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STRONGLY STATISTICAL CONVERGENCE

СИЛЬНО СТАТИСТИЧНА ЗБІЖНІСТЬ

We introduce A-strongly statistical convergence for sequences of complex numbers, where $A = (a_{nk})_{n,k\in\mathbb{N}}$ is an infinite matrix with nonnegative entries. A sequence (x_n) is called strongly convergent to L if $\lim_{n\to\infty}\sum_{k=1}^{\infty} a_{nk}\left|x_k - L\right| = 0$ in the ordinary sense. In the definition of A-strongly statistical limit, we use the statistical limit instead of the ordinary limit via a convenient density. We study some densities and show that the (a_{nk})-strongly statistical limit is a $(a_{m_n k})$-strong limit, where the density of the set $\{m_n \in \mathbb{N} : n \in \mathbb{N}\}$ is equal to 1. We introduce the notion of dense positivity for nonnegative sequences. A nonnegative sequence (r_n) is dense positive provided the limit superior of a subsequence (r_{m_n}) is positive for all (m_n) with density equal to 1. We show that the dense positivity of (r_n) is a necessary and sufficient condition for the uniqueness of A-strongly statistical limit, where $A = (a_{nk})$ and $r_n = \sum_{k=1}^{\infty} a_{nk}$. Furthermore, necessary conditions for the regularity, linearity and multiplicativity of A-strongly statistical limit are established.

Введено поняття A-сильно статистичної збіжності для послідовностей комплексних чисел, де $A = (a_{nk})_{n,k\in\mathbb{N}}$ — нескінченна матриця з невід'ємними елементами. Послідовність (x_n) називається сильно збіжною до L, якщо $\lim_{n\to\infty}\sum_{k=1}^{\infty} a_{nk}\left|x_k - L\right| = 0$ у звичайному сенсі. У визначенні A-сильно статистичної границі застосовується поняття статистичної границі замість звичайної границі з відповідною щільністю. Вивчено деякі щільності і показано, що (a_{nk})-сильно статистична границя — це $(a_{m_n k})$-сильна границя, де щільність множини $\{m_n \in \mathbb{N} : n \in \mathbb{N}\}$ дорівнює 1. Введено поняття щільної позитивності для невід'ємних послідовностей. Невід'ємна послідовність (r_n) є щільно позитивною за умови, що верхня границя підпослідовності (r_{m_n}) є додатною для всіх (m_n) з щільністю, що дорівнює 1. Показано, що щільна позитивність (r_n) є необхідною та достатньою умовою для єдиності A-сильно статистичної границі, де $A = (a_{nk})$ та $r_n = \sum_{k=1}^{\infty} a_{nk}$. Крім того, встановлено необхідні умови регулярності, лінійності та мультиплікативності A-сильно статистичної границі.

1. Introduction. The usual limit concept has many useful applications in several fields of mathematics, statistics, physics, engineering and so on. It is well known that a complex sequence is convergent to a point if and only if every neighbourhood of the given point includes all the elements of the sequence but a finite number. If a sequence (x_n) is convergent to L, we write

$$\lim_{n\to\infty} \left|x_n - L\right| = 0. \tag{1.1}$$

Hamilton and Hill [8] developed this concept by introducing strong summability in 1938. They generalized equality (1.1) as follows:

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk}\left|x_k - L\right|^p = 0, \tag{1.2}$$

where $A = (a_{nk})$ is an infinite matrix and $p > 0$. If equality (1.2) holds, (x_n) is said to be strongly summable to L. In the case that A is identity matrix and $p = 1$ in (1.2), then we get usual convergence in (1.1).

Whenever a new convergence method is introduced, mathematicians investigate the typical properties of it, such as uniqueness of limit point, regularity, linearity and so on. Under some conditions, Hamilton and Hill studied these typical properties of strong convergence.

In 1963, Wlodarski [15] generalized the strong summability into strong continuous summability method. He considered a sequence of continuous functions $(a_k(t))$ instead of an infinite matrix

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(a_{nk}) and gave his definition as follows:

$$\lim_{t\to T}\sum_{k=1}^{\infty} a_k(t)\left|x_k-L\right|^p=0.$$

Besides, he defined some pseudonormed, normed and Banach spaces by using the new convergence method.

Maddox [10] generalized the strong convergence in 1966 by introducing A-strong convergence of order $(p_k)_{k\in\mathbb{N}}$ for a positive sequence $(p_k)_{k\in\mathbb{N}}$ as follows:

$$\lim_{n\to\infty}\sum_{k=1}^{\infty} a_{nk}\left|x_k-L\right|^{p_k}=0.$$

Maddox also investigated the uniqueness, regularity, linearity of this concept and studied its application to some Cesàro-type space.

