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SOLUTION OF THE BOUNDARY-VALUE PROBLEM OF HEAT CONDUCTION WITH PERIODIC BOUNDARY CONDITIONS

РОЗВ'ЯЗОК ГРАНИЧНОЇ ЗАДАЧІ ТЕПЛОПРОВІДНОСТІ З ПЕРІОДИЧНИМИ ГРАНИЧНИМИ УМОВАМИ

We investigate the solution of the inverse problem for a linear two-dimensional parabolic equation with periodic boundary and integral overdetermination conditions. Under certain natural regularity and consistency conditions imposed on the input data, we establish the existence, uniqueness of the solution and its continuous dependence on the data by using the generalized Fourier method. In addition, an iterative algorithm is constructed for the numerical solution of this problem.

Вивчається розв'язок оберненої задачі для лінійного двовимірного параболічного рівняння з періодичними граничними умовами та інтегральними умовами перевизначення. За деяких природних умов регулярності й узгодженості, що накладені на початкові дані, встановлено існування, єдиність розв'язку та його неперервну залежність від даних за допомогою узагальненого методу Фур'є. Крім того, побудовано ітеративний алгоритм для побудови чисельного розв'язку цієї проблеми.

1. Introduction. The study of mathematical models for many important applications such as chemical diffusion, applications in heat conduction processes [5, 8], population dynamics, thermoelasticity, medical science, electrochemistry, engineering, wide scope, chemical engineering [9] and control theory give rise in the two-dimensional parabolic partial differential equation with nonlocal boundary conditions [13, 14, 17].

Inverse problems are the problems that consist of finding an unknown property of an object, or a medium, from the observation of a response of this object, or medium, to a probing signal. Thus, the theory of inverse problems yields a theoretical basis for remote sensing and nondestructive evaluation. For example, if an acoustic plane wave is scattered by an obstacle, and one observes the scattered field far from the obstacle, or in some exterior region, then the inverse problem is to find the shape and material properties of the obstacle. Such problems are important in identification of flying objects (airplanes missiles, etc.), objects immersed inwater (submarines, paces of fish, etc.) and in many other situations. In geophysics one sends an acoustic wave from the surface of the earth and collects the scattered field on the surface for various positions of the source of the field for a fixed frequency, or for several frequencies. The inverse problem is to find the subsurface inhomogeneities. In technology one measures the eigenfrequencies of a piece of a material, and the inverse problem is to find a defect in this material, for example, a hole in a metal. In geophysics the inhomogeneity can be an oil deposit, a cave, a mine. In medicine it may be a tumor or some abnormality in a human body. If one is able to find inhomogeneities in a medium by processing the scattered field on the surface, then one does not have to drill a hole in a medium. This, in turn, avoids expensive and destructive evaluation. The practical advantages of remote sensing are what make the inverse problems important in [20].

There are several methods for the numerical approximation of two-dimensional parabolic inverse problem. In [13], three different implicit finite difference schemes for solving the two-dimensional

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parabolic inverse problem with temperature overspecification are considered. These schemes are developed for identifying the control parameter which produces, at any given time, a desired temperature distribution at a given point in the spatial domain. The numerical methods discussed, are based on the second-order Backward Time Centered Space (BTCS) implicit formula, and the second-order Crank – Nicolson implicit finite difference formula and the fourth-order implicit scheme. These finite difference schemes are unconditionally stable. The implicit formula takes a huge amount of central processor (CPU) time, but its fourth-order accuracy is significant. The results of a numerical experiment are presented, and the accuracy and CPU times needed for each of the methods are discussed and compared. The implicit finite difference schemes use more central processor times than the explicit finite difference techniques, but they are stable for every diffusion number.

Over the last couple of years, considerable efforts have been put in to develop either approximate analytical solution or purely numerical solution to nonlocal boundary-value problems [5, 10 – 12, 15], implemented finite difference scheme to obtain the numerical solution of the one dimensional nonlocal boundary-value problem [12, 13, 16].

The periodic boundary conditions arise from many important applications in heat transfer, life sciences [1 – 4].

In this paper, we prove the existence, the uniqueness and the continuous dependence on the data of the solution and we will develop the numerical solution of two-dimensional diffusion problem with periodic boundary conditions. We will use Fourier method and the finite difference method for two-dimensional inverse parabolic equation [1 – 3].

The paper is organized as follows. In Section 2, the existence and uniqueness of the solution of the problem are proved by using the Fourier method. In Section 3, stability of the solution is shown. In Section 4, the numerical procedure for the solution of the problem is given.

Let $T > 0$ be fixed number and denote by $\Omega := \big\{(x, y, t) : 0 < x < \pi, 0 < y < \pi, 0 < t < T\big\}$.

