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ESTIMATES FOR λ-SPIRALLIKE FUNCTION OF COMPLEX ORDER ON THE BOUNDARY

ОЦІНКИ ДЛЯ СПІРАЛЕПОДІБНОЇ λ-ФУНКЦІЇ КОМПЛЕКСНОГО ПОРЯДКУ НА ГРАНИЦІ

We give some results for λ-spirallike function of complex order at the boundary of the unit disc U. The sharpness of these results is also proved. Furthermore, three examples for our results are considered.

Наведено деякі результати для спіралеподібної λ-функції комплексного порядку на границі одиничного диска U, а також доведено точність цих результатів. Крім того, розглянуто три приклади для ілюстрації цих результатів.

1. Introduction. Let f be a holomorphic function in the unit disc $U = \{z : |z| < 1\}$, $f(0) = 0$ and $|f(z)| < 1$ for $|z| < 1$. In accordance with the classical Schwarz lemma, for any point z in the disc U, we have $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$. Equality in these inequalities (in the first one, for $z \neq 0$) occurs only if $f(z) = ze^{i\theta}$, where θ is a real number [7, p. 329]. The study of generalizations and variations of Schwarz's lemma as well as Littlewood's theorem is of fundamental significance in the area of geometric function theory and attracts many authors' interest during the last years (see, for example, [3, 7, 20] and the references therein). The generalization of the Schwarz lemma as follows:

$$|f(z)| \leq |z| \frac{|z| + |f'(0)|}{1 + |z||f'(0)|}, \quad z \in U. \tag{1.1}$$

Inequality (1.1) and its generalizations have important applications in geometric theory of functions (see, e.g., [7, 16, 19]). Therefore, the interest in such type results has also continued in recent years (see, e.g., [2, 3, 5, 6, 11, 12, 16–19] and references therein).

Let $\mathcal{A}$ denote the class of functions

$$f(z) = z + c_2 z^2 + c_3 z^3 + \ldots$$

which are holomorphic in the unit disc U. Let $\mathcal{M}$ denote the class of bounded holomorphic functions $h(z)$ in U, satisfying the condition $h(0) = 0$ and $|h(z)| \leq |z|$ for $z \in U$. For a function belonging to the class $\mathcal{A}$ we say that $f(z)$ is λ-*spirallike* function of complex order in U if and only if

$$\Re\left(\frac{1}{b\cos\lambda}\left[e^{i\lambda}\frac{zf'(z)}{f(z)} - (1-b)\cos\lambda - i\sin\lambda\right]\right) > 0, \tag{1.2}$$

for some real λ, $|\lambda| < \dfrac{\pi}{2}$, $b \neq 0$, complex.We denote this class by $\mathcal{N}(\lambda)$. It was introduced and studied by Al-Oboudi and Haidan [1].

It is easy to show that $f(z) \in \mathcal{N}(\lambda)$ if and only if there is an $h \in \mathcal{M}$ such that

$$\sec\lambda e^{i\lambda}\frac{zf'(z)}{f(z)} - i\tan\lambda = \frac{1 + (2b-1)h(z)}{1 - h(z)} \tag{1.3}$$

for $z \in U$ and for some λ, $|\lambda| < \dfrac{\pi}{2}$.

Therefore, from (1.3), we take

$$h(z) = e^{i\lambda} \frac{\dfrac{zf'(z)}{f(z)} - 1}{e^{i\lambda}\left(\dfrac{zf'(z)}{f(z)} - 1\right) + 2b\cos\lambda}.$$

