g \in \mathcal{A}$ given by

$$g(z) = z + \sum_{n=1}^{\infty} b_{n+1} z^{n+1}, \quad z \in \mathbb{U},$$

the Hadamard (or convolution) product of f and g is defined by

$$(f * g)(z) = z + \sum_{n=1}^{\infty} a_{n+1} b_{n+1} z^{n+1} = (g * f)(z), \quad z \in \mathbb{U}.$$

For the functions f and g analytic in $\mathbb{U}$, we say that f is subordinate to g, written $f(z) \prec g(z)$ if there exists a Schwarz function w (which is analytic in $\mathbb{U}$, with $w(0) = 0$, and $|w(z)| < 1$ $(z \in \mathbb{U})$ such that $f(z) = g\big(w(z)\big)$ for all $z \in \mathbb{U}$. Furthermore, if g is univalent in $\mathbb{U}$, then we have the following equivalence (see [1, 6, 8]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \qquad \text{and} \qquad f(\mathbb{U}) \subset g(U).$$

Consider the first-order differential subordination

$$\mathcal{H}\big(g(z), zg'(z)\big) \prec h(z).$$

A univalent function q is called dominant, if $g(z) \prec q(z)$ for all analytic functions g that satisfies this differential subordination. A dominant $\widetilde{q}$ is called the best dominant, if $\widetilde{q}(z) \prec q(z)$ for all dominants q (see [1, 8]).

For $v \in \mathbb{C}$, $k \in \mathbb{R}$ and $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$, the Pochhammer k-symbol $(v)_{n,k}$ is given by (see [3])

$$(v)_{n,k} = v(v+k)(v+2k)\dots\big(v+(n-1)k\big) = \prod_{i=1}^{n}\big(v+(i-1)k\big), \tag{1.2}$$

we define the function $F_{\delta,k}(z)$ by

$$F_{\delta,k}(z) = \frac{z}{(1-z)^{\frac{\delta+k}{k}}}, \qquad \delta > -k, \quad k > 0, \quad z \in \mathbb{U}. \tag{1.3}$$

Corresponding to the function $F_{\delta,k}(z)$, we consider a linear operator $\mathcal{R}_k^\delta\colon \mathcal{A} \to \mathcal{A}$ $(\delta > -k,\ k > 0)$, which is defined by means of the following Hadamard product (or convolution):

$$\mathcal{R}_k^\delta f(z) = F_{\delta,k}(z) * f(z) = z + \sum_{n=1}^{\infty} \frac{(\delta+k)_{n,k}}{(k)_{n,k}} a_{n+1} z^{n+1}, \quad z \in \mathbb{U}. \tag{1.4}$$

It is easily verified from (1.4) that

$$z\big(\mathcal{R}_k^\delta f(z)\big)' = \left(\frac{\delta+k}{k}\right)\mathcal{R}_k^{\delta+k} f(z) - \frac{\delta}{k}\mathcal{R}_k^\delta f(z), \quad k > 0. \tag{1.5}$$

Moreover, for $f \in \mathcal{A}$, we observe that:

(1) $\mathcal{R}_1^\delta f(z) = \mathcal{R}^\delta f(z)\,(\delta > -1)$, where $\mathcal{R}^\delta$ denotes the Ruscheweyh derivative of order δ (see [10]);

(2) $\mathcal{R}_k^0 f(z) = f(z)$ and $\mathcal{R}_k^k f(z) = zf'(z)$.

By using k-Ruscheweyh derivative operator $\mathcal{R}_k^\delta$, we introduce the following subclass of univalent functions.

Definition 1. *For fixed parameters A, B with $-1 \le B < A \le 1$ and $0 \le \lambda < 1$, we say that a function $f \in \mathcal{A}$ is in the class $\mathcal{S}_k^\delta(\lambda; A, B)$ if it satisfies the subordination condition*

$$\frac{1}{1-\lambda}\left(\frac{z\big(\mathcal{R}_k^\delta f(z)\big)'}{\mathcal{R}_k^\delta f(z)} - \lambda\right) \prec \frac{1+Az}{1+Bz},$$

which is equivalent to

$$\left| \frac{\dfrac{z\left(\mathcal{R}_k^\delta f(z)\right)'}{\mathcal{R}_k^\delta f(z)} - 1}{B\dfrac{z\big(\mathcal{R}_k^\delta f(z)\big)'}{\mathcal{R}_k^\delta f(z)} - [B + (A-B)(1-\lambda)]} \right| < 1, \quad z \in \mathbb{U}.$$

We note that

(1) $\mathcal{S}_k^\delta(\lambda; 1, -1) = \mathcal{S}_k^\delta(\lambda) = \left\{ f \in \mathcal{A} \colon \Re\left\{\dfrac{z\big(\mathcal{R}_k^\delta f(z)\big)'}{\mathcal{R}_k^\delta f(z)}\right\} > \lambda \right\}$;

(2) $\mathcal{S}_k^0(\lambda; A, B) = \mathcal{S}^*(\lambda; A, B) = \left\{ f \in \mathcal{A} \colon \dfrac{1}{1-\lambda}\left\{\dfrac{zf'(z)}{f(z)} - \lambda\right\} \prec \dfrac{1+Az}{1+Bz} \right\}$ and $\mathcal{S}_k^0(\lambda; 1, -1) = \mathcal{S}^*(\lambda) = \left\{ f \in \mathcal{A} \colon \Re\left\{\dfrac{zf'(z)}{f(z)}\right\} > \lambda \right\}$;

(3) $\mathcal{S}_k^k(\lambda; A, B) = \mathcal{C}(\lambda; A, B) = \left\{ f \in \mathcal{A} \colon \dfrac{1}{1-\lambda}\left\{1 + \dfrac{zf''(z)}{f'(z)} - \lambda\right\} \prec \dfrac{1+Az}{1+Bz} \right\}$ and $\mathcal{S}_k^k(\lambda; 1, -1) = \mathcal{C}(\lambda) = \left\{ f \in \mathcal{A} \colon \Re\left\{1 + \dfrac{zf''(z)}{f'(z)}\right\} > \lambda \right\}$.