Consider the problem of finding a pair of functions $\big\{r(t), u(x, y, t)\big\}$ satisfying the following equations:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + r(t) f(x, y, t), \quad (x, y, t) \in \Omega, \tag{1}$$

$$\begin{gathered} u(0, y, t) = u(\pi, y, t), \qquad y \in [0, \pi], \quad t \in [0, T], \\ u(x, 0, t) = u(x, \pi, t), \qquad x \in [0, \pi], \quad t \in [0, T], \end{gathered} \tag{2}$$

$$\begin{gathered} u_x(0, y, t) = u_x(\pi, y, t), \qquad y \in [0, \pi], \quad t \in [0, T], \\ u_y(x, 0, t) = u_y(x, \pi, t), \qquad x \in [0, \pi], \quad t \in [0, T], \end{gathered} \tag{3}$$

$$u(x, y, 0) = \varphi(x, y), \qquad x \in [0, \pi], \quad y \in [0, \pi], \tag{4}$$

$$E(t) = \int\limits_0^\pi \int\limits_0^\pi xyu(x, y, t)\, dx\, dy, \quad t \in [0, T], \tag{5}$$

for a two-dimensional parabolic equation with the periodic boundary conditions. The functions $\varphi(x,y)$ and $f(x,y,t)$ are given functions on $[0,\pi]\times[0,\pi]$ and $\overline{\Omega}$, respectively. In heat propagation in a thin rod in which the law of variation $E(t)$ of the total quantity of heat in the rod is given in [18]. This integral condition in parabolic problems is also called heat moments which are analyzed in [19].

Condition (4) is initial condition, conditions (2) and (3) are periodic Dirichlet and Neumann conditions, respectively.

Problem (1)–(5) will be called an inverse problem, the pair $\{r(t), u(x,y,t)\}$ from the class $C[0,T]\times\big(C^{2,2,1}(\Omega)\cap C^{1,1,0}(\overline{\Omega})\big)$ for which conditions (1)–(5) are satisfied, is called a classical solution of the inverse problem (1)–(5).

The inverse problem of finding the heat source in a parabolic equation has been investigated in many studies for the cases when the unknown heat source is space-dependent in [6, 7] and time-dependent in [5].

Nomenclature:

$$\varphi(x,y) \text{ — initial temprature,}$$

$$r(t) \text{ — unknown coefficient,}$$

$$E(t) \text{ — energy,}$$

$$u(x,y,t) \text{ — temperature distribution,}$$

$$f(x,y,t) \text{ — source function,}$$

$$u_0(t), u_{cmn}(t), u_{csmn}(t), u_{scmn}(t), u_{smn}(t) \text{ — Fourier coefficients,}$$

$$M \text{ — arbitrary constant,}$$

$$M_1, M_2, M_3, M_4, M_5, M_6 \text{ — dimensionless constants,}$$

$$F(t) \text{ — continuous function,} \quad K(t,\tau) \text{ — kernel function,}$$

$$\Omega := \big\{(x,y,t)\colon 0<x<\pi, 0<y<\pi, 0<t<T\big\} \text{ — domain of } x,y,t.$$

2. Existence and uniqueness of the solution of inverse problem. Let us look for the solution of (1)–(5) in the form:

$$u(x,y,t)=\frac{u_0(t)}{4}+\sum_{m,n=1}^{\infty}u_{cmn}(t)\cos 2mx\cos 2ny=$$

$$+\sum_{m,n=1}^{\infty}u_{csmn}(t)\cos 2mx\sin 2ny+\sum_{m,n=1}^{\infty}u_{scmn}(t)\sin 2mx\cos 2ny+$$

$$+\sum_{m,n=1}^{\infty}u_{smn}(t)\sin 2mx\sin 2ny.$$

By applying the standard procedure of Fourier method, we obtain Fourier coefficients:

$$u_0(t) = \varphi_0 + \frac{4}{\pi^2}\int_0^t\int_0^\pi\int_0^\pi r(\tau) f\left(\xi,\eta,\tau\right) d\xi\, d\eta\, d\tau,$$

$$u_{cmn}(t) = \varphi_{cmn} e^{-\left((2m)^2+(2n)^2\right)t} + \frac{4}{\pi^2}\int_0^t\int_0^\pi\int_0^\pi r(\tau) e^{-\left((2m)^2+(2n)^2\right)(t-\tau)} \times$$

$$\times f(\xi,\eta,\tau) \cos 2m\xi \cos 2n\eta\, d\xi\, d\eta\, d\tau,$$

$$u_{csmn}(t) = \varphi_{csmn} e^{-\left((2m)^2+(2n)^2\right)t} + \frac{4}{\pi^2}\int_0^t\int_0^\pi\int_0^\pi r(\tau) e^{-\left((2m)^2+(2n)^2\right)(t-\tau)} \times$$