Since $h(z) \in \mathcal{M}$, from Schwarz lemma, we obtain

$$h(z) = e^{i\lambda} \frac{c_2 z + \left(2c_3 - c_2^2\right) z^2 + \ldots}{e^{i\lambda}\left(c_2 z + \left(2c_3 - c_2^2\right) z^2 + \ldots\right) + 2b\cos\lambda},$$

$$\frac{h(z)}{z} = e^{i\lambda} \frac{c_2 + \left(2c_3 - c_2^2\right) z + \ldots}{e^{i\lambda}\left(c_2 z + \left(2c_3 - c_2^2\right) z^2 + \ldots\right) + 2b\cos\lambda},$$

$$|h'(0)| = \frac{|c_2|}{2|b|\cos\lambda} \leq 1$$

and

$$|c_2| \leq 2|b|\cos\lambda. \tag{1.4}$$

Moreover, the equality in (1.4) occurs for the function

$$f(z) = \frac{z}{(1-z)^{2be^{-i\lambda}\cos\lambda}}.$$

It is an elementary consequence of Schwarz lemma that if f extends continuously to some boundary point c with $|c| = 1$, and if $|f(c)| = 1$ and $f'(c)$ exists, then $|f'(c)| \geq 1$, which is known as the Schwarz lemma on the boundary. Passing to the angular limit in (1.1), we obtain the boundary Schwarz lemma [16]

$$|f'(c)| \geq \frac{2}{1 + |f'(0)|}. \tag{1.5}$$

For $c = 1$, the equality in (1.5) occurs for the function

$$f(z) = z\frac{z+a}{1+az},$$

where $0 \leq a \leq 1$. It follow that

$$|f'(c)| \geq 1 \tag{1.6}$$

with equality only if f is of the form $f(z) = ze^{i\theta}$, θ is real.

V. N. Dubinin has continued this line and has made a refinement on the boundary Schwarz lemma by considering the function $f(z) = c_p z^p + c_{p+1} z^{p+1} + \ldots$ with a zero set $\{z_k\}$ (see [5]).

S. G. Krantz and D. M. Burns [10] and D. Chelst [4] has studied the uniqueness part of the Schwarz lemma. P. R. Mercer [11] has proved a version of the Schwarz lemma where the images of two points are known. Also, he has considered some Schwarz and Carathéodory inequalities at the boundary, as consequences of a lemma due to Rogosinski [12]. For more general results and related estimates, we refer to the papers [13 – 15].

Also, M. Jeong [9] has obtained some inequalities at a boundary point for different form of holomorphic functions and has found the condition for equality and also in [8], has defined a holomorphic self map on the closed unit disc with fixed points only on the boundary of the unit disc.

In the proofs of our main results, we will resort to the following lemma due to Julia – Wolff [19].

Lemma 1 (Julia – Wolff lemma). *Let f be a holomorphic function in U, $f(0)=0$ and $f(U)\subset \subset U$. If, in addition, the function f has an angular limit $f(c)$ at $c\in\partial U$, $|f(c)|=1$, then the angular derivative $f'(c)$ exists and $1\leq |f'(c)|\leq\infty$.*

***Corollary* 1.** *The holomorphic function f has a finite angular derivative $f'(c)$ if and only if f' has the finite angular limit $f'(c)$ at $c\in\partial U$.*

2. Main results. In this section, for holomorphic function $f(z)=z+c_2z^2+c_3z^3+\ldots$ belonging to the class of $\mathcal{N}(\lambda)$, the modulus of the angular derivative of the function $\dfrac{zf'(z)}{f(z)}$ will be estimated from below on the boundary point of the unit disc. The sharpness of these results is also proved. Furthermore, examples will be presented for the inequalities obtained.

Theorem 1. *Let $f(z)\in\mathcal{N}(\lambda)$. Assume that, for some $c\in\partial U$, f has angular limit $f(c)$ at c and $\dfrac{cf'(c)}{f(c)}=1-b\dfrac{\cos\lambda}{e^{i\lambda}}$. Then we have the inequality*

$$\left|\left(\frac{zf'(z)}{f(z)}\right)'_{z=c}\right|\geq\frac{|b|}{2}\cos\lambda. \tag{2.1}$$