The concept of statistical convergence which is an extension of the usual concept of sequential limit was independently introduced by Fast [4] and Steinhaus [14]. This new method was not as strict as usual convergence, i.e., it is easy that a sequence is statistical convergent in comparison with usual convergence. Indeed, a sequence is statistically convergent to a point if and only if it is convergent to this point in a subset of naturals $\mathbb{N}$ with the asymptotic density of 1. Here, we must define the asymptotic density. Let $K\subset\mathbb{N}$. If the limit

$$\delta(K)=\lim_{n\to\infty}\frac{1}{n}\left|\{k\le n\colon k\in K\}\right|$$

exists, then $\delta(K)$ is said to be the asymptotic density of K, where the notation $|\cdot|$ denotes the cardinality of a set. By means of the asymptotic density, the statistical limit can be defined by the condition

$$\lim_{n\to\infty}\frac{1}{n}\left|\{k\le n\colon\ |x_k-L|\ge\varepsilon\}\right|=0 \tag{1.3}$$

for every $\varepsilon>0$.

In 1980, Salat [13] showed that the statistical limit can be considered as a linear operator and every statistical convergent sequence has a unique limit. Also, Salat proved that the space of the bounded statistical convergent sequences is nowhere dense in the space of bounded sequences and the space of statistically convergent sequences of real numbers is a dense subset in the first Baire category in the Fréchet space.

Fridy [6], in 1985, proved that if a sequence is convergent to L, then it is also statistically convergent to the number L, i.e., statistical convergence is regular. He introduced the concept of statistically Cauchy sequence and proved that it is equivalent to statistical convergence. Finally, he proved some Tauberian theorems.

The relation between A-strong convergence and statistical convergence was studied by Connor [2]. Freedman and Sember [5] investigated densities on natural numbers and showed that the density used in statistical convergence can be defined by Cesàro matrix. Also, they generalized this concept for arbitrary regular matrix.

Karakaya and Chishti [9] introduced the concept of weighted statistical convergence in 2009. Later, Mursaleen et al. [12] modified this concept in 2012. Let (p_n) be a nonnegative sequence such that $p_0 > 0$ and $P_n = \sum_{k=1}^{n} p_k \to \infty$ as $n \to \infty$. A complex sequence (x_n) is said to be weighted statistically convergent to a number L if

$$\lim_{n\to\infty} \frac{1}{P_n} \left|\{k \le P_n : p_k \left|x_k - L\right| \ge \varepsilon\}\right| = 0$$

for every $\varepsilon > 0$. Also, they found its relationship with the concept of statistical summability $\left(\overline{N}, p\right)$ and gave its applications to Korovkin-type approximation theorems.

Belen and Mohiuddine [1] introduced weighted λ-statistical convergence and statistical summability $\left(\overline{N}_\lambda, p\right)$ in 2013. They determined a Korovkin-type approximation theorem through the statistical summability $\left(\overline{N}_\lambda, p\right)$ and showed that their approximation theorem was stronger than classical Korovkin theorem by using classical Bernstein polynomials.

Edely et al. [3], in 2013, used the weighted statistical convergence to give a Korovkin-type approximation theorem for 2π-periodic functions.

In 2014, Ghosal [7] modified the concept of weighted λ-statistical convergence by adding the condition

$$\liminf_{n\to\infty} p_n > 0, \tag{1.4}$$

where (p_n) is the weight sequence. Ghosal proved that (1.4) is a sufficient condition for the uniqueness of the limit.

In this work, we introduce a new convergence method which we call strongly statistical convergence. Let

$$A = (a_{nk}) = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots \\ a_{21} & a_{22} & a_{23} & \cdots \\ a_{31} & a_{32} & a_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

be a nonnegative matrix, $r_n = \sum_{k=1}^{\infty} a_{nk} < +\infty$ for every $n \in \mathbb{N}$ and $S_n = \sum_{i=1}^{n} r_i \to \infty$ as $n \to \infty$. We call that $(x_n) \subset \mathbb{C}$ is A-strongly statistically convergent to $L \in \mathbb{C}$ if

$$\lim_{n\to\infty} \frac{1}{S_n} \left| \left\{ k \le S_n : \sum_{i=1}^{\infty} a_{ki} \left|x_i - L\right| \ge \varepsilon \right\} \right| = 0$$

for all $\varepsilon > 0$.

This new method is generalizations of the following:

1) statistical convergence,

2) weighted statistical convergence,

3) strong convergence.

Relation (1.4) is a sufficient condition for the uniqueness of the weighted statistical limit point. In this study, we give a necessary and sufficient condition, which we call dense positivity, for the uniqueness of the strongly statistical limit point (and naturally for the uniqueness of the weighted statistical limit point).

Finally, we investigate the regularity, linearity and multiplicativity of this new concept. The strongly statistical convergence is a new concept and it can be applied the approximation theory, Fourier analysis, topology and so on. For example, it can be given a Korovkin-type approximation theorem. Also, it can be determined whether the space of the strongly statistically convergent sequences is a subspace of l_∞ or not. It can be described strongly statistically Cauchy sequence and studied its properties.