$$\times f(\xi,\eta,\tau) \cos 2m\xi \sin 2n\eta\, d\xi\, d\eta\, d\tau,$$

$$u_{scmn}(t) = \varphi_{scmn} e^{-\left((2m)^2+(2n)^2\right)t} + \frac{4}{\pi^2}\int_0^t\int_0^\pi\int_0^\pi r(\tau) e^{-\left((2m)^2+(2n)^2\right)(t-\tau)} \times$$

$$\times f(\xi,\eta,\tau) \sin 2m\xi \cos 2n\eta\, d\xi\, d\eta\, d\tau,$$

$$u_{smn}(t) = \varphi_{smn} e^{-\left((2m)^2+(2n)^2\right)t} + \frac{4}{\pi^2}\int_0^t\int_0^\pi\int_0^\pi r(\tau) e^{-\left((2m)^2+(2n)^2\right)(t-\tau)} \times$$

$$\times f(\xi,\eta,\tau) \sin 2m\xi \sin 2n\eta\, d\xi\, d\eta\, d\tau,$$

where

$$\varphi_0 = \frac{4}{\pi^2}\int_0^\pi\int_0^\pi \varphi(x,y)\, dx\, dy, \qquad \varphi_{cmn} = \frac{4}{\pi^2}\int_0^\pi\int_0^\pi \varphi(x,y) \cos 2mx \cos 2ny\, dx\, dy,$$

$$\varphi csmn = \frac{4}{\pi^2}\int_0^\pi\int_0^\pi \varphi(x,y) \cos 2mx \sin 2ny\, dx\, dy,$$

$$\varphi_{scmn} = \frac{4}{\pi^2}\int_0^\pi\int_0^\pi \varphi(x,y) \sin 2mx \cos 2ny\, dx\, dy,$$

$$\varphi_{smn} = \frac{4}{\pi^2}\int_0^\pi\int_0^\pi \varphi(x,y) \sin 2mx \sin 2ny\, dx\, dy.$$

We obtain the solution of the problem (1)–(4) for arbitrary $r(t) \in C[0,T]$ by

$$u(x,y,t) = \frac{1}{4}\left(\varphi_0 + \frac{4}{\pi^2}\int_0^t r(\tau) f_0(\tau) d\xi d\eta d\tau\right) +$$

$$+\sum_{m,n=1}^{\infty}\left(\varphi_{cmn}e^{-((2m)^2+(2n)^2)t}+\frac{4}{\pi^2}\int\limits_0^t r(\tau)e^{-\left((2m)^2+(2n)^2\right)(t-\tau)}f_{cmn}(\tau)d\tau\right)\cos 2mx\cos 2ny+$$

$$+\sum_{m,n=1}^{\infty}\left(\varphi_{csmn}e^{-((2m)^2+(2n)^2)t}+\frac{4}{\pi^2}\int\limits_0^t r(\tau)e^{-((2m)^2+(2n)^2)(t-\tau)}f_{csmn}(\tau)\,d\tau\right)\cos 2mx\sin 2ny+$$

$$+\sum_{m,n=1}^{\infty}\left(\varphi_{scmn}e^{-((2m)^2+(2n)^2)t}+\frac{4}{\pi^2}\int\limits_0^t r(\tau)e^{-\left((2m)^2+(2n)^2\right)(t-\tau)}f_{scmn}(\tau)\,d\tau\right)\sin 2mx\cos 2ny+$$

$$+\sum_{m,n=1}^{\infty}\left(\varphi_{smn}e^{-((2m)^2+(2n)^2)t}+\frac{4}{\pi^2}\int\limits_0^t r(\tau)e^{-\left((2m)^2+(2n)^2\right)(t-\tau)}f_{smn}(\tau)\,d\tau\right)\sin 2mx\sin 2ny, \tag{6}$$

where

$$f_0(t)=\frac{4}{\pi^2}\int\limits_0^{\pi}\int\limits_0^{\pi}f(x,y,t)\,dx\,dy,\qquad f_{cmn}(t)=\frac{4}{\pi^2}\int\limits_0^{\pi}\int\limits_0^{\pi}f(x,y,t)\cos 2mx\cos 2ny\,dx\,dy,$$

$$f_{csmn}(t)=\frac{4}{\pi^2}\int\limits_0^{\pi}\int\limits_0^{\pi}f(x,y,t)\cos 2mx\sin 2ny\,dx\,dy,$$

$$f_{scmn}(t)=\frac{4}{\pi^2}\int\limits_0^{\pi}\int\limits_0^{\pi}f(x,y,t)\sin 2mx\cos 2ny\,dx\,dy,$$

$$f_{smn}(t)=\frac{4}{\pi^2}\int\limits_0^{\pi}\int\limits_0^{\pi}f(x,y,t)\sin 2mx\sin 2ny\,dx\,dy.$$