To prove main results, we need the following lemmas.

Lemma 1 [4]. *Let h be a convex function in $\mathbb{U}$ with $h(0)=1$. Suppose also that the function g of the form $g(z)=1+c_nz^n+c_{n+1}z^{n+1}+\ldots$ is analytic in $\mathbb{U}$. Then*

$$g(z)+\frac{zg'(z)}{\sigma}\prec h(z),\quad \Re\{\sigma\}\geq 0,\quad \sigma\neq 0, \tag{1.6}$$

implies

$$g(z)\prec Q(z)=\frac{\sigma}{n}z^{-\frac{\sigma}{n}}\int\limits_0^z t^{\frac{\sigma}{n}-1}h(t)\,dt\prec h(z),$$

and Q is the best dominant of (1.6).

Lemma 2 [12] (see also [8]). *Let ν be a positive measure on the unit interval $[0,1]$. Let $h(z,t)$ be a complex-valued function defined on $\mathbb{U}\times[0,1]$ such that $h(\cdot,t)$ is analytic in $\mathbb{U}$ for each $t\in[0,1]$, and $h(z,\cdot)$ is ν-integrable on $[0,1]$ for all $z\in\mathbb{U}$. In addition, suppose that $\Re\big\{h(z,t)\big\}>0$, $h(-r,t)$ is real, and*

$$\Re\left\{\frac{1}{h(z,t)}\right\}\geq\frac{1}{h(-r,t)},\qquad |z|\leq r<1,\quad t\in[0,1].$$

If the function H is defined by

$$H(z)=\int\limits_0^1 h(z,t)\,d\nu(t),$$

then

$$\Re\left\{\frac{1}{H(z)}\right\}\geq\frac{1}{H(-r)},\quad |z|\leq r<1.$$

Lemma 3 [5]. *Suppose $\lambda\neq 0$ be a real number, $\dfrac{\gamma}{\lambda}>0$, $\eta\in[0,1)$, and let g be an analytic function in $\mathbb{U}$ of the form $g(z)=1+a_nz^n+a_{n+1}z^{n+1}+\ldots$, $z\in\mathbb{U}$, with*

$$P(z)\prec 1+Rz,\qquad R=\frac{\gamma M}{n\lambda+\gamma},$$

where

$$M=M_n(\lambda,\gamma,\eta)=\frac{(1-\eta)|\lambda|\left(1+\dfrac{n\lambda}{\gamma}\right)}{\big|1-\lambda+\lambda\eta\big|+\sqrt{1+\left(1+\dfrac{n\lambda}{\gamma}\right)^2}}.$$

If p is an analytic function in $\mathbb{U}$ of the form $p(z)=1+b_nz^n+b_{n+1}z^{n+1}+\ldots$, $z\in\mathbb{U}$, and satisfies the subordination

$$g(z)\big[1-\lambda+\lambda\left((1-\eta)p(z)+\eta\right)\big]\prec 1+Mz,$$

then $\Re\big\{p(z)\big\}>0$ for all $z\in\mathbb{U}$.

Lemma 4 ([7], Corollary 3.2). *If* $-1 \le B < A \le 1,\ \eta > 0$, *and the complex number* ζ *satisfies*

$$\Re\{\zeta\} \ge -\frac{\eta(1-A)}{1-B},$$

then the following differential equation:

$$q(z) + \frac{zq'(z)}{\eta q(z)+\zeta} = \frac{1+Az}{1+Bz} \quad \textit{with} \quad q(0)=1$$

has a univalent solution in $\mathbb{U}$ *given by*

$$q(z) = \begin{cases} \dfrac{z^{\eta+\zeta}(1+Bz)^{\eta(A-B)/B}}{\eta \displaystyle\int_0^z t^{\eta+\zeta-1}(1+Bt)^{\eta(A-B)/B}\,dt} - \dfrac{\zeta}{\eta}, & \textit{if} \quad B \ne 0, \\ \dfrac{z^{\eta+\zeta}\exp(\eta Az)}{\eta \displaystyle\int_0^z t^{\eta+\zeta-1}\exp(\eta At)\,dt} - \dfrac{\zeta}{\eta}, & \textit{if} \quad B = 0. \end{cases}$$

Moreover, if the function g *is analytic in* $\mathbb{U}$ *and satisfies the following subordination*:

$$g(z) + \frac{zg'(z)}{\eta g(z)+\zeta} \prec \frac{1+Az}{1+Bz}, \tag{1.7}$$

then

$$g(z) \prec q(z) \prec \frac{1+Az}{1+Bz},$$

and q *is the best dominant of* (1.7).

For real or complex numbers $\alpha_1,\ \alpha_2$ and β_1 with $\beta_1 \notin \mathbb{Z}_0^- = \{0,-1,-2,\ldots\}$, the Gauss hypergeometric function ${}_2F_1$ is defined by

$${}_2F_1(\alpha_1,\alpha_2;\beta_1;z) = 1 + \frac{\alpha_1\alpha_2}{\beta_1}\frac{z}{1!} + \frac{\alpha_1(\alpha_1+1)\,\alpha_2(\alpha_2+1)}{\beta_1(\beta_1+1)}\frac{z^2}{2!} + \ldots .$$

We note that the above series converges absolutely for $z \in \mathbb{U}$ and hence represents an analytic function in $\mathbb{U}$ (see, for details, [11], Chapter 14).

