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PALEY – WIENER TYPE THEOREM FOR FUNCTIONS WITH VALUES IN BANACH SPACES*

ТЕОРЕМА ТИПУ ПЕЛІ – ВІНЕРА ДЛЯ ФУНКЦІЙ ІЗ ЗНАЧЕННЯМИ У БАНАХОВИХ ПРОСТОРАХ

Let $(\mathbb{X}, \|.\|_{\mathbb{X}})$ denote a complex Banach space and $L(\mathbb{X}) = BC(\mathbb{R} \to \mathbb{X})$ be the set of all $\mathbb{X}$-valued bounded continuous functions $f : \mathbb{R} \to \mathbb{X}$. For $f \in L(\mathbb{X})$ we define $\|f\|_{L(\mathbb{X})} = \sup\{\|f(x)\|_{\mathbb{X}} : x \in \mathbb{R}\}$. Then $(L(\mathbb{X}), \|.\|_{L(\mathbb{X})})$ itself is a Banach space. The Beurling spectrum $\mathrm{Spec}(f)$ of a function $f \in L(\mathbb{X})$ is defined by

$$\mathrm{Spec}(f) = \{\zeta \in \mathbb{R} : \forall\epsilon > 0\ \exists\varphi \in \mathcal{S}(\mathbb{R}) : \mathrm{supp}\,\widehat{\varphi} \subset (\zeta - \epsilon, \zeta + \epsilon), \varphi * f \not\equiv 0\}\,.$$

We obtain the following Paley – Wiener type theorem for functions with values in Banach spaces:

Let $f \in L(\mathbb{X})$ and K be an arbitrary compact set in $\mathbb{R}$. Then $\mathrm{Spec}(f) \subset K$ if and only if for any $\tau > 0$ there exists a constant $C_\tau < \infty$ such that

$$\|P(D)f\|_{L(\mathbb{X})} \le C_\tau \|f\|_{L(\mathbb{X})} \sup_{x \in K^{(\tau)}} |P(x)|$$

for all polynomials with complex coefficients $P(x)$, where the differential operator $P(D)$ is obtained from $P(x)$ by substituting $x \to -i\dfrac{d}{dx}$, $\dfrac{d}{dx}$ is the usual derivative in $L(\mathbb{X})$ and $K^{(\tau)}$ is the τ-neighborhood in $\mathbb{C}$ of K.

Moreover, Paley – Wiener type theorem for integral operators and one for some special compacts K are also given.

Нехай $(\mathbb{X}, \|.\|_{\mathbb{X}})$ — комплексний простір Банаха і $L(\mathbb{X}) = BC(\mathbb{R} \to \mathbb{X})$ — множина всіх обмежених неперервних $\mathbb{X}$-значних функцій $f : \mathbb{R} \to \mathbb{X}$. Для $f \in L(\mathbb{X})$ вводиться позначення $\|f\|_{L(\mathbb{X})} = \sup\{\|f(x)\|_{\mathbb{X}} : x \in \mathbb{R}\}$. Тоді $(L(\mathbb{X}), \|.\|_{L(\mathbb{X})})$ сам є банаховим простором. Спектр Берлінга $\mathrm{Spec}(f)$ функції $f \in L(\mathbb{X})$ визначається як

$$\mathrm{Spec}(f) = \{\zeta \in \mathbb{R} : \forall\epsilon > 0\ \exists\varphi \in \mathcal{S}(\mathbb{R}) : \mathrm{supp}\,\widehat{\varphi} \subset (\zeta - \epsilon, \zeta + \epsilon), \varphi * f \not\equiv 0\}\,.$$

Отримано таку теорему типу Пелі – Вінера для функцій із значеннями у просторах Банаха:

Нехай $f \in L(\mathbb{X})$ і K — довільна компактна множина в $\mathbb{R}$. У цьому випадку $\mathrm{Spec}(f) \subset K$ тоді й лише тоді, коли для будь-якого $\tau > 0$ існує стала $C_\tau < \infty$ така, що

$$\|P(D)f\|_{L(\mathbb{X})} \le C_\tau \|f\|_{L(\mathbb{X})} \sup_{x \in K^{(\tau)}} |P(x)|$$

для всіх поліномів з комплексними коефіцієнтами $P(x)$, де диференціальний оператор $P(D)$ отримано з $P(x)$ заміною $x \to -i\dfrac{d}{dx}$, $\dfrac{d}{dx}$ — звичайна похідна у $L(\mathbb{X})$ і $K^{(\tau)}$ — τ-окіл для K у $\mathbb{C}$.