Theorem 1. *Suppose that the following conditions hold*:

(A_1) $E(t)\in C^1[0,T]$,

(A_2) $\varphi(x,y)\in C^{2,2}\big([0,\pi]\times[0,\pi]\big)$, $\varphi(0,y)=\varphi(\pi,y)$, $\varphi_x(0,y)=\varphi_x(\pi,y)$, $\varphi(x,0)=\varphi(x,\pi)$, $\varphi_y(x,0)=\varphi_y(x,\pi)$ *and*

$$\int\limits_0^{\pi}\int\limits_0^{\pi}xy\varphi(x,y)\,dx\,dy=E(0),$$

(A_3) $f(x,y,t)\in C^{2,2,0}(\overline{\Omega})$, $f(0,y,t)=f(\pi,y,t)$, $f_x(0,y,t)=f_x(\pi,y,t)$, $f(x,0,t)=$ $=f(x,\pi,t)$, $f_y(x,0,t)=f_y(x,\pi,t)$ *and* $\displaystyle\int_0^{\pi}\int_0^{\pi}xyf(x,y,t)\,dx\,dy\neq 0$,

then solution of the system (1)–(5) *has a unique solution.*

Proof. The assumptions $\varphi(0,y)=\varphi(\pi,y)$, $\varphi(x,0)=\varphi(x,\pi)$, $f(0,y,t)=f(\pi,y,t)$, $f(x,0,t)=f(x,\pi,t)$ are consistency conditions which are necessary for solution $u(x,y,t)$ to be in

$C^{2,2,1}(\Omega) \cap C^{1,1,0}(\overline{\Omega})$. Further, under the smoothness assumptions $\varphi(x,y) \in C^{2,2}([0,\pi] \times [0,\pi])$ and $f(x,y,t) \in C^{2,2}([0,\pi] \times [0,\pi])\, \forall t \in [0,T]$, the series (6) and its x,y-partial derivative converge uniformly in $\overline{\Omega}$ since their majorizing sums are absolutely convergent. Therefore, their sums $u(x,y,t), u_x(x,y,t)$ and $u_y(x,y,t)$ are continuous in $\overline{\Omega}$. In addition, the t-partial derivative and xx, yy-second order partial derivative series are uniformly convergent for $t > 0$. Thus, $u(x,y,t) \in C^{2,2,1}(\Omega) \cap C^{1,1,0}(\overline{\Omega})$ and satisfies conditions (1)–(5). In addition, $u_t(x,y,t)$ is continuous in $\overline{\Omega}$ because the majorizing sum of t-partial derivative series is absolutely convergent under the conditions $\varphi_x(0,y) = \varphi_x(\pi,y)$, $\varphi_y(x,0) = \varphi_y(x,\pi)$, $f_x(0,y,t) = f_x(\pi,y,t)$ and $f_y(x,0,t) = f_y(x,\pi,t)$ in $\overline{\Omega}$.

We differentiate equation (5) under the condition (A_1) to obtain

$$E'(t) = \int_0^\pi \int_0^\pi xyu_t(x,y,t)\,dx\,dy. \tag{7}$$

Further, under the consistency assumption $\int_0^\pi \int_0^\pi xy\varphi(x,y)\,dx\,dy = E(0)$, formulas (6), (7) yield the following Volterra integral equation of the second kind:

$$r(t) = F(t) + \int_0^t K(t,\tau) r(\tau)\,d\tau, \quad t \in [0,T],$$

where

$$F(t) = \frac{E'(t) + \dfrac{\pi^2}{4} \sum_{m,n=1}^{\infty} \dfrac{(2m)^2 + (2n)^2}{mn} \varphi_{smn} e^{-\left((2m)^2+(2n)^2\right)t}}{f_0(t) + \dfrac{\pi^2}{4} \sum_{m,n=1}^{\infty} \dfrac{(2m)^2 + (2n)^2}{mn} f_{smn}(t)}, \tag{8}$$

$$K(t,\tau) = \frac{\dfrac{\pi^2}{4} \sum_{m,n=1}^{\infty} \dfrac{(2m)^2 + (2n)^2}{mn} f_{smn}(\tau) e^{-\left((2m)^2+(2n)^2\right)(t-\tau)}}{f_0(t) + \dfrac{\pi^2}{4} \sum_{m,n=1}^{\infty} \dfrac{(2m)^2 + (2n)^2}{mn} f_{smn}(t)}. \tag{9}$$

Under the assumption (A_1)–(A_3) the function $F(t)$ and the kernel function $K(t,\tau)$ are continuous functions in $[0,T]$ and $[0,T] \times [0,T]$, respectively. We obtain a unique function $r(t)$ continuous on $[0,T]$ which, together with the solution of the problem (1)–(4) given by the Fourier series $u(x,y,t)$, form the unique solution of the inverse problem (1)–(5).