The equality in (2.1) *occurs for the function*

$$f(z)=\frac{z}{(1-z)^{2be^{-i\lambda}\cos\lambda}}.$$

Proof. Since $f(z)\in\mathcal{N}(\lambda)$, we write

$$\sec\lambda e^{i\lambda}\frac{zf'(z)}{f(z)}-i\tan\lambda=\frac{1+(2b-1)h(z)}{1-h(z)}$$

for $z\in U$ and for some λ, $|\lambda|<\dfrac{\pi}{2}$ and $h(z)\in\mathcal{M}$. Thus, we get

$$h(z)=e^{i\lambda}\frac{\dfrac{zf'(z)}{f(z)}-1}{e^{i\lambda}\left(\dfrac{zf'(z)}{f(z)}-1\right)+2b\cos\lambda};$$

$h(z)$ is a holomorphic function in the unit disc U, $|h(z)|<1$ for $|z|<1$ and $h(0)=0$. For $c\in\partial U$ and $\dfrac{cf'(c)}{f(c)}=1-b\dfrac{\cos\lambda}{e^{i\lambda}}$, we take

$$|h(c)|=\left|e^{i\lambda}\frac{\dfrac{cf'(c)}{f(c)}-1}{e^{i\lambda}\left(\dfrac{cf'(c)}{f(c)}-1\right)+2b\cos\lambda}\right|=\left|\frac{1-b\dfrac{\cos\lambda}{e^{i\lambda}}-1}{e^{i\lambda}\left(1-b\dfrac{\cos\lambda}{e^{i\lambda}}-1\right)+2b\cos\lambda}\right|=$$

$$=\left|\frac{-b\cos\lambda}{-b\cos\lambda+2b\cos\lambda}\right|=\left|\frac{b\cos\lambda}{b\cos\lambda}\right|=1.$$

Let

$$g(z)=\frac{zf'(z)}{f(z)}.$$

So,

$$h(z)=e^{i\lambda}\frac{g(z)-1}{e^{i\lambda}\left(g(z)-1\right)+2b\cos\lambda}.$$

From (1.6), we obtain

$$1\leq |h'(c)|=\left|e^{i\lambda}\frac{2b\cos\lambda g'(c)}{\left(e^{i\lambda}\left(g(c)-1\right)+2b\cos\lambda\right)^2}\right|=$$

$$=\left|\frac{2b\cos\lambda g'(c)}{\left(e^{i\lambda}\left(1-b\dfrac{\cos\lambda}{e^{i\lambda}}-1\right)+2b\cos\lambda\right)^2}\right|=$$

$$=\left|\frac{2b\cos\lambda g'(c)}{\left(-b\cos\lambda+2b\cos\lambda\right)^2}\right|=\left|\frac{2b\cos\lambda g'(c)}{\left(b\cos\lambda\right)^2}\right|=\frac{2|g'(c)|}{|b|\cos\lambda}$$

and

$$|g'(c)|\geq\frac{|b|}{2}\cos\lambda.$$

Now, we shall show that inequality (2.1) is sharp. Let

$$f(z)=\frac{z}{(1-z)^{2be^{-i\lambda}\cos\lambda}}. \tag{2.2}$$

Differentiating (2.2) logarithmically, we have

$$\ln f(z)=\ln\frac{z}{(1-z)^{2be^{-i\lambda}\cos\lambda}},$$

$$\frac{f'(z)}{f(z)}=\frac{1}{z}+\frac{2be^{-i\lambda}\cos\lambda}{1-z}$$

and

$$g(z)=\frac{zf'(z)}{f(z)}=1+2be^{-i\lambda}\cos\lambda\frac{z}{1-z}.$$

Therefore, we take

$$g'(z)=\frac{1}{(1-z)^2}2be^{-i\lambda}\cos\lambda$$

and

$$\left|g'(-1)\right|=\frac{|b|}{2}\cos\lambda.$$

Theorem 1 is proved.

Inequality (2.1) can be strengthened as below by taking into account c_2 which is the second coefficient in the expansion of the function $f(z)$.