2. Preliminaries.

Definition 2.1 [5]. *Let $\underline{\delta}$ is a function from all the subsets of natural numbers to the closed interval $[0,1]$. If the following conditions hold, then $\underline{\delta}$ is said to be a lower density in the sense of Freedman and Sember:*

1) *if the symmetric difference of the sets A and B is finite, then $\underline{\delta}(A) = \underline{\delta}(B)$,*
2) *if $A \cap B = \varnothing$, then $\underline{\delta}(A) + \underline{\delta}(B) \le \underline{\delta}(A \cup B)$,*
3) *$\underline{\delta}(A) + \underline{\delta}(B) \le 1 + \underline{\delta}(A \cap B)$ for all A and B,*
4) *$\underline{\delta}(\mathbb{N}) = 1$.*

If $\underline{\delta}$ is a lower density, then $\overline{\delta}$ is called an upper density associated with $\underline{\delta}$ defined by the equality

$$\overline{\delta}(A) = 1 - \underline{\delta}(\mathbb{N} \setminus A).$$

Proposition 2.1. *Assume that $(S_n) \subset \mathbb{R}$ is a nondecreasing, nonnegative and unbounded sequence. Then*

$$\underline{\delta}_{S_n}(K) := \liminf_{n\to\infty} \frac{1}{S_n} \left|\{k \le S_n : k \in K\}\right| \tag{2.1}$$

is a lower density, where $|E|$ denotes the cardinality of a set E, and $K \subset \mathbb{N}$.

Proof. Let

$$\underline{\delta}_{[\![S_n]\!]}(K) := \liminf_{n\to\infty} \frac{1}{[\![S_n]\!]} \left|\{k \le [\![S_n]\!] : k \in K\}\right|, \tag{2.2}$$

where $[\![S_n]\!]$ denotes the integral part of S_n, and $K \subset \mathbb{N}$. We now show that $\underline{\delta}_{S_n}(K) = \underline{\delta}_{[\![S_n]\!]}(K)$ for each $K \subset \mathbb{N}$.

Obviously, the relations $S_n - 1 < [\![S_n]\!] \le S_n$ and $\{k \le S_n : k \in K\} = \{k \le [\![S_n]\!] : k \in K\}$ hold for each $n \in \mathbb{N}$. By these relations, we have the inequality

$$\frac{1}{S_n}\left|\{k \le S_n : k \in K\}\right| \le \frac{1}{[\![S_n]\!]}\left|\{k \le [\![S_n]\!] : k \in K\}\right| <$$

$$< \frac{1}{S_n - 1}\left|\{k \le S_n : k \in K\}\right|. \tag{2.3}$$

Since $\lim_{n\to\infty} \dfrac{S_n - 1}{S_n} = 1$, the limit inferiors of

$$\frac{1}{S_n}\left|\{k \le S_n : k \in K\}\right| \text{ and } \frac{1}{S_n - 1}\left|\{k \le S_n : k \in K\}\right| \tag{2.4}$$

coincide. Therefore, we get $\underline{\delta}_{S_n}(K) = \underline{\delta}_{[\![S_n]\!]}(K)$ for each $K \subset \mathbb{N}$.

Now, we will prove that $\underline{\delta}_{[\![S_n]\!]}$ is a lower density. Let

$$M=(a_{nk})=\begin{cases}1, & \text{if} \quad 0\le S_n<1 \quad \text{and} \quad k=1,\\ \dfrac{1}{[\![S_n]\!]}, & \text{if} \quad S_n\ge 1 \quad \text{and} \quad k\le S_n,\\ 0, & \text{otherwise.}\end{cases}$$

Note that M is a nonnegative Toeplitz matrix (see [11], Chapter 7.1, Theorem 3). Moreover, the equality

$$\underline{\delta}_{S_n}(K)=\liminf_{n\to\infty}(M\chi_K)_n$$

holds for every $K\subset\mathbb{N}$, where χ_K is the characteristic function of the set K, i.e.,

$$\chi_K(j)=\begin{cases}1, & \text{if} \quad j\in K,\\ 0, & \text{if} \quad j\in\mathbb{N}\backslash K.\end{cases}$$

Hence, $\underline{\delta}_{S_n}$ is a lower density by [5] (Proposition 3.1).

Proposition 2.1 is proved.

***Remark* 2.1.** In (2.1), (2.2), (2.3), and (2.4), some of the numbers S_n, $[\![S_n]\!]$ and S_n-1 may be zero for some n. In this case, since $\lim_{n\to\infty}S_n=+\infty$, there exists $n_0\in\mathbb{N}$ such that $S_n>0$, $[\![S_n]\!]>0$ and $S_n-1>0$ for each $n\ge n_0$. Thus, we will assume $n\ge n_0$.