Each of the identities (asserted by Lemma 5 below) is well-known (cf., e.g., [11], Chapter 14).

Lemma 5 ([11], Chapter 14). *For real or complex numbers* $\alpha_1,\ \alpha_2$ *and* β_1 *with* $\beta_1 \notin \{0,-1,-2,\ldots\}$, *we have*

$$\int_0^1 t^{\alpha_2-1}(1-t)^{\beta_1-\alpha_2-1}(1-tz)^{-\alpha_1}\,dt = \frac{\Gamma(\alpha_2)\Gamma(\beta_1-\alpha_2)}{\Gamma(\beta_1)}\,{}_2F_1(\alpha_1,\alpha_2;\beta_1;z), \tag{1.8}$$

$$\Re\{\beta_1\} > \Re\{\alpha_2\} > 0,$$

$${}_2F_1(\alpha_1,\alpha_2;\beta_1;z) = (1-z)^{-\alpha_1}\,{}_2F_1\left(\alpha_1,\beta_1-\alpha_2;\beta_1;\frac{z}{z-1}\right), \tag{1.9}$$

and

$${}_2F_1(\alpha_1,\alpha_2;\beta_1;z) = {}_2F_1(\alpha_2,\alpha_1;\beta_1;z). \tag{1.10}$$

2. Properties involving the operator $\mathcal{R}_k^\delta$. Unless otherwise mentioned, we assume throughout this paper that $-1 \le B < A \le 1$, $\delta > -k$, $k > 0$, $\theta > 0$, $0 \le \lambda < 1$ and all the powers are considered the principal ones.

Theorem 1. *If the function $f \in \mathcal{A}$ satisfies the subordination condition*

$$(1-\theta)\frac{\mathcal{R}_k^\delta f(z)}{z} + \theta\frac{\mathcal{R}_k^{\delta+k} f(z)}{z} \prec \frac{1+Az}{1+Bz}, \tag{2.1}$$

then

$$\frac{\mathcal{R}_k^\delta f(z)}{z} \prec Q(z) \prec \frac{1+Az}{1+Bz}, \tag{2.2}$$

where the function Q given by

$$Q(z) = \begin{cases} \dfrac{A}{B} + \left(1 - \dfrac{A}{B}\right)(1+Bz)^{-1}\,{}_2F_1\left(1,1;\dfrac{\delta+(1+\theta)k}{\theta k};\dfrac{Bz}{1+Bz}\right), & B \ne 0, \\ 1 + \dfrac{\delta+k}{\delta+(1+\theta)k}Az, & B = 0, \end{cases}$$

is best dominant of (2.1). *Furthermore,*

$$\Re\left\{\frac{\mathcal{R}_k^\delta f(z)}{z}\right\} > M, \quad z \in \mathbb{U}, \tag{2.3}$$

where

$$M = \begin{cases} \dfrac{A}{B} + \left(1 - \dfrac{A}{B}\right)(1-B)^{-1}\,{}_2F_1\left(1,1;\dfrac{\delta+(1+\theta)k}{\theta k};\dfrac{B}{B-1}\right), & B \ne 0, \\ 1 - \dfrac{\delta+k}{\delta+(1+\theta)k}A, & B = 0. \end{cases}$$

The estimate in (2.3) *is the best possible.*

Proof. Letting

$$g(z) = \frac{\mathcal{R}_k^\delta f(z)}{z}, \quad z \in \mathbb{U}, \tag{2.4}$$

then g is analytic in $\mathbb{U}$. Differentiating (2.4) with respect to z and using identity (1.5) in the resulting relation, we get

$$(1-\theta)\frac{\mathcal{R}_k^\delta f(z)}{z} + \theta\frac{\mathcal{R}_k^{\delta+k} f(z)}{z} = g(z) + \frac{\theta k}{\delta+k}zg'(z) \prec \frac{1+Az}{1+Bz} = h(z).$$

Now, by using Lemma 1 for $\sigma = \dfrac{\delta+k}{\theta k}$, and making a change of variables followed by the use of identities (1.8), (1.9), and (1.10) with $\alpha_1 = 1$, $\alpha_2 = \dfrac{\delta+k}{\theta k}$, and $\beta_1 - \alpha_2 = 1$, we deduce that

$$\frac{\mathcal{R}_k^\delta f(z)}{z} \prec Q(z) = \frac{\delta+k}{\theta k}z^{-\frac{\delta+k}{\theta k}}\int\limits_0^z t^{\frac{\delta+k}{\theta k}-1}\frac{1+At}{1+Bt}\,dt =$$

$$
= \begin{cases} \dfrac{A}{B} + \left(1 - \dfrac{A}{B}\right)(1+Bz)^{-1}\,{}_2F_1\left(1,1;\dfrac{\delta+(1+\theta)k}{\theta k};\dfrac{Bz}{1+Bz}\right), & B \neq 0, \\ 1 + \dfrac{\delta+k}{\delta+(1+\theta)k}Az, & B = 0, \end{cases}
$$

which proves assertion (2.2). From here, to prove inequality (2.3) it is sufficient to show that

$$
\inf\left\{\Re\{Q(z)\}\colon |z|<1\right\} = Q(-1).
$$

Indeed, we have

$$
\Re\left\{\frac{1+Az}{1+Bz}\right\} \geq \frac{1-Ar}{1-Br}, \quad |z| \leq r < 1.
$$

Setting

$$
h\,(z,s) = \frac{1+Asz}{1+Bsz} \qquad \text{and} \qquad d\nu(s) = \frac{\delta+k}{\theta k} s^{\frac{\delta+k}{\theta k}-1}\,ds, \quad 0 \leq s \leq 1,
$$

which is a positive measure on the closed interval $[0,1]$, we get

$$
Q(z) = \int\limits_0^1 h(z,s)\,d\nu(s)
$$

and

$$
\Re\{Q(z)\} \geq \int\limits_0^1 \frac{1-Asr}{1-Bsr}\,d\nu(s) = Q(-r), \quad |z| \leq r < 1.
$$

Letting $r \to 1^-$ in the above inequality we obtain (2.3). Finally, the estimate (2.3) is the best possible as the function Q is the best dominant of (2.1).

