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NONHOMOGENEOUS ELLIPTIC KIRCHHOFF EQUATIONS OF THE p-LAPLACIAN TYPE

НЕОДНОРІДНІ ЕЛІПТИЧНІ РІВНЯННЯ КІРХГОФФА ТИПУ p-ЛАПЛАСІАНА

We use variational methods to study the existence and multiplicity of solutions for an nonhomogeneous p-Kirchhoff equation involving the critical Sobolev exponent.

Варіаційні методи застосовуються для вивчення існування та кратності розв'язків неоднорідного еліптичного p-рівняння Кірхгоффа з критичним показником Соболева.

1. Introduction. This paper deals with the existence and multiplicity of solutions to the following Kirchhoff problem with the critical Sobolev exponent

$$\begin{gathered} -\left(a\|u\|^p+b\right)\Delta_p u=u^{p^*-1}+\lambda g(x) \text{ in } \mathbb{R}^N, \\ u\in W^{1,p}\left(\mathbb{R}^N\right), \end{gathered} \tag{$\mathcal{P}_\lambda$}$$

where $N\geq 3,\ 1<p<N,\ \Delta_p$ is the p-Laplacian operator, $\|.\|$ is the usual norm in $W^{1,p}\left(\mathbb{R}^N\right)$ given by

$$\|u\|^p=\int\limits_{\mathbb{R}^N}|\nabla u|^p dx,$$

$p^*=pN/(N-p)$ is the critical Sobolev exponent of the embedding

$$\left(W^{1,p}(\mathbb{R}^N),\|.\|\right)\hookrightarrow\left(L^q\left(\mathbb{R}^N\right),\|.\|_q\right)$$

with $q\in[p,\,p^*]$ and $\|u\|_q^q=\int_{\mathbb{R}^N}|u|^q dx$ is the norm in $L^q\left(\mathbb{R}^N\right)$, a and b are two positive constants, λ is a positive parameter and g belongs to $\left(W^{1,p}\left(\mathbb{R}^N\right)\right)^*$ such that $\int_{\mathbb{R}^N} gu_*\,dx\neq 0$, where u_* is a function defined below in (1), $\left(\left(W^{1,p}(\mathbb{R}^N)\right)^*\right.$ is the dual of $\left.W^{1,p}(\mathbb{R}^N)\right)$.

In recent years, the Kirchhoff-type problems in bounded or unbounded domaine have been studied in many papers by using variational methods. Some interesting studies can be found in [1, 4–6, 8]. Since the Sobolev embedding $\left(W^{1,p}(\mathbb{R}^N),\|.\|\right)\hookrightarrow\left(L^q\left(\mathbb{R}^N\right),\|.\|_q\right)$ is not compact for all $q\in[p,$ $p^*]$, many authors considered the following Kirchhoff-type problem without the critical Sobolev exponent

$$-\left(a\|u\|^p+b\right)\Delta_p u+V(x)\,u=h(x,u) \text{ in } \mathbb{R}^N, \tag{$\mathcal{P}_V$}$$

where $V\in C\left(\mathbb{R}^N,\mathbb{R}\right)$ and $h\in C\left(\mathbb{R}^N\times\mathbb{R},\mathbb{R}\right)$ is subcritical, satisfies sufficiently conditions to show the boundedness of any Palais Smale or Cerami sequence. They imposed some conditions on the weight function $V(x)$ for recovering the compactness of Sobolev embedding (see, for example, [11]). We should mention here that, to the best of our knowledge, there is no result concerning Kirchhoff equations of p-Laplacian type with the presence of nonlinear term of critical growth and without potential term in higher dimension.

Main result of this paper is the following theorem.

Theorem 1.1. *Assume that* $a > 0,\ b > 0,\ N = 3k,\ p = 2k$, *and* $k \in \mathbb{N}^*$. *Then there exists* $\Lambda_* > 0$ *such that problem* $(\mathcal{P}_\lambda)$ *has at least two nontrivial solutions for any* $\lambda \in (0, \Lambda_*)$.

This paper is organized as follows. In Section 2, we give some technical results which allow us to give a variational approach of our main result that we prove in Section 3.