By Proposition 2.1,

$$\overline{\delta}_{S_n}(K):=\limsup_{n\to\infty}(M\chi_K)_n$$

is an upper density. If $\overline{\delta}_{S_n}(K)=\underline{\delta}_{S_n}(K)$, then we say that K has density (see [5]). In this case, $\underline{\delta}_{S_n}(K)=\overline{\delta}_{S_n}(K)$ will be denoted by $\delta_{S_n}(K)$ and we say that $\delta_{S_n}(K)$ is the density of K with respect to (S_n). When $S_n=n$, we will say that δ_{S_n} is Cesàro density and write δ instead of δ_{S_n}. K is said to be an S_n-null set in case $\delta_{S_n}(K)=0$. K is called a Cesàro-null set when $S_n=n$ and $\delta(K)=0$.

It is easy to observe the following:

$$0\le\underline{\delta}_{S_n}(K)\le\overline{\delta}_{S_n}(K)\le 1.$$

In addition, if $\delta_{S_n}(K)$ exists, then $0\le\delta_{S_n}(K)\le 1$. We say that K is dense with respect to (S_n) provided $\delta_{S_n}(K)=1$.

Now, we give a proposition about the intersection of dense subsets of natural numbers.

Proposition 2.2. *The intersection of two dense subsets of* $\mathbb{N}$ *is dense, i.e., if* $\delta_{S_n}(K_1)=$ $=\delta_{S_n}(K_2)=1$, *then* $\delta_{S_n}(K_1\cap K_2)=1$.

By using [5] (Propositions 2.1 – 2.3), one can easily prove this assertion.

We now give an example for an S_n-null set that fails to be Cesàro-null set.

***Example* 2.1.** Let $K_n=\left(2^{n^2},2^{n^2+1}\right]\cap\mathbb{N}$ and $K=\bigcup_{n=0}^{\infty}K_n$. Since

$$\frac{\left|\left\{k\le 2^{n^2+1}: k\in K\right\}\right|}{2^{n^2+1}}=\frac{\sum_{k=0}^{n}\left(2^{k^2+1}-2^{k^2}\right)}{2^{n^2+1}}=\frac{\sum_{k=0}^{n}2^{k^2}}{2^{n^2+1}}\ge\frac{2^{n^2}}{2^{n^2+1}}=\frac{1}{2},$$

then $\overline{\delta}(K)\ge\dfrac{1}{2}$. Similarly, by the following:

$$\frac{\left|\left\{k \leq 2^{n^2} : k \in K\right\}\right|}{2^{n^2}} = \frac{\sum_{k=0}^{n-1}\left(2^{k^2+1} - 2^{k^2}\right)}{2^{n^2}} = \frac{\sum_{k=0}^{n-1} 2^{k^2}}{2^{n^2}} \leq$$

$$\leq \frac{\sum_{k=0}^{(n-1)^2} 2^k}{2^{n^2}} = \frac{2^{(n-1)^2+1} - 1}{2^{n^2}} = \left(2^{2(1-n)} - 2^{-n^2}\right),$$

we get

$$\lim_{n\to\infty} \frac{\left|\left\{k \leq 2^{n^2} : k \in K\right\}\right|}{2^{n^2}} = 0, \tag{2.5}$$

i.e., $\underline{\delta}(K) = 0$.

Consequently, K has no Cesàro density. However, by (2.5), we obtain $\delta_{S_n}(K) = 0$, where $S_n = 2^{n^2}$. K is 2^{n^2}-null set but is not Cesàro-null.

We now give some definitions.

Definition 2.2 [4]. *Let $(x_n) \subset \mathbb{C}$ and $L \in \mathbb{C}$. If $\delta\left(\{k \in \mathbb{N}\colon |x_k - L| \geq \varepsilon\}\right) = 0$ for each $\varepsilon > 0$, then (x_n) is called statistically convergent to L.*

Definition 2.3 [12]. *Let (p_k) be a sequence of nonnegative numbers such that $p_1 > 0$, $P_n = \sum_{k=1}^{n} p_k \to \infty$ as $n \to \infty$. A sequence $x = (x_k)$ is called weighted statistically convergent to L if the set $\{k \in \mathbb{N}\colon p_k\,|x_k - L| \geq \varepsilon\}$ is P_n-null set for every $\varepsilon > 0$.*

Definition 2.4 [8]. *Let A be an infinite matrix, $(x_n) \subset \mathbb{C}$ and $L \in \mathbb{R}$. Then (x_n) is called A-strongly convergent to L if $\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk}\,|x_k - L| = 0$. This convergence is denoted by $x_n \to L\,[A]$.*

3. A-strongly statistical convergence.

Definition 3.1. *Let*

$$A = (a_{nk}) = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots \\ a_{21} & a_{22} & a_{23} & \cdots \\ a_{31} & a_{32} & a_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

be a nonnegative matrix, $r_n = \sum_{k=1}^{\infty} a_{nk}$ and $S_n = \sum_{i=1}^{n} r_i$. Assume that the matrix A satisfies the following conditions:

(i) $r_n < +\infty$ *for each* $n \in \mathbb{N}$,

(ii) $\lim_{n\to\infty} S_n = +\infty$.