Theorem 1 is proved.

For a function $f \in \mathcal{A}$ the generalized Bernardi – Libera – Livingston integral operator $J_\mu\colon \mathcal{A} \to \to \mathcal{A}$ is defined by (see [2])

$$
J_\mu f(z) = \frac{\mu+1}{z^\mu}\int\limits_0^z t^{\mu-1} f(t)\,dt, \quad \mu > -1. \tag{2.5}
$$

It is easy to verify that, for all $f \in \mathcal{A}$, we have

$$
z\left(\mathcal{R}_k^\delta J_\mu f(z)\right)' = (1+\mu)\mathcal{R}_k^\delta f(z) - \mu\mathcal{R}_k^\delta J_\mu f(z). \tag{2.6}
$$

Theorem 2. *If the function $f \in \mathcal{A}$ satisfies the subordination condition*

$$
\frac{\mathcal{R}_k^\delta f(z)}{z} \prec \frac{1+Az}{1+Bz} \tag{2.7}
$$

and J_μ is the integral operator defined by (2.5), *then*

$$\frac{\mathcal{R}_k^\delta J_\mu f(z)}{z} \prec K(z) \prec \frac{1+Az}{1+Bz},$$

where the function K *given by*

$$K(z) = \begin{cases} \dfrac{A}{B} + \left(1 - \dfrac{A}{B}\right)(1+Bz)^{-1}\,{}_2F_1\left(1,1;\mu+2;\dfrac{Bz}{1+Bz}\right), & B \neq 0, \\ 1 + \dfrac{\mu+1}{\mu+2}Az, & B = 0, \end{cases}$$

is the best dominant of (2.7). *Furthermore,*

$$\Re\left\{\frac{\mathcal{R}_k^\delta J_\mu f(z)}{z}\right\} > L, \quad z \in \mathbb{U}, \tag{2.8}$$

where

$$L = \begin{cases} \dfrac{A}{B} + \left(1 - \dfrac{A}{B}\right)(1-B)^{-1}\,{}_2F_1\left(1,1;\mu+2;\dfrac{B}{B-1}\right), & B \neq 0, \\ 1 - \dfrac{\mu+1}{\mu+2}A, & B = 0. \end{cases}$$

The estimate in (2.8) *is the best possible.*

Proof. Let

$$\varphi(z) = \frac{\mathcal{R}_k^\delta J_\mu f(z)}{z}, \quad z \in \mathbb{U}. \tag{2.9}$$

Then φ is analytic in $\mathbb{U}$. Differentiating (2.9) with respect to z and using identity (2.6) in the resulting relation, we get

$$\frac{\mathcal{R}_k^\delta f(z)}{z} = \varphi(z) + \frac{z\varphi'(z)}{\mu+1} \prec \frac{1+Az}{1+Bz}.$$

Employing the same technique that we used in the proof of Theorem 1, the remaining part of the theorem can be proved similarly.

Theorem 3. *If the function* $f \in \mathcal{A}$ *satisfies the condition*

$$\left|(1-\theta)\frac{\mathcal{R}_k^\delta f(z)}{z} + \theta\frac{\mathcal{R}_k^{\delta+k} f(z)}{z} - 1\right| < M_1 = \frac{\delta + (1+\theta)k}{\delta+k}N_1, \quad z \in \mathbb{U}, \tag{2.10}$$

with

$$\frac{\delta + (1+\theta)k}{\delta+k}N_1 \leq 1, \tag{2.11}$$

where

$$N_1 = \min\left\{x \in (0,1) : \Phi(x) = 0\right\} \tag{2.12}$$

and

$$\Phi(x) = \left[\left(\frac{\theta k(1-\lambda)}{\delta+k}\right)^2 - 2\frac{\theta k(1-\lambda)}{\delta+k} - \left(\frac{\delta+(1+\theta)\,k}{\delta+k}\right)^2\right]x^2 -$$

$$-2\frac{\theta k(1-\lambda)}{\delta+k}\left|1-\frac{\theta k(1-\lambda)}{\delta+k}\right|x + \left(\frac{\theta k(1-\lambda)}{\delta+k}\right)^2, \tag{2.13}$$

then $f \in \mathcal{S}_k^{\delta}(\lambda)$.

Proof. Let

$$g(z) = \frac{\mathcal{R}_k^{\delta} f(z)}{z}, \quad z \in \mathbb{U}.$$

Then g is analytic in $\mathbb{U}$. From assumption (2.11), according to Theorem 1 for the special case $A = M_1$ and $B = 0$, we obtain that assumption (2.10) implies

$$g(z) \prec 1 + \frac{\delta+k}{\delta+(1+\theta)k} M_1 z = 1 + N_1 z,$$

which is equivalent to

$$|g(z)-1| < N_1, \quad z \in \mathbb{U}. \tag{2.14}$$

Setting

$$p(z) = \frac{1}{1-\lambda}\left(\frac{z\big(\mathcal{R}_k^{\delta} f(z)\big)'}{\mathcal{R}_k^{\delta} f(z)} - \lambda\right), \tag{2.15}$$

assumption (2.10) could be written as

$$\left|\left(1-\frac{\theta k(1-\lambda)}{\delta+k}\right)g(z) + \frac{\theta k(1-\lambda)}{\delta+k}p(z)g(z) - 1\right| < \frac{\delta+(1+\theta)\,k}{\delta+k}N_1, \quad z \in \mathbb{U}. \tag{2.16}$$

Now, we will show that (2.16) implies $\Re\{p(z)\} > 0$ for all $z \in \mathbb{U}$, that is, $f \in \mathcal{S}_k^{\delta}(\lambda)$.