Theorem 1 is proved.

3. Continuous dependence of (r,u) upon the data. The following result on continuous dependence on the data of the solution of the inverse problem (1)–(5) holds.

Theorem 2. *If* $\Phi = \{\varphi, E, f\}$ *satisfies the assumptions* (A_1)–(A_3) *of Theorem* 1, *then the solution* (r,u) *of problem* (1)–(5) *depends continuously upon the data* φ, E, f.

Proof. Let $\Phi = \{\varphi, E, f\}$ and $\overline{\Phi} = \{\overline{\varphi}, \overline{E}, \overline{f}\}$ be two sets of the data, which satisfy the assumptions (A1)–(A3). Suppose that there exist positive constants M_i, $i = 1,2,3,4,5$, such that

$$\|f\|_{C^{2,2,0}(\Omega)} \le M, \qquad \|\overline{f}\|_{C^{2,2,0}(\Omega)} \le M,$$

$$\|\varphi\|_{C^{2,2}([0,\pi]\times[0,\pi])} \leq M_1, \qquad \|\overline{\varphi}\|_{C^{2,2}([0,\pi]\times[0,\pi])} \leq M_1,$$

$$\|E\|_{C^1[0,T]} \leq M_2, \qquad \|\overline{E}\|_{C^1[0,T]} \leq M_2, \qquad \|F\|_{C[0,T]} \leq M_3, \qquad \|\overline{F}\|_{C[0,T]} \leq M_3,$$

$$\|K\|_{C([0,T]\times[0,T])} \leq M_4, \qquad 0 < M_5 \leq \min_{(x,y,t)\in\overline{\Omega}} |f(x,y,t)|, \qquad 0 < M_5 \leq \min_{(x,y,t)\in\overline{\Omega}} \left|\overline{f}(x,y,t)\right|.$$

Let us denote $\|\Phi\| = (\|E\|_{C^1[0,T]} + \|\varphi\|_{C^{2,2}\left([0,\pi]\times[0,\pi]\right)} + \|f\|_{C^{2,2,0}(\overline{\Omega})})$. Let (r,u) and $(\overline{r},\overline{u})$ be solutions of inverse problems (1)–(5) corresponding to the data $\Phi = \{\varphi, E, f\}$ and $\overline{\Phi} = \left\{\overline{\varphi}, \overline{E}, \overline{f}\right\}$, respectively, where

$$\|\varphi - \overline{\varphi}\|_{C^{2,2}\left([0,\pi]\times[0,\pi]\right)} \leq \frac{\|\varphi_0 - \overline{\varphi_0}\|_{C^{2,2}\left([0,\pi]\times[0,\pi]\right)}}{4} +$$

$$+\frac{1}{6}\sum_{m,n=1}^{\infty} \left\|(\varphi_{xy})_{cmn} - \overline{(\varphi_{xy})_{cmn}}\right\|_{C^{2,2}\left([0,\pi]\times[0,\pi]\right)} + \frac{1}{6}\left\|(\varphi_{xy})_{csmn} - \overline{(\varphi_{xy})_{csmn}}\right\|_{C^{2,2}\left([0,\pi]\times[0,\pi]\right)} +$$

$$+\frac{1}{6}\left\|(\varphi_{xy})_{scmn} - \overline{(\varphi_{xy})_{scmn}}\right\|_{C^{2,2}\left([0,\pi]\times[0,\pi]\right)} + \frac{1}{6}\left\|(\varphi_{xy})_{smn} - \overline{(\varphi_{xy})_{smn}}\right\|_{C^{2,2}\left([0,\pi]\times[0,\pi]\right)},$$

$$(\varphi_{xy})_{cmn} = \frac{4}{\pi^2 mn}\int_0^{\pi}\int_0^{\pi} \varphi_{xy}(x,y)\sin 2mx \sin 2ny \, dx\, dy,$$

$$(\varphi_{xy})_{csmn} = \frac{4}{\pi^2 mn}\int_0^{\pi}\int_0^{\pi} \varphi_{xy}(x,y)\sin 2mx \cos 2ny \, dx\, dy,$$

$$(\varphi_{xy})_{scmn} = \frac{4}{\pi^2 mn}\int_0^{\pi}\int_0^{\pi} \varphi_{xy}(x,y)\cos 2mx \sin 2ny \, dx\, dy,$$

$$(\varphi_{xy})_{smn} = \frac{4}{\pi^2 mn}\int_0^{\pi}\int_0^{\pi} \varphi_{xy}(x,y)\cos 2mx \cos 2ny \, dx\, dy.$$