Theorem 2. *Under the same assumptions as in* Theorem 1, *we have the inequality*

$$\left|\left(\frac{zf'(z)}{f(z)}\right)'_{z=c}\right|\geq\frac{2|b|^2\cos^2\lambda}{2|b|\cos\lambda+|c_2|}. \tag{2.3}$$

Inequality (2.3) *is sharp with equality for the function*

$$f(z)=\frac{z}{\left(z^2+2dz+1\right)^{e^{-i\lambda}b\cos\lambda}},$$

where $d=\dfrac{|c_2|}{2|b|\cos\lambda}$ *is an arbitrary number on* $[0,1]$ (*see* (1.4)).

Proof. Let $h(z)$ be as in the proof of Theorem 1. By using inequality (1.5) for the function $h(z)$, we obtain

$$\frac{2}{1+|h'(0)|}\leq |h'(c)|=\left|e^{i\lambda}\frac{2b\cos\lambda g'(c)}{\left(e^{i\lambda}\left(g(c)-1\right)+2b\cos\lambda\right)^2}\right|=\frac{2|g'(c)|}{|b|\cos\lambda},$$

where $g(z)=\left(\dfrac{zf'(z)}{f(z)}\right)$.

Since

$$h'(z)=e^{i\lambda}\frac{2b\cos\lambda g'(z)}{\left(e^{i\lambda}\left(g(z)-1\right)+2b\cos\lambda\right)^2},$$

$$h'(0)=e^{i\lambda}\frac{2b\cos\lambda g'(0)}{\left(e^{i\lambda}\left(g(0)-1\right)+2b\cos\lambda\right)^2}$$

and

$$\left|h'(0)\right|=\frac{|c_2|}{2|b|\cos\lambda},$$

we take

$$\frac{2}{1+\dfrac{|c_2|}{2|b|\cos\lambda}}\leq\frac{2|g'(c)|}{|b|\cos\lambda}.$$

Thus, we obtain inequality (2.3).

Now, we shall show that inequality (2.3) is sharp. Choose arbitrary $d\in[0,1]$. Let

$$f(z)=\frac{z}{\left(z^2+2dz+1\right)^{e^{-i\lambda}b\cos\lambda}}. \tag{2.4}$$

Differentiating (2.4) logarithmically, we get

$$\ln f(z)=\ln\frac{z}{\left(z^2+2dz+1\right)^{e^{-i\lambda}b\cos\lambda}},$$

$$\frac{f'(z)}{f(z)}=\frac{1}{z}-2e^{-i\lambda}b\cos\lambda\frac{z+d}{z^2+2dz+1}$$

and

$$g(z)=\frac{zf'(z)}{f(z)}=1-2e^{-i\lambda}b\cos\lambda\frac{z^2+dz}{z^2+2dz+1}.$$

Thus, since $d=\dfrac{|c_2|}{2|b|\cos\lambda}$, we obtain

$$|g'(1)| = \frac{2|b|^2 \cos^2 \lambda}{2|b| \cos \lambda + |c_2|}.$$

Theorem 2 is proved.

Now, inequality (2.3) can be strengthened as below by taking into account c_3 which is the third coefficient in the expansion of the function $f(z)$.

Theorem 3. *Let $f(z)$ belong to $\mathcal{N}(\lambda)$. Assume that, for some $c \in \partial U$, f has angular limit $f(c)$ at c and $\dfrac{cf'(c)}{f(c)} = 1 - b\dfrac{\cos \lambda}{e^{i\lambda}}$. Then we have the inequality*

$$\left| \left(\frac{zf'(z)}{f(z)} \right)'_{z=c} \right| \geq \frac{|b| \cos \lambda}{2} \left(1 + \frac{2\big(2|b| \cos \lambda - |c_2|\big)^2}{4|b|^2 \cos^2 \lambda - |c_2|^2 + |(2c_3 - c_2^2)2b \cos \lambda - e^{i\lambda} c_2^2|} \right). \tag{2.5}$$

Inequality (2.5) *is sharp with equality for the function*

$$f(z) = \frac{z}{(1 - z^2)^{e^{-i\lambda} b \cos \lambda}}.$$

Proof. Let $h(z)$ be as in the proof of Theorem 1 and $\eta(z) = z$. By the maximum principle for each $z \in U$, we have $|h(z)| \leq |\eta(z)|$. So,

$$p(z) = \frac{h(z)}{\eta(z)}$$

is a holomorphic function in U and $|p(z)| < 1$ for $|z| < 1$.