Також наведено теорему типу Пелі – Вінера для інтегральних операторів та деяких спеціальних компактів K.

1. Introduction. The relation between properties of functions and their spectrum (the support of their Fourier transform) are interested for many mathematicians. The Paley – Wiener theorem is one of the well-known results belonging to this direction. The initial Paley – Wiener theorem was proved for L^2-functions, was extended to generalized functions by L. Schwartz and has many generalizations

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(see, for example, [1 – 8, 10 – 14, 16, 17]. In this paper we provide Paley – Wiener type theorem for functions with values in Banach space.

Let $f \in L^1(\mathbb{R})$ and $\widehat{f} = \mathcal{F}f$ be the Fourier transform of f

$$\widehat{f}(\zeta) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} e^{-ix\zeta} f(x)\, dx,$$

and $\check{f} = \mathcal{F}^{-1}f$ denote its inverse Fourier transform

$$\check{f}(\zeta) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} e^{ix\zeta} f(x)\, dx.$$

Let $(\mathbb{X}, \|.\|_{\mathbb{X}})$ denote a complex Banach space and $L(\mathbb{X}) = BC(\mathbb{R} \to \mathbb{X})$ be the set of all $\mathbb{X}$-valued bounded continuous functions $f\colon \mathbb{R} \to \mathbb{X}$. For a given function $f \in L(\mathbb{X})$ we define $\|f\|_{L(\mathbb{X})} = \sup\{\|f(x)\|_{\mathbb{X}} : x \in \mathbb{R}\}$. Then $(L(\mathbb{X}), \|.\|_{L(\mathbb{X})})$ itself is a Banach space. We define the derivative Df of $f \in L(\mathbb{X})$, as usual,

$$Df(\zeta) = \lim_{x\to 0} \frac{f(\zeta + x) - f(\zeta)}{x}.$$

Given $f \in L(\mathbb{X})$. It was shown in [15] that for every $\lambda \in \mathbb{C}\backslash i\mathbb{R}$ the equation $\lambda u(x) - Du(x) = f(x)$ has a unique solution $u_{f,\lambda} \in L(\mathbb{X})$. That means the operator $\lambda - D$ is invertible and $(\lambda - D)^{-1} f = u_{f,\lambda}$. More exactly,

$$(\lambda - D)^{-1} f(\zeta) = \begin{cases} \int_0^{\infty} e^{-\lambda x} f(\zeta + x)\, dx, & \text{if} \quad \operatorname{Re}\lambda > 0, \\ -\int_0^{\infty} e^{-\lambda x} f(\zeta + x)\, dx, & \text{if} \quad \operatorname{Re}\lambda < 0. \end{cases}$$

Hence, the spectrum of the differential operator $\lambda - D$ is $i\mathbb{R}$ and

$$\int\limits_{-\infty}^{+\infty} \varphi(x)(\lambda - D)^{-n} f(x)\, dx = \int\limits_{-\infty}^{+\infty} \left[(\lambda - D)^{-n}\varphi(x)\right] f(x)\, dx$$

for any $f \in L(\mathbb{X})$ and $\varphi \in \mathcal{S}(\mathbb{R})$, where $\mathcal{S}(\mathbb{R})$ is the Schwartz space. The convolution $\varphi * f$ of f with a Schwartz function φ is defined as follows:

$$\varphi * f(\zeta) = \int\limits_{-\infty}^{+\infty} \varphi(\zeta - x) f(x)\, dx.$$

The Beurling spectrum $\operatorname{Spec}(f)$ of a function $f \in L(\mathbb{X})$ is defined by

$$\operatorname{Spec}(f) = \{\zeta \in \mathbb{R} : \forall\epsilon > 0\ \exists\varphi \in \mathcal{S}(\mathbb{R}) : \operatorname{supp}\widehat{\varphi} \subset (\zeta - \epsilon, \zeta + \epsilon),\ \varphi * f \not\equiv 0\}.$$

Note that $\operatorname{Spec}(f)$ is always a closed subset of $\mathbb{R}$. Let $K \subset \mathbb{R}$ and $\tau > 0$. We put

$$K^{(\tau)} := \{\zeta \in \mathbb{C} : \exists x \in K : |x - \zeta| < \tau\},$$

which is the τ-neighborhood in $\mathbb{C}$ of K and $K^\tau := \{\zeta \in \mathbb{R} : \exists x \in K : |x - \zeta| < \tau\}$, $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$.