Supposing that this last inequality is false, since $p(0) = 1$, there exist $z_0 \in \mathbb{U}$ and a number $x \in \mathbb{R}$ such that $p(z_0) = ix$. Therefore, in order to show that (2.16) implies $\Re\{p(z)\} > 0$ for all $z \in \mathbb{U}$, it is sufficient to obtain a contradiction with (2.16), for example,

$$E = \left|\left(1-\frac{\theta k(1-\lambda)}{\delta+k}\right)g(z_0) + \frac{\theta k(1-\lambda)}{\delta+k}p(z_0)g(z_0) - 1\right| \geq \frac{\delta+(1+\theta)k}{\delta+k}N_1. \tag{2.17}$$

Thus, if we put $g(z_0) = u + iv$, then

$$E^2 = \left|\left(1-\frac{\theta k(1-\lambda)}{\delta+k}\right)\varphi(z_0) + \frac{\theta k(1-\lambda)}{\delta+k}P(z_0)\varphi(z_0) - 1\right|^2 =$$

$$= \left(u^2+v^2\right)\left(\frac{\theta k(1-\lambda)}{\delta+k}\right)^2 x^2 + 2xv\frac{\theta k(1-\lambda)}{\delta+k} + \left|\left(1-\frac{\theta k(1-\lambda)}{\delta+k}\right)\varphi(z_0) - 1\right|^2. \tag{2.18}$$

By using (2.14), we have

$$\left|\left(1-\frac{\theta k(1-\lambda)}{\delta+k}\right)\varphi(z_0)-1\right|=\left|\left(1-\frac{\theta k(1-\lambda)}{\delta+k}\right)(\varphi(z_0)-1)-\frac{\theta k(1-\lambda)}{\delta+k}\right|\geq$$

$$\geq\frac{\theta k(1-\lambda)}{\delta+k}-\left|1-\frac{\theta k(1-\lambda)}{\delta+k}\right|N_1. \tag{2.19}$$

Now, we will prove that under our assumptions the next inequality holds:

$$\frac{\theta k(1-\lambda)}{\delta+k}-\left|1-\frac{\theta k(1-\lambda)}{\delta+k}\right|N_1\geq 0. \tag{2.20}$$

Thus, if we denote

$$l=\frac{\theta k(1-\lambda)}{\delta+k}>0,\qquad m=\left(\frac{\delta+(1+\theta)k}{\delta+k}\right)^2>0,$$

then the function Φ given by (2.13) becomes

$$\Phi(x)=(l^2-2l-m)x^2-2l|1-l|x+l^2,$$

and

$$\Phi(0)=l^2>0,\qquad \Phi(1)=-2(\delta+k)(|1-l|+1-l)-m<0.$$

If $l=\dfrac{\theta k(1-\lambda)}{\delta+k}=1$, it is obvious that (2.20) holds for any number N_1. If $l\neq 1$, since

$$\Phi\left(\frac{l}{|1-l|}\right)=-\frac{l^2(1+m)}{|1-l|^2}<0,$$

we deduce that if N_1 is given by (2.12), then inequality (2.20) is also true.

Hence, from (2.18), (2.19) and (2.20), we obtain that

$$E^2-M_1^2\geq(u^2+v^2)\left(\frac{\theta k\,(1-\lambda)}{\delta+k}\right)^2x^2+2xv\frac{\theta k\,(1-\lambda)}{\delta+k}+$$

$$+\left(\frac{\theta k(1-\lambda)}{\delta+k}-\left|1-\frac{\theta k(1-\lambda)}{\delta+k}\right|N_1\right)^2-\left(\frac{\delta+(1+\theta)k}{\delta+k}\right)^2N_1^2.$$

Denoting

$$F(x)=(u^2+v^2)\left(\frac{\theta k(1-\lambda)}{\delta+k}\right)^2x^2+2xv\frac{\theta k(1-\lambda)}{\delta+k}+$$

$$+\left(\frac{\theta k(1-\lambda)}{\delta+k}-\left|1-\frac{\theta k(1-\lambda)}{\delta+k}\right|N_1\right)^2-\left(\frac{\delta+(1+\theta)k}{\delta+k}\right)^2N_1^2,$$

since $(u^2+v^2)\left(\dfrac{\theta k(1-\lambda)}{\delta+k}\right)^2>0$, it follows that the inequality $F(x)\geq 0$ holds for all $x\in\mathbb{R}$, if and only if the discriminant $\Delta\leq 0$, that is,

$$\Delta = 4\left\{v^2\left(\frac{\theta k(1-\lambda)}{\delta+k}\right)^2 - (u^2+v^2)\left(\frac{\theta k\,(1-\lambda)}{\delta+k}\right)^2 \times\right.$$

$$\left.\times\left[\left(\frac{\theta k(1-\lambda)}{\delta+k} - \left|1-\frac{\theta k(1-\lambda)}{\delta+k}\right| N_1\right)^2 - \left(\frac{\delta+(1+\theta)k}{\delta+k}\right)^2 N_1^2\right]\right\} \le 0,$$

which is equivalent to

$$v^2\left[1-\left(\frac{\theta k(1-\lambda)}{\delta+k} - \left|1-\frac{\theta k(1-\lambda)}{\delta+k}\right| N_1\right)^2 + \left(\frac{\delta+(1+\theta)k}{\delta+k}\right)^2 N_1^2\right] \le$$

$$\le u^2\left[\left(\frac{\theta k(1-\lambda)}{\delta+k} - \left|1-\frac{\theta k(1-\lambda)}{\delta+k}\right| N_1\right)^2 - \left(\frac{\delta+(1+\theta)k}{\delta+k}\right)^2 N_1^2\right]. \tag{2.21}$$