2. Auxiliary results. In this paper we use the following notation: $X = W^{1,2k}\left(\mathbb{R}^{3k}\right)$ where $k \in \mathbb{N}^*$, $||.||_*$ denotes the norm in X^*, B_ρ is the ball centred at 0 and of radius ρ, $\to$ (resp., $\rightharpoonup$) denotes strong (resp., weak) convergence and $\circ_n(1)$ denotes $\circ_n(1) \to 0$ as $n \to \infty$. S is the best Sobolev constant defined by

$$S = \inf_{u \in W^{1,p}(\mathbb{R}^N)} \frac{\|u\|^p}{\|u\|_{p^*}^p}, \tag{1}$$

it is well known that S is attained in $\mathbb{R}^N$ by a function u_* (see [10]).

Since our approach is variational, we define the functional I_λ by

$$I_\lambda(u) = \frac{a}{4k}||u||^{4k} + \frac{b}{2k}||u||^{2k} - \frac{1}{6k}\|u\|_{6k}^{6k} - \lambda \int\limits_{\mathbb{R}^{3k}} gu\,dx$$

for all $k \in \mathbb{N}^*$ and $u \in X$. It is clear that I_λ is well defined in X and belongs to $C^1(X, \mathbb{R})$.
$u \in X$ is said to be a weak solution of problem $(\mathcal{P}_\lambda)$ if it satisfies

$$\left(a||u||^{2k} + b\right) \int\limits_{\mathbb{R}^{3k}} |\nabla u|^{2k-2} \nabla u \nabla\varphi\,dx - \int\limits_{\mathbb{R}^{3k}} u^{6k-1}\varphi\,dx - \lambda \int\limits_{\mathbb{R}^{3k}} g\varphi\,dx = 0 \quad \text{for all} \quad \varphi \in X.$$

To prove our main result, we need following lemmas.

Lemma 2.1. *Let* $(u_n) \subset X$ *be a* $(PS)_c$ *sequence of* I_λ *for some* $c \in \mathbb{R}$. *Then* $u_n \rightharpoonup u$ *in* X *for some* u *with* $I'_\lambda(u) = 0$.

Proof. We have

$$c + \circ_n(1) = I_\lambda(u_n) \quad \text{and} \quad \circ_n(1) = \left\langle I'_\lambda(u_n), u_n \right\rangle, \tag{2}$$

then

$$c + \circ_n(1) = I_\lambda(u_n) - \frac{1}{6k}\left\langle I'_\lambda(u_n), u_n\right\rangle \geq$$

$$\geq \frac{a}{12k}||u_n||^{4k} + \frac{b}{3k}||u_n||^{2k} - \lambda\frac{6k-1}{6k}\|g\|_*\|u_n\|.$$

Hence, (u_n) is bounded in X. Up to a subsequence if necessary, we obtain

$$u_n \rightharpoonup u \text{ in } X, u_n \rightharpoonup u \text{ in } L^{6k}\left(\mathbb{R}^{3k}\right), u_n \to u \text{ a.e. in } \mathbb{R}^{3k}, \text{ and } \nabla u_n \to \nabla u \text{ a.e. in } \mathbb{R}^{3k}.$$

Thus, $\langle I'_\lambda(u_n), \varphi\rangle = 0$ for all $\varphi \in C_0^\infty\left(\mathbb{R}^{3k}\right)$, which means that $I'_\lambda(u) = 0$.

Lemma 2.2. *There exist positive constants $\Lambda_1,\ \rho_1$ and δ_1 such that for all $\lambda \in (0,\Lambda_1)$ we have*

$$I_\lambda(u)|_{\partial B_{\rho_1}} \geq \delta_1 \text{ and } I_\lambda(u)|_{B_{\rho_1}} \geq -\frac{2k-1}{2k}\lambda^{(4k-1)/(4k-2)}\|g\|_*^{2k/(2k-1)}.$$

Proof. Let $u \in X\setminus\{0\}$ and $\rho = \|u\|$. Then by Sobolev and Hölder inequalities, we have

$$I_\lambda(u) \geq \frac{a}{4k}\rho^{4k} + \frac{b}{2k}\rho^{2k} - \frac{S^{-3}}{6k}\rho^{6k} - \lambda\|g\|_*\rho.$$