From (10), the following equality can be written:

$$F(t) - \overline{F(t)} = \frac{E'(t) + \frac{\pi^2}{4}\sum_{m,n=1}^{\infty} \frac{(2m)^2+(2n)^2}{mn}\varphi_{smn} e^{-\left((2m)^2+(2n)^2\right)t}}{f_0(t) + \frac{\pi^2}{4}\sum_{m,n=1}^{\infty}\frac{(2m)^2+(2n)^2}{mn} f_{smn}(t)} -$$

$$-\frac{\overline{E}'(t) + \frac{\pi^2}{4}\sum_{m,n=1}^{\infty} \frac{(2m)^2+(2n)^2}{mn}\overline{\varphi}_{smn} e^{-\left((2m)^2+(2n)^2\right)t}}{\overline{f}_0(t) + \frac{\pi^2}{4}\sum_{m,n=1}^{\infty}\frac{(2m)^2+(2n)^2}{mn} \overline{f}_{smn}(t)}.$$

Applying Hölder inequality and taking maximum of both sides of the last inequality, we obtain

$$\left\|F-\overline{F}\right\| \leq \frac{2M}{M_5^2}\left\|E'(t)-\overline{E'(t)}\right\| + \frac{2M\pi^3}{\sqrt{6}M_5^2}\left\|\varphi-\overline{\varphi}\right\| + M_0\left\|f-\overline{f}\right\|,$$

and, similarly, we have

$$\left\|K-\overline{K}\right\| \leq M_6\left\|f-\overline{f}\right\|,$$

$$\left\|r-\overline{r}\right\| \leq \left\|F-\overline{F}\right\| + M_4T\left\|r-\overline{r}\right\| + \frac{MT}{1-TM_4}\left\|K-\overline{K}\right\|,$$

$$\left\|r-\overline{r}\right\| \leq \frac{2M}{M_5^2\left(1-TM_4\right)}\left\|E'(t)-\overline{E'(t)}\right\| + \frac{M_0}{1-TM_4}\left\|f-\overline{f}\right\| +$$

$$+\frac{TM}{\left(1-TM_4\right)^2}\left\|f-\overline{f}\right\| + \frac{2M\pi^3}{\sqrt{6}M_5^2\left(1-TM_4\right)}\left\|\varphi-\overline{\varphi}\right\|,$$

$$\left\|u-\overline{u}\right\| \leq M_7\left\|E'(t)-\overline{E'(t)}\right\| + M_8\left\|\varphi-\overline{\varphi}\right\| + M_9\left\|f-\overline{f}\right\|,$$

$$\left\|u-\overline{u}\right\| \leq M_{10}\left\|\Phi-\overline{\Phi}\right\|,$$

where

$$M_0 = \max\left\{\frac{\pi^4 M_2}{6M_5^2}, \frac{\pi^3 M_2}{\sqrt{6}M_5^2}\right\}, \qquad M_6 = \max\left\{\frac{\pi^3 2M}{\sqrt{6}M_5^2}, \qquad \frac{\pi^3}{\sqrt{6}M_5^2}\right\},$$

$$M_7 = \frac{2MT}{3M_5^2(1-TM_4)}, \qquad M_8 = \max\left\{1, \frac{4M^2\pi^3 T}{3\sqrt{6}M_5^2(1-TM_4)}\right\},$$

$$M_9 = \max\left\{\frac{2MT}{3M_5^2(1-TM_4)}, \frac{2M^2M_6T^2}{3(1-TM_4)}\right\}, \qquad M_{10} = \max\{1, M_7, M_8, M_9\}.$$

If $\Phi \to \overline{\Phi}$ then $r \to \overline{r}$ and $u \to \overline{u}$.

Theorem 2 is proved.

4. Numerical method for the problem (1) – (4). In this section, we use implicit finite-difference approximation for the discretizing problem (1) – (5):

$$\frac{1}{\tau}\left(u_{i,j}^{k+1}-u_{i,j}^{k}\right) = \frac{1}{h^2}\left(u_{i-1,j}^{k+1}-2u_{i,j}^{k+1}+u_{i+1,j}^{k+1}\right) +$$

$$+\frac{1}{h^2}\left(u_{i,j-1}^{k+1}-2u_{i,j}^{k+1}+u_{i,j+1}^{k+1}\right) + r^{k+1}f_{i,j}^{k+1},$$

$$u_{i,j}^{0} = \phi_i,$$

$$(10)$$

$$u_{0,j}^{k} = u_{M+1,j}^{k}, \qquad u_{M+1,j}^{k} = \frac{u_{1,j}^{k}-u_{M,j}^{k}}{2},$$

$$u_{i,0}^{k} = u_{i,M+1}^{k}, \qquad u_{i,M+1}^{k} = \frac{u_{i,1}^{k}-u_{i,M}^{k}}{2},$$

where we discretize the computing domain $[0,\pi]\times[0,\pi]\times[0,T]$ by $x_i = ih,\ i = 0,1,\ldots,M,$ $y_j = jh,\ j = 0,1,\ldots,M,$ and $t_k = k\tau,\ k = 0,1,\ldots,N,$ where $h = \pi/M$ and $\tau = T/N$ are the space and time steps, respectively, and $M,\ N$ are two positive integers, $u^k_{i,j} = u(x_i, y_j, t_k),$ $f^k_{i,j} = f(x_i, y_j, t_k),\ r^k = r(t_k).$