Since

$$h(z) = e^{i\lambda} \frac{c_2 z + \big(2c_3 - c_2^2\big) z^2 + \ldots}{e^{i\lambda} \left(c_2 z + \big(2c_3 - c_2^2\big) z^2 + \ldots\right) + 2b \cos \lambda}$$

and

$$p(z) = \frac{h(z)}{z} = e^{i\lambda} \frac{z \left(c_2 + \big(2c_3 - c_2^2\big) z + \ldots\right)}{\left(e^{i\lambda} \left(c_2 z + \big(2c_3 - c_2^2\big) z^2 + \ldots\right) + 2b \cos \lambda\right) z} =$$

$$= e^{i\lambda} \frac{\left(c_2 + \big(2c_3 - c_2^2\big) z + \ldots\right)}{\left(e^{i\lambda} \left(c_2 z + \big(2c_3 - c_2^2\big) z^2 + \ldots\right) + 2b \cos \lambda\right)},$$

in particular, we have

$$|p(0)| = \frac{|c_2|}{2|b| \cos \lambda} \leq 1 \tag{2.6}$$

and

$$|p'(0)| = \frac{\left|(2c_3 - c_2^2)2b \cos \lambda - e^{i\lambda} c_2^2\right|}{4|b|^2 \cos^2 \lambda}.$$

Moreover, it can be seen that

$$\frac{ch'(c)}{h(c)} = |h'(c)| \geq |\eta'(c)| = \frac{c\eta'(c)}{\eta(c)}.$$

The function

$$\Gamma(z)=\frac{p(z)-p(0)}{1-\overline{p(0)}p(z)}$$

is holomorphic in the unit disc U, $|\Gamma(z)|<1$ for $|z|<1$, $\Gamma(0)=0$ and $|\Gamma(c)|=1$ for $c\in\partial U$.

From (1.5), we obtain

$$\frac{2}{1+|\Gamma'(0)|}\le\left|\Gamma'(c)\right|=\frac{1-|p(0)|^2}{|1-\overline{p(0)}p(c)|^2}|p'(c)|\le\frac{1+|p(0)|}{1-|p(0)|}|p'(c)|=$$
$$=\frac{1+|p(0)|}{1-|p(0)|}\left\{|h'(c)|-1\right\}.$$

Since

$$\Gamma'(z)=\frac{1-|p(0)|^2}{\left(1-\overline{p(0)}p(z)\right)^2}p'(z),$$

$$|\Gamma'(0)|=\frac{|p'(0)|}{1-|p(0)|^2}=\frac{\dfrac{\left|\left(2c_3-c_2^2\right)2b\cos\lambda-e^{i\lambda}c_2^2\right|}{4|b|^2\cos^2\lambda}}{1-\left(\dfrac{|c_2|}{2|b|\cos\lambda}\right)^2}=\frac{\left|\left(2c_3-c_2^2\right)2b\cos\lambda-e^{i\lambda}c_2^2\right|}{4|b|^2\cos^2\lambda-|c_2|^2},$$

we take

$$\frac{2}{1+\dfrac{\left|\left(2c_3-c_2^2\right)2b\cos\lambda-e^{i\lambda}c_2^2\right|}{4|b|^2\cos^2\lambda-|c_2|^2}}\le\frac{1+\dfrac{|c_2|}{2|b|\cos\lambda}}{1-\dfrac{|c_2|}{2|b|\cos\lambda}}\left\{\frac{2|g'(c)|}{|b|\cos\lambda}-1\right\}=$$
$$=\frac{2|b|\cos\lambda+|c_2|}{2|b|\cos\lambda-|c_2|}\left\{\frac{2|g'(c)|}{|b|\cos\lambda}-1\right\}.$$