Putting $\varphi(z_0) = 1+\rho\, e^{i\varepsilon}$ for some $\varepsilon \in \mathbb{R}$, it is easy to show that

$$\frac{v^2}{u^2} = \frac{\rho^2\sin^2\varepsilon}{(1+\rho\cos\varepsilon)^2} \le \frac{\rho^2}{1-\rho^2}, \quad \varepsilon\in\mathbb{R}.$$

From (2.14) we have $\rho \le N_1 < 1$ and, by using the above inequality, we obtain

$$\frac{v^2}{u^2} \le \frac{\rho^2}{1-\rho^2} \le \frac{N_1^2}{1-N_1^2}. \tag{2.22}$$

Since the function

$$T(\rho) = \frac{\rho^2}{1-\rho^2}, \quad \rho\in[0,1),$$

is a strictly increasing function on $[0,1)$, we need to determine the maximum value of $N_1 \in [0,1)$ (this condition follows from the previous comments) such that

$$N_1^2 \le \left(\frac{\theta k(1-\lambda)}{\delta+k} - \left|1-\frac{\theta k(1-\lambda)}{\delta+k}\right| N_1\right)^2 - \left(\frac{\delta+(1+\theta)k}{\delta+k}\right)^2 N_1^2.$$

A simple computation shows that this value is given by (2.11), where Φ has the form (2.13). According to the above reasons, from (2.22) it follows that

$$\frac{v^2}{u^2} \le \frac{\rho^2}{1-\rho^2} \le \frac{N_1^2}{1-N_1^2} \le \frac{\left(\frac{\theta k(1-\lambda)}{\delta+k} - \left|1-\frac{\theta k(1-\lambda)}{\delta+k}\right| N_1\right)^2 - \left(\frac{\delta+(1+\theta)k}{\delta+k}\right)^2 N_1^2}{1-\left(\frac{\theta k\,(1-\lambda)}{\delta+k} - \left|1-\frac{\theta k(1-\lambda)}{\delta+k}\right| N_1\right)^2 + \left(\frac{\delta+(1+\theta)\,k}{\delta+k}\right)^2 N_1^2},$$

hence

$$\frac{v^2}{u^2} \le \frac{\left(\frac{\theta k(1-\lambda)}{\delta+k} - \left|1-\frac{\theta k(1-\lambda)}{\delta+k}\right| N_1\right)^2 - \left(\frac{\delta+(1+\theta)k}{\delta+k}\right)^2 N_1^2}{1-\left(\frac{\theta k\,(1-\lambda)}{\delta+k} - \left|1-\frac{\theta k(1-\lambda)}{\delta+k}\right| N_1\right)^2 + \left(\frac{\delta+(1+\theta)\,k}{\delta+k}\right)^2 N_1^2},$$

which is equivalent to (2.21), that is, $\Delta \le 0$. Therefore, (2.17) holds.

Theorem 3 is proved.

Theorem 4. *If the function* $f \in \mathcal{A}$ *such that* $\dfrac{\mathcal{R}_k^\delta f(z)}{z} \neq 0$ *for all* $z \in \mathbb{U}$ *and satisfies the subordination condition*

$$(1-\theta)\left(\frac{\mathcal{R}_k^\delta f(z)}{z}\right)^\mu + \theta\frac{\mathcal{R}_k^{\delta+k} f(z)}{\mathcal{R}_k^\delta f(z)}\left(\frac{\mathcal{R}_k^\delta f(z)}{z}\right)^\mu \prec 1 + M_2 z, \tag{2.23}$$

where

$$M_2 = \begin{cases} \dfrac{\dfrac{(1-\lambda)\theta k}{\delta+k}\left(1+\dfrac{\theta k}{\mu(\delta+k)}\right)}{\left|1-\theta\dfrac{(1-\lambda)k}{\delta+k}\right| + \sqrt{1+\left(1+\dfrac{\theta k}{\mu(\delta+k)}\right)^2}}, & \mu > 0, \\ \dfrac{(1-\lambda)\theta k}{\delta+k}, & \mu = 0, \end{cases}$$

then $f \in \mathcal{S}_k^\delta(\lambda)$.

Proof. If $\mu = 0$, then assumption (2.23) is equivalent to

$$\frac{z\big(\mathcal{R}_k^\delta f(z)\big)'}{\mathcal{R}_k^\delta f(z)} - \lambda \prec (1-\lambda)(1+z),$$

which implies that $f \in \mathcal{S}_k^\delta(\lambda)$.

If we consider $\mu > 0$, let define the function

$$g(z) = \left(\frac{\mathcal{R}_k^\delta f(z)}{z}\right)^\mu, \quad z \in \mathbb{U}, \tag{2.24}$$

where we choose the principal value of the power function. Then g is analytic in $\mathbb{U}$ with $g(0) = 1$, and differentiating (2.24) with respect to z, we obtain

$$(1-\theta)\left(\frac{\mathcal{R}_k^\delta f(z)}{z}\right)^\mu + \theta\frac{\mathcal{R}_k^{\delta+k} f(z)}{\mathcal{R}_k^\delta f(z)}\left(\frac{\mathcal{R}_k^\delta f(z)}{z}\right)^\mu = g(z) + \frac{\theta k}{\mu(\delta+k)} z g'(z),$$

which in view of Lemma 1 yields

$$g(z) \prec 1 + \frac{\mu(\delta+k)}{\mu(\delta+k)+\theta k} M_2 z.$$

Also, the subordination assumption (2.23) can be written as

$$\left[(1-\theta) + \theta\left(\frac{(1-\lambda)k}{\delta+k} p(z) + \frac{\lambda k+\delta}{\delta+k}\right)\right] g(z) \prec 1 + M_2 z,$$

where $p(z)$ is given by (2.15). Therefore, from Lemma 3 we deduce that $\Re\{p(z)\} > 0,\ z \in \mathbb{U}$.