Let us integrate equation (1) with respect to x and y from 0 to π and use (2), (3) and (5), we obtain

$$r(t) = \frac{E'(t) - \displaystyle\int_0^\pi y u_x(\pi, y, t)\,dy - \int_0^\pi x u_y(x, \pi, t)\,dx}{\displaystyle\int_0^\pi\int_0^\pi xyf(x,y,t)\,dx\,dy}. \qquad (11)$$

The finite difference approximation of (11) is

$$r^{k+1} = \frac{(E^{k+2} - E^k)/\tau - \left(\displaystyle\int_0^\pi y u_x(\pi, y, t)\,dy\right)^k - \left(\displaystyle\int_0^\pi x u_y(x, \pi, t)\,dx\right)^k}{\left(\displaystyle\int_0^\pi\int_0^\pi xyf(x,y,t)\,dx\,dy\right)^k},$$

where $E^k = E(t_k),\ k = 0,1,\ldots,N.$ We mention that the integrals are numerically calculated using Simpson's rule of integration and also the first derivatives are calculated using central difference scheme.

$r^{k(s)},\ u^{k(s)}_{i,j}$ are the values of $r^k,\ u^k_{i,j}$ at the sth iteration step, respectively. At each $(s+1)$th iteration step, $r^{k+1(s+1)}$ is as follows:

$$r^{k+1(s+1)} = \frac{(E^{k+2} - E^k)/\tau - \left(\displaystyle\int_0^\pi y u_x(\pi, y, t)\,dy\right)^{k(s)} - \left(\displaystyle\int_0^\pi x u_y(x, \pi, t)\,dx\right)^{k(s)}}{\left(\displaystyle\int_0^\pi\int_0^\pi xyf(x,y,t)\,dx\,dy\right)^k}.$$

The iteration of (10) is

$$\begin{gathered}
\frac{1}{\tau}\left(u^{k+1(s+1)}_{i,j} - u^{k+1(s)}_{i,j}\right) = \frac{1}{h^2}\left(u^{k+1(s+1)}_{i-1,j} - 2u^{k+1(s+1)}_{i,j} + u^{k+1(s+1)}_{i+1,j}\right) + \\
+\frac{1}{h^2}\left(u^{k+1(s+1)}_{i,j-1} - 2u^{k+1(s+1)}_{i,j} + u^{k+1(s+1)}_{i,j+1}\right) + r^{k+1(s+1)} f^{k+1}_{i,j}, \\
u^0_{i,j} = \phi_i, \\
u^{k+1(s)}_{0,j} = u^{k+1(s)}_{M+1,j}, \qquad u^{k+1(s)}_{M+1,j} = \frac{u^{k+1(s)}_{1,j} - u^{k+1(s)}_{M,j}}{2}, \\
u^{k+1(s)}_{i,0} = u^{k+1(s)}_{i,M+1}, \qquad u^{k+1(s)}_{i,M+1} = \frac{u^{k+1(s)}_{i,1} - u^{k+1(s)}_{i,M}}{2}.
\end{gathered} \qquad (12)$$

The system of the equations (12) is solved and $u^{k+1(s+1)}_{i,j}$ is determined. If the difference of values between two iterations reaches the prescribed tolerance, the iteration is stopped.

In order to illustrate the behavior of our numerical method, an example is considered.