We say that $(x_n) \subset \mathbb{C}$ is A-strongly statistically convergent to $L \in \mathbb{C}$ if

$$\lim_{n\to\infty} \frac{1}{S_n} \left|\left\{k \leq S_n\colon \sum_{i=1}^{\infty} a_{ki}\,|x_i - L| \geq \varepsilon\right\}\right| = 0$$

for all $\varepsilon > 0$. We write $x_n \overset{st}{\to} L\,[A]$ when (x_n) is A-strongly statistically convergent to L.

We will say that A matrix A satisfying (i) and (ii) in Definition 3.1 is an S-type matrix.

If

$$A = I = (a_{nk}) = \begin{cases} 1, & \text{if} \quad n = k, \\ 0, & \text{if} \quad n \neq k, \end{cases}$$

then A-strongly statistical convergence coincides with the statistical convergence which was introduced in [4].

If

$$A=(a_{nk})=\begin{cases} p_n, & \text{if} \quad n=k, \\ 0, & \text{if} \quad n\neq k, \end{cases}$$

then A-strongly statistical convergence coincides with the weighted statistical convergence which was introduced in [12], where $p_n \geq 0,\ p_1>0$ and $\sum_{n=1}^{\infty} p_n=+\infty$.

Theorem 3.1. *Let A be an S-type matrix, (x_n) be a complex sequence, $L\in\mathbb{C}$ and $A_k = \sum_{i=1}^{\infty} a_{ki}\,|x_i-L|$. Hence, $x_n \overset{st}{\to} L\,[A]$ if and only if there exist two nonnegative sequences $(B_k)\,,\,(C_k)$ such that $A_k=B_k+C_k$, $\lim_{n\to\infty} B_n=0$ and $\delta_{S_n}\left(\{k\in\mathbb{N}\colon C_k\neq 0\}\right)=0$.*

Proof. *Necessity.* Assume that $x_n \overset{st}{\to} L\,[A]$. Then we get

$$\lim_{n\to\infty}\frac{1}{S_n}\left|\left\{k\leq S_n:\ \sum_{i=1}^{\infty} a_{ki}\,|x_i-L|\geq\varepsilon\right\}\right|=0$$

for each $\varepsilon>0$. Therefore, if $\varepsilon>0$ and $r>0$, then there exists $n_{r,\varepsilon}\in\mathbb{N}$ such that the inequality

$$\frac{1}{S_n}\left|\left\{k\leq S_n:\ \sum_{i=1}^{\infty} a_{ki}\,|x_i-L|\geq\varepsilon\right\}\right|<r \tag{3.1}$$

holds for every $n>n_{r,\varepsilon}$. We choose $\varepsilon=r=\dfrac{1}{j}$ and $N_j=n_{r,\varepsilon}$ for $j\in\mathbb{N}$. Then, the relation (3.1) turns into the inequality

$$\frac{1}{S_n}\left|\left\{k\leq S_n:\ \sum_{i=1}^{\infty} a_{ki}\,|x_i-L|\geq\frac{1}{j}\right\}\right|<\frac{1}{j}, \tag{3.2}$$

where $n>N_j$. Note that the sequence (N_j) of naturals can be constructed as strictly increasing. We define the sequences (B_k) and (C_k) as follows:

$$B_k:=\begin{cases} A_k, & \text{if} \quad 1\leq k\leq N_1 \quad \text{or if} \quad N_j<k\leq N_{j+1} \quad \text{and} \quad A_k<\dfrac{1}{j}, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$C_k:=A_k-B_k,$$

where $k,j\in\mathbb{N}$. It is obvious that $A_k=B_k+C_k$ and $B_k,C_k\geq 0$ for each $k\in\mathbb{N}$.

Given $\varepsilon>0$, there exists $j\in\mathbb{N}$ such that $\dfrac{1}{j}<\varepsilon$. Let $k>N_j$. Since (N_j) is a strictly increasing sequence, there exists $M\geq j$ such that $N_M<k\leq N_{M+1}$. There are two cases for A_k: $A_k<\dfrac{1}{M}$ or $A_k\geq\dfrac{1}{M}$. In the former case, $B_k=A_k<\dfrac{1}{M}\leq\dfrac{1}{j}<\varepsilon$. In the latter case, $B_k=0<\varepsilon$. Therefore, $\lim_{n\to\infty} B_n=0$.