By applying the inequality

$$\alpha\beta < \frac{1}{p_1}\alpha^{p_1} + \frac{1}{p_2}\beta^{p_2}$$

for any $\alpha,\ \beta,\ p_1,\ p_2 > 0$ such that $\frac{1}{p_1} + \frac{1}{p_2} = 1$, we get

$$\lambda\|g\|_*\rho = \left(\lambda^{(4k-1)/4k}\|g\|_*\right)\left(\lambda^{1/4k}\rho\right) \leq$$

$$\leq \frac{2k-1}{2k}\lambda^{(4k-1)/(4k-2)}\|g\|_*^{2k/(2k-1)} + \frac{1}{2k}\lambda^{1/2}\rho^{2k}.$$

Therefore, if $\lambda \leq b^2$, then

$$I_\lambda(u) \geq \Psi(\rho) - \frac{2k-1}{2k}\lambda^{(4k-1)/(4k-2)}\|g\|_*^{2k/(2k-1)},$$

where

$$\Psi(\rho) = \frac{a}{4k}\rho^{4k} - \frac{S^{-3}}{6k}\rho^{6k} \text{ and, hence, } \max_{\rho\geq 0}\Psi(\rho) = \Psi\left(\left(aS^3\right)^{1/2k}\right).$$

Taking

$$\Lambda_1 = \min\left\{b^2, \left[\frac{k}{2k-1}\Psi\left(\left(aS^3\right)^{1/2k}\right)\right]^{\frac{4k-2}{4k-1}}\|g\|_*^{\frac{4k}{1-4k}}\right\} \text{ and } \delta_1 = \frac{1}{2}\Psi\left(\left(aS^3\right)^{1/2k}\right).$$

Then the conclusion holds.

Lemma 2.3. *Let $(u_n) \subset X$ be a $(PS)_c$ sequence of I_λ for some $c \in \mathbb{R}$ such that $u_n \rightharpoonup u$ in X. Then*

$$\textit{either } u_n \to u \textit{ or } c \geq I_\lambda(u) + C_{a,b,k,S},$$

where $C_{a,b,k,S} = \left[a + \left(a^2 + 4bS^3\right)^{1/2}\right]\left[\frac{aS^6}{48k}\left[a + \left(a^2+4bS^3\right)^{1/2}\right] + \frac{bS^3}{6k}\right].$

Proof. By the proof of Lemma 2.1, we obtain (u_n) is a bounded sequence in X. Furthermore, if $v_n = u_n - u$, we derive $v_n \rightharpoonup 0$ in X. Then, by using Brezis–Lieb lemma [3], we have

$$\|u_n\|^{2k} = \|v_n\|^{2k} + \|u\|^{2k} + o_n(1) \text{ and } \|u_n\|_{6k}^{6k} = \|v_n\|_{6k}^{6k} + \|u\|_{6k}^{6k} + o_n(1). \tag{3}$$

Putting together (2) and (3), we get

$$c + o_n(1) = I_\lambda(u) + \frac{a}{4k}\|v_n\|^{4k} + \frac{b}{2k}\|v_n\|^{2k} + \frac{a}{2k}\|v_n\|^{2k}\|u\|^{2k} - \frac{1}{6k}\|v_n\|_{6k}^{6k}$$

and

$$o_n(1) = a\|v_n\|^{4k} + b\|v_n\|^{2k} + 2a\|v_n\|^{2k}\|u\|^{2k} - \|v_n\|_{6k}^{6k}. \tag{4}$$

Therefore,

$$c + o_n(1) = I_\lambda(u_n) - \frac{1}{6k}\left\langle I'_\lambda(u_n), u_n\right\rangle =$$

$$= I_\lambda(u) + \frac{a}{12k}\|v_n\|^{4k} + \frac{b}{3k}\|v_n\|^{2k} + \frac{a}{6k}\|v_n\|^{2k}\|u\|^{2k}. \tag{5}$$

Assume that $\|v_n\| \to l > 0$, then by (5) and the Sobolev inequality we obtain

$$S^{-3}l^{6k} \ge al^4 + bl^{2k},$$

this implies that

$$l^{2k} \ge \frac{a}{2}S^3 + \frac{1}{2}S^3\left(a^2 + 4bS^{-3}\right)^{1/2}.$$

From the above inequality and (5), we conclude

$$c \ge I_\lambda(u) + \frac{a}{12k}l^{4k} + \frac{b}{3k}l^{2k} \ge$$

$$\ge I_\lambda(u) + \frac{aS^6}{48k}\left[a + \left(a^2 + 4bS^3\right)^{1/2}\right]^2 + \frac{bS^3}{6k}\left[a + \left(a^2 + 4bS^3\right)^{1/2}\right] =$$

$$= I_\lambda(u) + C_{a,b,k,S}.$$

Lemma 2.3 is proved.