***Example* 1.** This example investigates finding the exact solution

$$\{r(t), u(x,y,t)\} = \{2\exp(t^2), (2+\cos 2x + \cos 2y)\exp(t^2)\}.$$

for the given functions

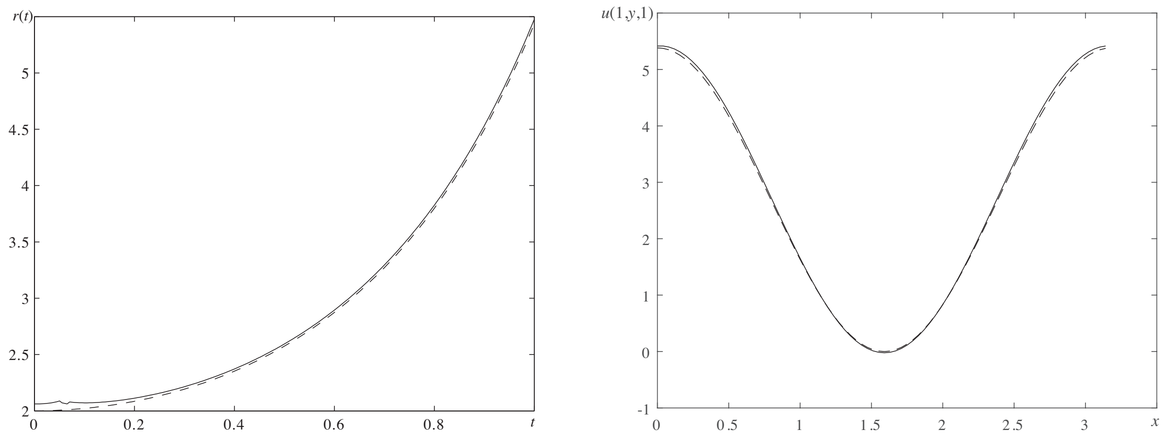
$$\varphi(x,y) = (2+\cos 2x+\cos 2y), \qquad E(t) = \frac{\pi^4}{2}\exp(t^2),$$

$$F(x,y,t) = 2t + (t+2)(\cos 2x + \cos 2y).$$

The step sizes are $h = 0.0393$, $\tau = 0.005$.

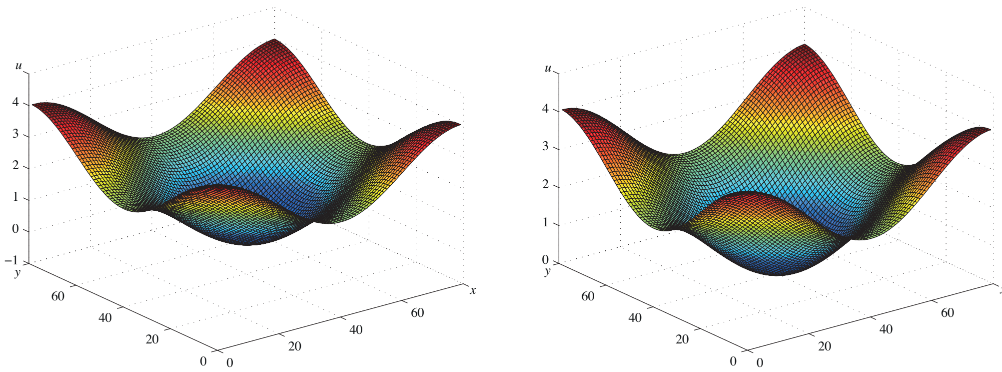
Note that the convergence criterion for $r(t)$, was $\left|r^{k+1(s+1)} - r^{k+1(s)}\right| \leq h/200$.

The comparisons between the exact solution and the numerical finite difference solution are shown in Figs. 1–3 and Table 1 when $T = 1$.



(*a*) (*b*)

Fig. 1. The exact and approximate solutions of $r(t)$ (*a*) and of $u(1,y,1)$ (*b*). The exact solution is shown with dashes line.



(*a*) (*b*)

Fig. 2. The approximate (*a*) and the numerical (*b*) solutions of $u(x,y,1/10)$.

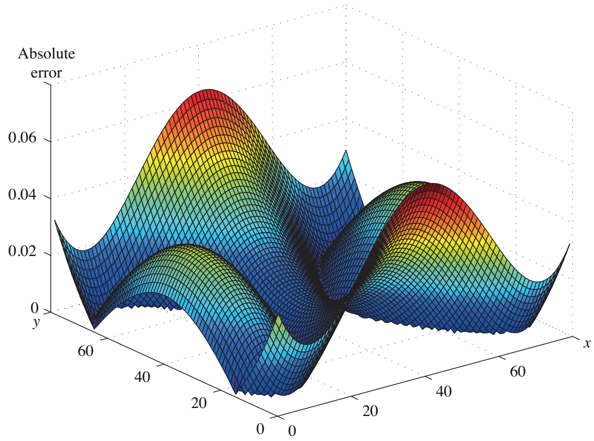


Fig. 3. The absolute error of $u(x, y, 1/10)$.

Table 1. The some values of $r(t)$

Exact	Approximate	Error	Relative error
2	2.0614	0.0614	0.0307
2.0201	2.0717	0.0516	0.0255
2.0816	2.1098	0.0282	0.0135
2.1883	2.2099	0.0216	0.0099
2.3470	2.3664	0.0194	0.0083
2.5681	2.5874	0.0193	0.0075
2.8667	2.8873	0.0206	0.0072
3.2646	3.2878	0.0232	0.0071
3.7930	3.8201	0.0272	0.0072
4.4958	4.5287	0.0410	0.0075

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