Theorem 4 is proved.

Remarks. **1.** Putting $\delta = 0$ in Theorem 4, we obtain the result of Liu [5] (Theorem 2.2).

2. Taking $\delta = 0$ and $\theta = 1$ in Theorem 4, we have the result of Liu [5] (Corollary 2.1).

3. Taking $\delta = 0$ and $\theta = \mu = 1$ in Theorem 4, we get the result of Mocanu and Oros [9] (Corollary 2.2, with $n = 1$).

4. Putting $\delta = 0$ and $\theta = \dfrac{1}{1-\lambda}(0 \leq \lambda < 1)$ in Theorem 4, we obtain the result of Liu [5] (Corollary 2.2).

5. Taking $\delta = 0$, $\theta = \dfrac{1}{1-\lambda}$ $(0 \leq \lambda < 1)$ and $\mu = 1$ in Theorem 4, we have the result of Mocanu and Oros [9] (Corollary 2.4).

3. Inclusion relationships for the class $\mathcal{S}_k^\delta(\lambda; A, B)$.

Theorem 5. *If $f \in \mathcal{S}_k^{\delta+k}(\lambda; A, B)$ such that $\mathcal{R}_k^\delta f(z) \neq 0$ for all $z \in \mathbb{U}^* = \mathbb{U} \setminus \{0\}$ and*

$$\left(\frac{\delta}{k} + \lambda\right)(1-B) + (1-\lambda)(1-A) \geq 0, \tag{3.1}$$

then

$$\frac{1}{1-\lambda}\left(\frac{z\big(\mathcal{R}_k^\delta f(z)\big)'}{\mathcal{R}_k^\delta f(z)} - \lambda\right) \prec q_1(z) \prec \frac{1+Az}{1+Bz}, \tag{3.2}$$

where

$$q_1(z) = \frac{1}{1-\lambda}\left(\frac{1}{Q_1(z)} - \frac{\delta}{k} - \lambda\right) \tag{3.3}$$

and

$$Q_1(z) = \begin{cases} \displaystyle\int_0^1 s^{\frac{\delta}{k}} \left(\frac{1+Bzs}{1+Bz}\right)^{\frac{(1-\lambda)(A-B)}{B}} ds, & B \neq 0, \\ \displaystyle\int_0^1 s^{\frac{\delta}{k}} \exp\big[(1-\lambda)(s-1)Az\big]\, ds, & B = 0, \end{cases}$$

and q_1 is the best dominant of (3.2). *If, in addition to* (3.1),

$$A \leq -\frac{\left(\dfrac{\delta}{k} + \lambda + 1\right)B}{1-\lambda}, \quad -1 \leq B < 0, \tag{3.4}$$

then $f \in \mathcal{S}_k^\delta(\rho_1)$, where

$$\rho_1 = \frac{\delta+k}{k}\left[{}_2F_1\left(1, \frac{(1-\lambda)(B-A)}{B}; \frac{\delta+2k}{k}; \frac{B}{B-1}\right)\right]^{-1} - \frac{\delta}{k}.$$

The result is the best possible.

Proof. Let

$$\phi(z) = \frac{1}{1-\lambda}\left(\frac{z\big(\mathcal{R}_k^\delta f(z)\big)'}{\mathcal{R}_k^\delta f(z)} - \lambda\right), \quad z \in \mathbb{U}. \tag{3.5}$$

Then ϕ is analytic in $\mathbb{U}$ with $\phi(0) = 1$. Using identity (1.5) in (3.5), we get

$$\frac{\delta+k}{k}\frac{\mathcal{R}_k^{\delta+k}f(z)}{\mathcal{R}_k^{\delta}f(z)}=(1-\lambda)\phi(z)+\frac{\delta}{k}+\lambda, \tag{3.6}$$

and differentiating (3.6) with respect to z, we obtain

$$\frac{1}{1-\lambda}\left(\frac{z\left(\mathcal{R}_k^{\delta+k}f(z)\right)'}{\mathcal{R}_k^{\delta+k}f(z)}-\lambda\right)=\phi(z)+\frac{z\phi'(z)}{(1-\lambda)\phi(z)+\dfrac{\delta}{k}+\lambda}\prec\frac{1+Az}{1+Bz}. \tag{3.7}$$

From the assumption (3.1), by using Lemma 4, we get

$$\phi(z)\prec q_1(z)\prec\frac{1+Az}{1+Bz},$$

where q_1 is given (3.3) is the best dominant of (3.7), and this proves (3.2).