Let $P(x)$ be a polynomial. The differential operator $P(D)$ is obtained from $P(x)$ by substituting $x \to -i\,\dfrac{d}{dx}$.

2. Paley – Wiener type theorem for differential operators. ***2.1. Paley – Wiener type theorem for any compact K.***

Theorem 2.1. *Let $f \in L(\mathbb{X})$ and K be an arbitrary compact set in $\mathbb{R}$. Then* $\operatorname{Spec}(f) \subset K$ *if and only if for any $\tau > 0$ there exists a constant $C_{\tau,K} < \infty$ independent of f such that*

$$\|P(D)f\|_{L(\mathbb{X})} \leq C_{\tau,K}\|f\|_{L(\mathbb{X})} \sup_{x \in K^{(\tau)}} |P(x)| \tag{1}$$

for all polynomials with complex coefficients $P(x)$.

To obtain the theorem, we need the following results.

Lemma 2.1 (Young inequality for Banach spaces)**.** *Let $f \in L(\mathbb{X})$ and $\varphi \in \mathcal{S}(\mathbb{R})$. Then $\varphi * f \in$ $\in L(\mathbb{X})$ and*

$$\|\varphi * f\|_{L(\mathbb{X})} \leq \|f\|_{L(\mathbb{X})}\|\varphi\|_{L^1}.$$

Proof. We see that

$$\|\varphi * f\|_{L(\mathbb{X})} = \sup_{s\in\mathbb{R}} \left\| \int_{-\infty}^{+\infty} \varphi(s-t)f(t)\,dt \right\|_{\mathbb{X}} \leq \sup_{s\in\mathbb{R}} \int_{-\infty}^{+\infty} \|\varphi(s-t)f(t)\|_{\mathbb{X}}\,dt \leq$$

$$\leq \|f\|_{L(\mathbb{X})} \sup_{s\in\mathbb{R}} \int_{-\infty}^{+\infty} |\varphi(s-t)|\,dt = \|f\|_{L(\mathbb{X})}\|\varphi\|_{L^1},$$

which completes the proof.

Lemma 2.2 [15]**.** *Let $f \in L(\mathbb{X})$ and φ, $\psi \in \mathcal{S}(\mathbb{R})$. Assume that $\widehat{\varphi} = 0$ on* $\operatorname{Spec}(f)$ *and $\widehat{\psi} = (2\pi)^{-1/2}$ on* $\operatorname{Spec}(f)$*. Then $\varphi * f = 0$ and $\psi * f = f$.*

It was proved in [9] the following radial spectral formula.

Lemma 2.3. *Let $f \in L(\mathbb{X})$ and $P(x)$ be a polynomial. Assume that* $\operatorname{Spec}(f)$ *is compact. Then there always exists the following limit:*

$$\lim_{m\to\infty} \|P^m(D)f\|_{L(\mathbb{X})}^{1/m}$$

and

$$\lim_{m\to\infty} \|P^m(D)f\|_{L(\mathbb{X})}^{1/m} = \sup\left\{|P(\zeta)| : \zeta \in \operatorname{Spec}(f)\right\}.$$

***Proof of Theorem* 2.1.** *Necessity.* We choose a function $\vartheta \in C_0^\infty(\mathbb{R})$ such that $\vartheta(\zeta) = (2\pi)^{-1/2}$ if $\zeta \in K^{\tau/4}$ and $\vartheta(\zeta) = 0$ if $\zeta \notin K^{\tau/2}$. Then it follows from $\operatorname{Spec}(f) \subset K$ and Lemma 2.2 that $f = \mathcal{F}^{-1}(\vartheta) * f$ and $D^n f = \left(D^n \mathcal{F}^{-1}(\vartheta)\right) * f = \mathcal{F}^{-1}\left(\vartheta(\zeta)(i\zeta)^n\right) * f$ for all $n \in \mathbb{Z}_+$. Combining these and the definition of the differential operator $P(D)$, we obtain