Therefore, we get

$$1+\frac{2\left(4|b|^2\cos^2\lambda-|c_2|^2\right)}{4|b|^2\cos^2\lambda-|c_2|^2+\left|\left(2c_3-c_2^2\right)2b\cos\lambda-e^{i\lambda}c_2^2\right|}\frac{2|b|\cos\lambda-|c_2|}{2|b|\cos\lambda+|c_2|}\le\frac{2|g'(c)|}{|b|\cos\lambda}$$

and

$$\frac{|b|\cos\lambda}{2}\left(1+\frac{2\left(2|b|\cos\lambda-|c_2|\right)^2}{4|b|^2\cos^2\lambda-|c_2|^2+\left|\left(2c_3-c_2^2\right)2b\cos\lambda-e^{i\lambda}c_2^2\right|}\right)\le\left|g'(c)\right|.$$

So, we obtain inequality (2.5).

To show that inequality (2.5) is sharp, take the holomorphic function

$$f(z)=\frac{z}{(1-z^2)^{e^{-i\lambda}b\cos\lambda}}. \tag{2.7}$$

Differentiating (2.7) logarithmically, we obtain

$$\ln f(z)=\ln\frac{z}{(1-z^2)^{e^{-i\lambda}b\cos\lambda}},$$

$$\frac{f'(z)}{f(z)} = \frac{1}{z} + e^{-i\lambda} b \cos\lambda \frac{2z}{1-z^2}$$

and

$$g(z) = \frac{zf'(z)}{f(z)} = 1 + e^{-i\lambda} b \cos\lambda \frac{2z^2}{1-z^2}.$$

Therefore, we take

$$g'(z) = e^{-i\lambda} b \cos\lambda \frac{4z}{(1-z^2)^2}$$

and

$$|g'(i)| = |b| \cos\lambda.$$

Since $|c_2| = 0$ and $|c_3| = |b| \cos\lambda$, (2.5) is satisfied with equality.

Theorem 3 is proved.

If $f(z) - z$ has no zeros different from $z = 0$ in Theorem 3, inequality (2.5) can be further strengthened. This is given by the following theorem.

Theorem 4. *Let $f(z) \in \mathcal{N}(\lambda)$, $f(z) - z$ has no zeros in U except $z = 0$ and $c_2 > 0$. Suppose that, for some $c \in \partial U$, f has angular limit $f(c)$ at c and $\dfrac{cf'(c)}{f(c)} = 1 - b\dfrac{\cos\lambda}{e^{i\lambda}}$. Then we have the inequality*

$$\left|\left(\frac{zf'(z)}{f(z)}\right)'_{z=c}\right| \geq \frac{|b|\cos\lambda}{2}\left(1 - \frac{4|c_2||b|\cos\lambda \ln^2\left(\dfrac{|c_2|}{2|b|\cos\lambda}\right)}{4|c_2||b|\cos\lambda \ln\left(\dfrac{|c_2|}{2|b|\cos\lambda}\right) - \left|\left(2c_3 - c_2^2\right) 2b\cos\lambda - e^{i\lambda}c_2^2\right|}\right). \tag{2.8}$$

Proof. Let $c_2 > 0$ in the expression of the function $f(z)$. Having in mind inequality (2.6) and the function $f(z) - z$ has no zeros in U except $z = 0$, we denote by $\ln p(z)$ the holomorphic branch of the logarithm normed by the condition

$$\ln p(0) = \ln\left(\frac{|c_2|}{2|b|\cos\lambda}\right) < 0.$$

The auxiliary function

$$\Upsilon(z) = \frac{\ln p(z) - \ln p(0)}{\ln p(z) + \ln p(0)}$$

is holomorphic in the unit disc U, $|\Upsilon(z)| < 1$, $\Upsilon(0) = 0$ and $|\Upsilon(c)| = 1$ for $c \in \partial U$.