3. Proof of the Theorem 1.1. The proof is given in two parts.

3.1. Existence of a local minimizer. By Lemma 2.2, we define

$$c_1 = \inf\left\{I_\lambda(u); u \in \bar{B}_{\rho_1}\right\}.$$

Since $g \not\equiv 0$, we can choose $\Phi \in C_0^\infty\left(\mathbb{R}^{3k}\setminus\{0\}\right)$ such that $\int_{\mathbb{R}^{3k}} g\Phi\, dx > 0$. Hence, there exists $t_0 > 0$ small enough such that $\|t_0\Phi\| < \rho_1$ and

$$I_\lambda(t_0\Phi) = \frac{a}{4k}t_0^{4k}\|\Phi\|^{4k} + \frac{b}{2k}t_0^{2k}\|\Phi\|^{2k} - \frac{t_0^{6k}}{6k}\|\Phi\|_{6k}^{6k} - \lambda t_0\int\limits_{\mathbb{R}^{3k}} g\Phi\, dx < 0,$$

which implies that $c_1 < 0 = I_\lambda(0)$. Using the Ekeland's variational principle [7], for the complete metric space $\bar{B}_{\rho_1}$ with respect to the norm of X, we obtain the result that there exists a $(\mathrm{PS})_{c_1}$ sequence $(u_n) \subset \bar{B}_{\rho_1}$ such that $u_n \rightharpoonup u_1$ in X for some u_1 with $\|u_1\| \le \rho_1$. Assume $u_n \not\to u_1$ in X, then it follows from Lemma 2.3 that

$$c_1 \ge I_\lambda(u) + C_{a,b,k,S} > c_1,$$

which is a contradiction. Thus u_1 is a nontrivial solution of $(\mathcal{P}_\lambda)$ with negative energy.

3.2. Existence of Mountain Pass type solution. The existence of a Mountain Pass type solution follows immediately from the following lemma.

Lemma 3.1. *Let* $\Lambda_2 > 0$ *such that*

$$-\frac{2k-1}{2k}\lambda^{(4k-1)/(4k-2)}\,\|g\|_*^{2k/(2k-1)} + C_{a,b,k,S} > 0 \qquad \forall\lambda\in(0,\Lambda_2).$$

Then there exist $z_*\in X$ *and* $0<\Lambda_*\le\Lambda_2$ *such that*

$$\sup_{t\ge 0} I_\lambda(tz_*) < c_1 + C_{a,b,k,S} \qquad \forall\lambda\in(0,\Lambda_*).$$

Proof. Since $\int_{\mathbb{R}^{3k}} gu_*\,dx \neq 0$, we can choose $z_*(x)=u_*(x)$ or $z_*(x)=-u_*(x)$ such that $\int_{\mathbb{R}^{3k}} gz_*\,dx > 0$.

We consider functions

$$\Phi_1(t) = \frac{at^{4k}}{4k}\,\|z_*\|^{4k} + \frac{bt^{2k}}{2k}\,\|z_*\|^{2k} - \frac{t^{6k}}{6k}\,\|z_*\|_{6k}^{6k}$$

and

$$\Phi_2(t) = \Phi_1(t) - \lambda t\int\limits_{\mathbb{R}^{3k}} gz_*\,dx.$$

So, for all $\lambda\in(0,\Lambda_2)$, we have

$$\Phi_2(0) = 0 < -\frac{2k-1}{2k}\lambda^{(4k-1)/(4k-2)}\,\|g\|_*^{2k/(2k-1)} + C_{a,b,k,S}.$$

Hence, by the continuity of $\Phi_2(t)$, there exists $t_1>0$ small enough such that

$$\Phi_2(t) < -\frac{2k-1}{2k}\lambda^{(4k-1)/(4k-2)}\,\|g\|_*^{2k/(2k-1)} + C_{a,b,k,S} \qquad \forall\, t\in(0,t_1)\,.$$

On the other hand, the function $\Phi_1(t)$ attains its maximum at

$$t_*^{2k} = \frac{a\,\|z_*\|^{4k} + \left(a^2\,\|z_*\|^{8k} + 4b\,\|z_*\|^{2k}\,\|z_*\|_{6k}^{6k}\right)^{1/2}}{2\,\|z_*\|_{6k}^{6k}}.$$