Now, we will show that $\delta_{S_n}\left(\{k\in\mathbb{N}\colon C_k\neq 0\}\right)=0$. By the definition of (C_k), the inclusion

$$\{k\leq S_n: N_j<k\leq N_{j+1}, C_k\neq 0\}\subset\left\{k\leq S_n: N_j<k\leq N_{j+1}, A_k\geq\frac{1}{j}\right\} \tag{3.3}$$

holds, where $n, j \in \mathbb{N}$. For $n \in \mathbb{N}$, there exists $j \in \mathbb{N} \cup \{0\}$ such that

$$N_j < S_n \leq N_{j+1}, \tag{3.4}$$

where $N_0 = 0$. Suppose that $C_k \neq 0$ and $k \leq S_n$. Since (N_j) is a strictly increasing sequence, (3.4) implies that there exists $M_1 \in \mathbb{N}$ such that $M_1 \leq j$ and $N_{M_1} < k \leq N_{M_1+1}$. By using (3.3), we have $A_k \geq \frac{1}{M_1}$ and, so, $A_k \geq \frac{1}{j}$. Thus, we get

$$\{k \leq S_n : C_k \neq 0\} \subset \left\{k \leq S_n : A_k \geq \frac{1}{j}\right\}. \tag{3.5}$$

By (3.2) and (3.5), we have

$$\frac{1}{S_n} |\{k \leq S_n : C_k \neq 0\}| \leq \frac{1}{S_n} \left|\left\{k \leq S_n : A_k \geq \frac{1}{j}\right\}\right| < \frac{1}{j}.$$

(3.4) implies that $j \to \infty$ as $n \to \infty$. Therefore,

$$\lim_{n\to\infty} \frac{1}{S_n} |\{k \leq S_n : C_k \neq 0\}| = 0.$$

This implies that $\delta_{S_n}(\{k \in \mathbb{N} : C_k \neq 0\}) = 0$.

Sufficiency. Assume that there exist (B_k) and (C_k) sequences which satisfy the following conditions:

(i) $B_k, C_k \geq 0$ for all $k \in \mathbb{N}$,

(ii) $A_k = B_k + C_k$ for all $k \in \mathbb{N}$,

(iii) $\lim_{n\to\infty} B_n = 0$,

(iv) $\delta_{S_n}(\{k \in \mathbb{N} : C_k \neq 0\}) = 0$.

Let $\varepsilon > 0$. Condition (iii) implies that the set $\{k \in \mathbb{N} : B_k \geq \varepsilon\}$ is finite. So, the equality

$$\delta_{S_n}(\{k \in \mathbb{N} : B_k \geq \varepsilon\}) = \lim_{n\to\infty} \frac{1}{S_n} |\{k \leq S_n : B_k \geq \varepsilon\}| = 0$$

holds. By (i) and (ii), if $A_k \geq \varepsilon$, then either $C_k = 0$ and $B_k \geq \varepsilon$ or $C_k \neq 0$. By using (iv), we have the following:

$$\lim_{n\to\infty} \frac{1}{S_n} |\{k \leq S_n : A_k \geq \varepsilon\}| \leq$$

$$\leq \lim_{n\to\infty} \frac{1}{S_n} |\{k \leq S_n : B_k \geq \varepsilon\}| + \lim_{n\to\infty} \frac{1}{S_n} |\{k \leq S_n : C_k \neq 0\}| = 0 + 0 = 0.$$

Theorem 3.1 is proved.

By choosing $B_k = A_k$ and $C_k = 0$, for $k \in \mathbb{N}$, we have the following corollary.

***Corollary* 3.1.** *Let A be an S-type matrix, $(x_n) \subset \mathbb{C}$ and $L \in \mathbb{C}$. Then $x_n \to L\,[A]$ implies $x_n \overset{st}{\to} L\,[A]$.*

The converse of Corollary 3.1 is not true. There exists a statistically convergent but not convergent sequence when $A = I$ (see [6]).

We will use $\delta_{S_n}(m_n)$ instead of $\delta_{S_n}(\{m_n \in \mathbb{N} : n \in \mathbb{N}\})$ for short notation.

***Corollary* 3.2.** *Let A be an S-type matrix, $(x_n) \subset \mathbb{C}$ and $L \in \mathbb{C}$. Hence, $x_n \overset{st}{\to} L\,[A]$ if and only if there exists a strictly increasing sequence (m_n) of naturals such that $\delta_{S_n}(m_n) = 1$, $B = (b_{nk}) = (a_{m_n k})$ and $x_n \to L\,[B]$.*

***Remark* 3.1.** Corollary 3.2 asserts that A-strongly statistical convergence is strong convergence on the matrix obtained by eliminating some rows from the matrix A, where the density of the indices of these rows is 0.

4. Main results.

Definition 4.1. *Let (S_n) be a nondecreasing, nonnegative and unbounded sequence. We say that a nonnegative sequence (r_n) is S_n-dense positive provided the condition*

$$\limsup_{n\to\infty} r_{m_n} > 0$$

holds for every indices of (m_n) satisfying $\delta_{S_n}(m_n) = 1$. When $S_n = n$, we write dense positive instead of n-dense positive for a sequence.