From (1.5), we get

$$\frac{2}{1 + |\Upsilon'(0)|} \leq |\Upsilon'(c)| = \frac{|2\ln p(0)|}{|\ln p(c) + \ln p(0)|^2}\left|\frac{p'(c)}{p(c)}\right| = \frac{-2\ln p(0)}{\ln^2 p(0) + \arg^2 p(c)}\left\{|h'(c)| - 1\right\}.$$

Replacing $\arg^2 p(c)$ by zero, then

$$\frac{1}{1 - \dfrac{1}{2\ln\left(\dfrac{|c_2|}{2|b|\cos\lambda}\right)} \dfrac{\left|\left(2c_3 - c_2^2\right) 2b\cos\lambda - e^{i\lambda}c_2^2\right|}{2|c_2||b|\cos\lambda}} \leq \frac{-1}{\ln\left(\dfrac{|c_2|}{2|b|\cos\lambda}\right)}\left\{\frac{2|g'(c)|}{|b|\cos\lambda} - 1\right\}$$

and

$$1-\frac{4|c_2||b|\cos\lambda\ln^2\left(\dfrac{|c_2|}{2|b|\cos\lambda}\right)}{4|c_2||b|\cos\lambda\ln\left(\dfrac{|c_2|}{2|b|\cos\lambda}\right)-\left|\left(2c_3-c_2^2\right)2b\cos\lambda-e^{i\lambda}c_2^2\right|}\leq\frac{2|g'(c)|}{|b|\cos\lambda}.$$

Thus, we obtain inequality (2.8).

The following inequality (2.9) is weaker, but is simpler than (2.8) and does not contain the coefficient c_3.

Theorem 5. *Under the same assumptions as in Theorem* 4, *we have the inequality*

$$\left|\left(\frac{zf'(z)}{f(z)}\right)'_{z=c}\right|\geq\frac{|b|\cos\lambda}{2}\left(1-\frac{1}{2}\ln\left(\frac{|c_2|}{2|b|\cos\lambda}\right)\right). \tag{2.9}$$

Proof. Let $c_2>0$. Using inequality (1.6) for the function $\Gamma(z)$, we obtain

$$1\leq|\Upsilon'(c)|=\frac{|2\ln p(0)|}{|\ln p(c)+\ln p(0)|^2}\left|\frac{p'(c)}{p(c)}\right|=\frac{-2\ln p(0)}{\ln^2 p(0)+\arg^2 p(c)}\left\{|h'(c)|-1\right\}.$$

Replacing $\arg^2 p(c)$ by zero, then

$$1\leq|\Upsilon'(c)|\leq\frac{-2}{\ln\left(\dfrac{|c_2|}{2|b|\cos\lambda}\right)}\left\{\frac{2|g'(c)|}{|b|\cos\lambda}-1\right\}.$$

Therefore, we obtain inequality (2.9).

Theorem 4 is proved.

3. Examples.

***Example* 1.** Let us consider the function $f(z)$ defined by

$$f(z)=\frac{z}{(1-z)^{2be^{-i\lambda}\cos\lambda}}.$$

From here, we have

$$\frac{zf'(z)}{f(z)}=1+2be^{-i\lambda}\cos\lambda\frac{z}{1-z}.$$

So, we take

$$\frac{1}{b\cos\lambda}\left[e^{i\lambda}\frac{zf'(z)}{f(z)}-(1-b)\cos\lambda-i\sin\lambda\right]=$$

$$=\frac{1}{b\cos\lambda}\left[e^{i\lambda}\left(1+2be^{-i\lambda}\cos\lambda\frac{z}{1-z}\right)-(1-b)\cos\lambda-i\sin\lambda\right]=$$

$$=\frac{1}{b\cos\lambda}\left[e^{i\lambda}+2b\cos\lambda\frac{z}{1-z}-\cos\lambda+b\cos\lambda-i\sin\lambda\right]=$$

$$=\frac{1}{b\cos\lambda}\left[e^{i\lambda}+2b\cos\lambda\frac{z}{1-z}-e^{i\lambda}+b\cos\lambda\right]=$$

$$=\frac{1}{b\cos\lambda}\left[b\cos\lambda\left(\frac{2z}{1-z}+1\right)\right]=\frac{1+z}{1-z}.$$