From the definition of S, we have

$$\frac{at_*^{4k}}{4k}\,\|z_*\|^{4k} = \frac{a}{4k}\,\|z_*\|^{4k}\left[\frac{a\,\|z_*\|^{4k} + \left(a^2\,\|z_*\|^{8k} + 4b\,\|z_*\|^{2k}\,\|z_*\|_{6k}^{6k}\right)^{1/2}}{2\,\|z_*\|_6^6}\right]^2 =$$

$$= \frac{a}{16k}\left[\frac{a\,\|z_*\|^{6k}}{\|z_*\|_{6k}^{6k}} + \left[\frac{a^2\,\|z_*\|^{12k} + 4b\,\|z_*\|^{6k}\,\|z_*\|_{6k}^{6k}}{\|z_*\|_{6k}^{12k}}\right]^{1/2}\right]^2 =$$

$$= \frac{a}{16k}\left[aS^3 + \left[a^2S^6 + 4bS^3\right]^{1/2}\right]^2.$$

Similarly, we obtain

$$\frac{bt_*^{2k}}{2k}\left\|z_*\right\|^{2k}=\frac{b}{4k}\left[aS^3+\left(a^2S^6+4bS^3\right)^{1/2}\right]$$

and

$$\frac{t_*^{6k}}{6k}\left\|z_*\right\|_{6k}^{6k}=\frac{S^{-3}}{48k}\left[aS^3+\left(a^2S^6+4bS^3\right)^{1/2}\right]^3.$$

By the above estimates, we deduce that $\sup_{t\geq 0}\Phi_1(t)\leq C_{a,b,k,S}$.

On the other hand, by using Lemma 2.2, we see that

$$c_1\geq -\frac{2k-1}{2k}\lambda^{(4k-1)/(4k-2)}\left\|g\right\|_*^{2k/(2k-1)}\qquad \forall\,\lambda\in(0,\Lambda_1),$$

furthermore, if

$$\lambda<\left(\frac{2kt_1}{2k-1}\left\|g\right\|_*^{\frac{-2k}{2k-1}}\int\limits_{\mathbb{R}^{3k}}gz_*\,dx\right)^{4k-2},$$

we get

$$c_1>-t_1\lambda\int\limits_{\mathbb{R}^{3k}}gz_*dx.$$

Taking

$$\Lambda_*=\min\left\{\Lambda_1,\Lambda_2,\left(\frac{2kt_1}{2k-1}\left\|g\right\|_*^{\frac{-2k}{2k-1}}\int\limits_{\mathbb{R}^{3k}}gz_*\,dx\right)^{4k-2}\right\}.$$

Then we deduce that

$$\sup_{t\geq 0}I_\lambda(tz_*)<c_1+C_{a,b,k,S}\qquad \forall\,\lambda\in(0,\Lambda_*)\,.$$

Note that $I_\lambda(0)=0$ and $I_\lambda(t_2z_*)<0$ for t_2 large enough, also from Lemma 2.2, we know that

$$I_\lambda\left(u\right)|_{\partial B_{\rho_1}}\geq\delta_1>0\qquad \forall\,\lambda\in(0,\Lambda_1).$$

Then, by the Mountain Pass theorem [2], there exists a $(\mathrm{PS})_{c_2}$ sequence, where

$$c_2=\inf_{\gamma\in\Gamma}\max_{t\in[0,1]}I_\lambda\left(\gamma\left(t\right)\right),$$

with

$$\Gamma=\left\{\gamma\in C(\left[0,1\right],X),\gamma(0)=0\text{ and }\gamma(1)=t_2z_*\right\}.$$

By using Lemma 2.1, we have (u_n) has a subsequence, still denoted by (u_n), such that $u_n\rightharpoonup u_2$ in X for some u_2. Furthermore, we know, by Lemma 3.1, that

$$\sup_{t\geq 0}I_\lambda(tz_*)<C_{a,b,k,S}+c_1\qquad \forall\,\lambda\in(0,\Lambda_*),$$

then, from Lemma 2.3, we deduce that $u_n\rightarrow u_2$ in X. Thus we obtain a critical point u_2 of I_λ satisfying $I_\lambda\left(u_2\right)>0$.

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