***Remark* 4.1.** S_n-dense positivity of a sequence (r_n) is weaker than the condition $\liminf_{n\to\infty}(r_n) > 0$ and stronger than the condition $\limsup_{n\to\infty}(r_n) > 0$. That is, if the limit inferior of a nonnegative sequence is positive, then it is S_n-dense positive. Similarly, if a nonnegative sequence is S_n-dense positive, then its limit superior is positive. However, the converses of these assertions are not true in general, as we show in next two examples.

***Example* 4.1.** Let

$$a_n = \begin{cases} 1, & \text{if} \quad k = n^2, \\ 0, & \text{if} \quad k \neq n^2, \end{cases}$$

and $S_n = n$. Obviously, $\limsup_{n\to\infty} a_n = 1$. However, if we choose (m_n) as all nonsquare numbers, the conditions $\delta(m_n) = 1$ and $\limsup_{n\to\infty} a_{m_n} = 0$ hold. So, $\limsup_{n\to\infty} a_n > 0$ but (a_n) is not dense positive.

***Example* 4.2.** Let

$$b_n = \begin{cases} 1, & \text{if} \quad k \neq n^2, \\ 0, & \text{if} \quad k = n^2, \end{cases}$$

and $S_n = n$. Obviously, $\liminf_{n\to\infty} b_n = 0$. However, for each $(m_n) \subset \mathbb{N}$ satisfying $\delta(m_n) = 1$, the equality $\limsup_{n\to\infty} b_{m_n} = 1$ holds. Hence, (b_n) is dense positive but $\liminf_{n\to\infty} b_n = 0$.

Theorem 4.1. *Let $A = (a_{nk})$ be an S-type matrix. A-strongly statistical limit is unique if and only if (r_n) is S_n-dense positive, where $r_n = \sum_{k=1}^{\infty} a_{nk}$.*

Proof. *Sufficiency.* Assume that (r_n) is S_n-dense positive and a sequence (x_n) of complex numbers strongly statistically tends to both L and R. By Corollary 3.2, since $x_n \overset{st}{\to} L\,[A]$ and $x_n \overset{st}{\to} R\,[A]$, there exist indices (m_n) and (j_n) such that $B = (a_{m_n k})$, $D = (a_{j_n k})$, $x_n \to L\,[B]$, $x_n \to R\,[D]$ and $\delta_{S_n}(m_n) = \delta_{S_n}(j_n) = 1$. By Proposition 2.2, we get $\delta_{S_n}((m_n) \cap (j_n)) = 1$. We now define (i_n) by $(i_n) := (m_n) \cap (j_n)$. Obviously, the inclusions $(i_n) \subset (m_n)$ and $(i_n) \subset (j_n)$ hold. From here, we have $x_n \to L\,[E]$ and $x_n \to R\,[E]$, where $E = (a_{i_n k})$. Since (r_n) is dense positive, we obtain $\limsup_{n\to\infty} r_{i_n} > 0$. Then, by [8] (Theorem 3), we have $L = R$.

Necessity. Assume that A-strongly statistical limit is unique. We claim that the sequence (r_n) is S_n-dense positive. Consider a set of indices $(m_n) \subset \mathbb{N}$ satisfying $\delta_{S_n}(m_n) = 1$. Let $B = (a_{m_n k})$.

We now define two classes of sequences as follows:

$$C_{[A]}^{st} := \left\{ (x_n) \subset \mathbb{C} \colon \exists L \in \mathbb{C} : x_n \overset{st}{\to} L\,[A] \right\},$$

$$C_{[B]} := \{ (x_n) \subset \mathbb{C} \colon \exists L \in \mathbb{C} : x_n \to L\,[B] \}.$$

By Corollary 3.2, we have $C_{[B]} \subset C_{[A]}^{st}$. From here, since A-statistically strong limit is unique, B-strong limit is also unique. So, $\limsup\limits_{n\to\infty} r_{m_n} > 0$ by [8] (Theorem 3).

Theorem 4.1 is proved.

***Corollary* 4.1.** *If A is nonnegative Toeplitz matrix, then A-strongly statistical limit is unique* (*see* [11], *Chapter* 7.1, *Theorem* 3).

***Remark* 4.2.** Recall the Definition 2. By Theorem 4.1, the uniqueness of weighted statistical limit requires that (p_n) is P_n-dense positive. Indeed, this condition is necessary and sufficient for uniqueness. A sequence may have infinitely many weighted statistical limits when (p_n) is not P_n-dense positive, as we show in next example.