Obviously, the function

$$k(z) = \frac{1+z}{1-z}$$

maps the unit disc U to the right half plane and, hence, the real part of the function $k(z)$ is nonnegative. Therefore, we obtain

$$\Re\left(\frac{1}{b\cos\lambda}\left[e^{i\lambda}\frac{zf'(z)}{f(z)} - (1-b)\cos\lambda - i\sin\lambda\right]\right) = \Re\left(\frac{1+z}{1-z}\right) > 0.$$

Thus, the function $f(z)$ satisfies condition (1.2) and Theorem 1. That is,

$$f(z) = \frac{z}{(1-z)^{2be^{-i\lambda}\cos\lambda}}$$

and, for some $c \in \partial U$, we have

$$\frac{(-1)f'(-1)}{f(-1)} = 1 + 2be^{-i\lambda}\cos\lambda\left(\frac{-1}{1-(-1)}\right) = 1 + 2be^{-i\lambda}\cos\lambda\left(\frac{-1}{2}\right)$$

and

$$\frac{(-1)f'(-1)}{f(-1)} = 1 - be^{-i\lambda}\cos\lambda.$$

***Example* 2.** Let us consider the function $f(z)$ defined by

$$f(z) = \frac{z}{\left(z^2 + 2dz + 1\right)^{e^{-i\lambda}b\cos\lambda}},$$

where $d = \dfrac{|c_2|}{2|b|\cos\lambda}$ is an arbitrary number from $[0,1]$ (see (1.4)). Hence, we get

$$\frac{zf'(z)}{f(z)} = 1 - 2e^{-i\lambda}b\cos\lambda\frac{z^2+dz}{z^2+2dz+1}.$$

Therefore, we obtain

$$\frac{1}{b\cos\lambda}\left[e^{i\lambda}\frac{zf'(z)}{f(z)} - (1-b)\cos\lambda - i\sin\lambda\right] =$$

$$= \frac{1}{b\cos\lambda}\left[e^{i\lambda}\left(1 - e^{-i\lambda}2b\cos\lambda\frac{z^2+dz}{z^2+2dz+1}\right) - (1-b)\cos\lambda - i\sin\lambda\right] =$$

$$= \frac{1}{b\cos\lambda}\left[e^{i\lambda} - 2b\cos\lambda\frac{z^2+dz}{z^2+2dz+1} - e^{i\lambda} + b\cos\lambda\right] =$$

$$= \frac{1}{b\cos\lambda}\left[-2b\cos\lambda\frac{z^2+dz}{z^2+2dz+1} + b\cos\lambda\right] =$$

$$= \frac{1}{b\cos\lambda}\left[b\cos\lambda\left(-2\frac{z^2+dz}{z^2+2dz+1} + 1\right)\right] = \frac{1-z^2}{z^2+2dz+1}.$$

Since $d \in [0,1]$, we see that

$$\Re\left(\frac{1}{b\cos\lambda}\left[e^{i\lambda}\frac{zf'(z)}{f(z)}-(1-b)\cos\lambda-i\sin\lambda\right]\right)=\Re\left(\frac{1-z^2}{z^2+2dz+1}\right)>0.$$

Thus, the function $f(z)$ satisfies condition (1.2) and Theorem 2. That is,

$$f(z)=\frac{z}{\left(z^2+2dz+1\right)^{e^{-i\lambda}b\cos\lambda}},$$

for some $c\in\partial U$, we have

$$\frac{f'(1)}{f(1)}=1-2e^{-i\lambda}b\cos\lambda\frac{1+d}{1+2d+1}=1-2e^{-i\lambda}b\cos\lambda\frac{1+d}{2(1+d)}$$

and

$$\frac{f'(1)}{f(1)}=1-e^{-i\lambda}b\cos\lambda.$$

***Example* 3.** Let us consider the function $f(z)$ given by

$$f(z)=\frac{z}{(1-z^2)^{e^{-i\lambda}b\cos\lambda}}.$$

Similar to other examples, it can be easily shown that the function $f(z)$ provides the properties of class $\mathcal{N}(\lambda)$.

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