Now, we will show that

$$\inf\left\{\Re\left\{q_1(z)\right\}\colon |z|<1\right\}=q_1(-1),$$

or, equivalently,

$$\inf\left\{\Re\left\{\frac{1}{Q_1(z)}\right\}:|z|<1\right\}=\frac{1}{Q_1(-1)}. \tag{3.8}$$

Denoting $\alpha_1=\dfrac{(1-\lambda)(B-A)}{B}$, $\alpha_2=\dfrac{\delta+k}{k}$ and $\beta_1=\dfrac{\delta+2k}{k}$, since $\beta_1>\alpha_2>0$, from (1.8), (1.9), and (1.10), we deduce that

$$Q_1(z)=(1+Bz)^{\alpha_1}\int\limits_0^1 s^{\alpha_2-1}(1+Bzs)^{-\alpha_1}\,ds=\frac{\Gamma(\alpha_2)}{\Gamma(\beta_1)}\,{}_2F_1\left(1,\alpha_1;\beta_1;\frac{Bz}{Bz+1}\right)=\frac{k}{\delta+k}H(z),$$

where

$$H(z)={}_2F_1\left(1,\alpha_1;\beta_1;\frac{Bz}{Bz+1}\right), \tag{3.9}$$

whenever $B\neq 0$. From (3.4), excepting the case of equality, we have $\beta_1>\alpha_1>0$, hence from (1.9) we get

$$H(z)=\int\limits_0^1 h(z,s)d\nu(s),$$

where

$$h(z,s)=\frac{1+Bz}{1+(1-s)Bz}$$

and

$$d\nu(s)=\frac{\Gamma(\beta_1)}{\Gamma(\alpha_1)\Gamma(\beta_1-\alpha_1)}s^{\alpha_1-1}(1-s)^{\beta_1-\alpha_1-1}\,ds,\quad 0\le s\le 1,$$

which is a positive measure on $[0,1]$. Using the fact $-1 \le B < 0$, it is easy to check that

$$\Re\left\{\frac{1}{H(z)}\right\} > 0, \quad z \in \mathbb{U},$$

$$h(-r,s) \in \mathbb{R}, \quad 0 \le r < 1; s \in [0,1],$$

$$\Re\left\{\frac{1}{h(z,s)}\right\} \ge \frac{1-(1-s)Br}{1-Br} = \frac{1}{h(-r,s)}, \quad |z| \le r < 1, \quad s \in [0,1].$$

Therefore, according to Lemma 2, we deduce that

$$\Re\left\{\frac{1}{H(z)}\right\} \ge \frac{1}{H(-r)}, \quad |z| \le r < 1,$$

and by letting $r \to 1^-$, taking into the account the relation (3.9), we obtain inequality (3.8).

Further, by taking $A \uparrow -\dfrac{(\delta/k+\lambda+1)\,B}{1-\lambda}$ for the case $A = -\dfrac{(\delta/k+\lambda+1)\,B}{1-\lambda}$, we conclude that (3.8) holds whenever the inequality (3.4) is satisfied, which prove $f \in \mathcal{S}_k^\delta(\rho_1)$. The result is the best possible as the function q_1 is the best dominant of (3.7).

Theorem 5 is proved.

Theorem 6. *If $f \in \mathcal{S}_k^\delta(\lambda; A, B)$ such that $\mathcal{R}_k^\delta J_\mu f(z) \ne 0$ for all $z \in \mathbb{U}^*$ and*

$$(\lambda+\mu)(1-B) + (1-\lambda)(1-A) \ge 0, \tag{3.10}$$

then $J_\mu f(z) \in \mathcal{S}_k^\delta(\lambda; A, B)$, where the operator J_μ is defined by (2.5). *Furthermore, if $f \in$ $\in \mathcal{S}_k^\delta(\lambda; A, B)$, then*

$$\frac{1}{1-\lambda}\left(\frac{z\left(\mathcal{R}_k^\delta J_\mu f(z)\right)'}{\mathcal{R}_k^\delta J_\mu f(z)} - \lambda\right) \prec q_3(z) \prec \frac{1+Az}{1+Bz}, \tag{3.11}$$

where

$$q_3(z) = \frac{1}{1-\lambda}\left(\frac{1}{Q_3(z)} - \lambda - \mu\right)$$

and

$$Q_3(z) = \begin{cases} \displaystyle\int_0^1 s^\mu \left(\frac{1+Bzs}{1+Bz}\right)^{\frac{(1-\lambda)(A-B)}{B}} ds, & B \ne 0, \\ \displaystyle\int_0^1 s^\mu \exp\left[(1-\lambda)(s-1)Az\right] ds, & B = 0, \end{cases}$$

and q_3 is the best dominant of (3.11). *If, in addition to* (3.10),

$$A \le -\frac{(\lambda+\mu+1)B}{1-\lambda} \quad \text{with} \quad -1 \le B < 0,$$

then $J_\mu f(z) \in \mathcal{S}_k^\delta(\rho_3)$, where

$$\rho_3 = (1+\mu)\left[{}_2F_1\left(1, \frac{(1-\lambda)(B-A)}{B};\ \mu+2;\ \frac{B}{B-1}\right)\right]^{-1} - \mu.$$

The result is best possible.

Proof. Let

$$\phi(z) = \frac{1}{1-\lambda}\left(\frac{z\left(\mathcal{R}_k^{\delta} J_{\mu} f(z)\right)'}{\mathcal{R}_k^{\delta} J_{\mu} f(z)} - \lambda\right), \quad z \in \mathbb{U}. \tag{3.12}$$

Then ϕ is analytic in $\mathbb{U}$ with $\phi(0) = 1$. Using the identity (2.6) in (3.12), we get

$$(1+\mu)\frac{\mathcal{R}_k^{\delta} f(z)}{\mathcal{R}_k^{\delta} J_{\mu} f(z)} = (1-\lambda)\phi(z) + \lambda + \mu. \tag{3.13}$$

Differentiating (3.13) with respect to z, we obtain

$$\frac{1}{1-\lambda}\left(\frac{z\left(\mathcal{R}_k^{\delta} f(z)\right)'}{\mathcal{R}_k^{\delta} f(z)} - \lambda\right) = \phi(z) + \frac{z\phi'(z)}{(1-\lambda)\phi(z) + \lambda + \mu} \prec \frac{1+Az}{1+Bz},$$

and employing the same technique that used in the proof of Theorem 5, the remaining part of the theorem can be proved similarly.

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