***Example* 4.3.** Consider the matrix

$$A = (a_{nk}) = \begin{cases} \dfrac{1}{n}, & \text{if} \quad n = k, \\ 0, & \text{if} \quad n \neq k, \end{cases}$$

and the constant sequence $(x_n) = (0, 0, \ldots)$. By Definition 3.1, we get $r_n = \dfrac{1}{n}$ and $S_n = \sum_{i=1}^{n} \dfrac{1}{i}$. Obviously, the equality $\lim_{n\to\infty} S_n = +\infty$ holds. So, A is an S-type matrix. Let $L \in \mathbb{C}$ and $\varepsilon > 0$. Since the sequences (S_n) and $\left(\dfrac{|L|}{n}\right)$ tend to infinity and zero, respectively, then the cardinality of the set

$$\left\{ k \le S_n \colon \frac{|L|}{k} \ge \varepsilon \right\}$$

is a constant for sufficiently large n. Thus, we have the following:

$$\lim_{n\to\infty} \frac{1}{S_n} \left| \left\{ k \le S_n \colon \frac{1}{k} |x_k - L| \ge \varepsilon \right\} \right| = \lim_{n\to\infty} \frac{1}{S_n} \left| \left\{ k \le S_n \colon \frac{|L|}{k} \ge \varepsilon \right\} \right| = 0.$$

This shows that each complex number L is A-strongly statistical limit or $\left(\dfrac{1}{n}\right)$-weighted statistical limit of (x_n).

Theorem 4.2. *Let $A = (a_{nk})$ be an S-type matrix. If there exist some indices $m_n \in \mathbb{N}$ such that $\delta_{S_n}(m_n) = 1$, $\lim_{n\to\infty} a_{m_n k} = 0$ for each $k \in \mathbb{N}$ and $\sup_{n\in\mathbb{N}} r_{m_n} < +\infty$, then A-strongly statistical convergence is a regular summability method, i.e., $x_n \to L$ implies $x_n \overset{st}{\to} L\,[A]$* $\left(r_{m_n} = \sum_{k=1}^{\infty} a_{m_n k} \right)$.

Proof. Let $x_n \to L$ and $B = (a_{m_n k})$. By the hypothesis and [8] (Theorem 1), $x_n \to L\,[B]$. Finally, by Corollary 3.2, we obtain $x_n \overset{st}{\to} L\,[A]$.

Proposition 4.1. *If $(x_n) \subset \mathbb{C}$ is statistically strongly summable to L by a matrix $A = (a_{nk})$ of which (r_n) is dense positive, then L must be a limit point of (x_n).*

The proof of this assertion is directly obtained from Corollary 3.2 and [8] (Theorem 4).

Theorem 4.3. *Let $A = (a_{nk})$ be an S-type matrix, (x_n), (y_n) be two complex sequences and $L, M, \alpha, \beta \in \mathbb{C}$. Then the followings hold*:

(i) $x_n \overset{st}{\to} L\,[A]$, $y_n \overset{st}{\to} M\,[A]$ *implies* $(\alpha x_n + \beta y_n) \overset{st}{\to} \alpha L + \beta M\,[A]$,

(ii) $x_n \overset{st}{\to} L\,[A]$, $y_n \overset{st}{\to} M\,[A]$ *implies* $(x_n y_n) \overset{st}{\to} LM\,[A]$ *if either* (x_n) *or* (y_n) *is bounded,*

(iii) $x_n \overset{st}{\to} L\,[A]$, $y_n \overset{st}{\to} M\,[A]$ *implies* $\left(\dfrac{x_n}{y_n}\right) \overset{st}{\to} \dfrac{L}{M}\,[A]$ *if there exists a positive number* d *such that* $|y_n| \geq d$ *for sufficiently large* n *and one of the conditions that either* $M \neq 0$ *or* (r_n) *of* A *is dense positive is true.*

Proof. By Corollary 3.2, $x_n \overset{st}{\to} L\,[A]$ implies $x_n \to L\,[B]$ and $y_n \overset{st}{\to} M\,[A]$ implies $y_n \to M\,[D]$, where $B = (a_{m_n k})$, $D = (a_{j_n k})$ and $\delta_{S_n}(m_n) = \delta_{S_n}(j_n) = 1$. Thus, we have $x_n \to L\,[E]$ and $y_n \to M\,[E]$, where $E = (a_{i_n k})$ and $(i_n) = (m_n) \cap (j_n)$. By Proposition 2.2, the equality $\delta_{S_n}(i_n) = 1$ is true. By [10] (Theorem 1), we have the followin:

(i) For each $\alpha, \beta \in \mathbb{C}$, $(\alpha x_n + \beta y_n) \to \alpha L + \beta M\,[E]$. Hence, we get, by Corollary 3.2, that $(\alpha x_n + \beta y_n) \overset{st}{\to} \alpha L + \beta M\,[A]$.

(ii) If either (x_n) or (y_n) is bounded sequence, then $(x_k y_k) \to LM\,[E]$. Similarly, $(x_n y_n) \overset{st}{\to}$ $\overset{st}{\to} LM\,[A]$.

(iii) If (r_n) of A is dense positive, we conclude $|M| \geq d$ by Proposition 4.1. So, we can assume that (r_n) of A is dense positive or $M \neq 0$. Since (y_n) satisfies the condition $|y_n| \geq d$, by (ii), we have $\left(\dfrac{x_n}{y_n}\right) \overset{st}{\to} \dfrac{L}{M}\,[A]$.

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