$$P(D)f = \mathcal{F}^{-1}(\vartheta(\zeta)P(\zeta)) * f.$$

Therefore, by Lemma 2.1, we have

$$\|P(D)f\|_{L(\mathbb{X})} \leq \|f\|_{L(\mathbb{X})} \left\|\mathcal{F}^{-1}\left(\vartheta(\zeta)P(\zeta)\right)\right\|_{L^1} =$$

$$= \|f\|_{L(\mathbb{X})} \left\|\mathcal{F}\left(\vartheta(\zeta)P(\zeta)\right)\right\|_{L^1} = \|f\|_{L(\mathbb{X})}\|\Psi\|_{L^1},$$

where

$$\Psi(x) := \left(\mathcal{F}\left(\vartheta(\zeta)P(\zeta)\right)\right)(x).$$

Hence, from

$$\int\limits_{\mathbb{R}} |\Psi(x)|\,dx \le \left(\sup_{x\in\mathbb{R}} |\left(1+x^2\right)\Psi(x)|\right)\left(\int\limits_{\mathbb{R}} \frac{dx}{1+x^2}\right) = \pi \sup_{x\in\mathbb{R}} \left|\left(1+x^2\right)\Psi(x)\right|,$$

we obtain

$$\|P(D)f\|_{L(\mathbb{X})} \le \pi \|f\|_{L(\mathbb{X})} \sup_{x\in\mathbb{R}} \left|\left(1+x^2\right)\Psi(x)\right|. \tag{2}$$

For $\beta \in \{0,1,2\}$ we get the following estimate:

$$\sup_{x\in\mathbb{R}} \left|x^\beta \Psi(x)\right| = (2\pi)^{-1/2} \sup_{x\in\mathbb{R}} \left|\int\limits_{\mathbb{R}} e^{-ix\zeta} D^\beta \left(\vartheta(\zeta)P(\zeta)\right)\,d\zeta\right| =$$

$$= (2\pi)^{-1/2} \sup_{x\in\mathbb{R}} \left|\int\limits_{\zeta\in K^{\tau/2}} e^{-ix\zeta} D^\beta \left(\vartheta(\zeta)P(\zeta)\right)\,d\zeta\right| \le$$

$$\le (2\pi)^{-1/2} \int\limits_{\zeta\in K^{\tau/2}} \left|D^\beta \left(\vartheta(\zeta)P(\zeta)\right)\right|\,d\zeta.$$

Then it follows from Leibniz's rule that

$$\sup_{x\in\mathbb{R}} \left|x^\beta \Psi(x)\right| \le (2\pi)^{-1/2} \int\limits_{\zeta\in K^{\tau/2}} \left|\sum_{\gamma\le\beta} \frac{\beta!}{\gamma!(\beta-\gamma)!} D^\gamma \vartheta(\zeta) D^{\beta-\gamma} P(\zeta)\right| d\zeta \le$$

$$\le (2\pi)^{-1/2} \sum_{\gamma\le\beta} \left(\frac{\beta!}{\gamma!(\beta-\gamma)!} \sup_{x\in K^{\tau/2}} \left|D^{\beta-\gamma}P(x)\right| \int\limits_{\zeta\in K^{\tau/2}} |D^\gamma \vartheta(\zeta)|\,d\zeta\right) \le$$

$$\le (2\pi)^{-1/2} \max_{\theta\le 2} \sup_{x\in K^{\tau/2}} \left|D^\theta P(x)\right| \sum_{\gamma\le\beta} \left(\frac{\beta!}{\gamma!(\beta-\gamma)!} \int\limits_{\zeta\in K^{\tau/2}} |D^\gamma \vartheta(\zeta)|\,d\zeta\right). \tag{3}$$

For each $x \in K^{\tau/2}$, we consider $\gamma_x = \{z \in \mathbb{C} \colon |z-x| = \tau/2\}$ as a simple closed curve oriented counterclockwise. Because $P(x)$ is a holomorphic function and by Cauchy's integral formula for derivatives, we obtain, for $n = 0,1,2,\ldots,$

$$D^n P(x) = \frac{n!}{2\pi i} \int\limits_{\gamma_x} \frac{P(z)dz}{(z-x)^{n+1}}.$$

Hence,

$$|D^n P(x)| \leq \frac{n! \sup_{z\in\gamma_x} |P(z)|}{(\tau/2)^n}.$$

Since $\gamma_x \subset K^{(\tau)}$ and above inequalities are true for all $x \in K^{\tau/2}$, we deduce

$$\sup_{x\in K^{\tau/2}} |D^n P(x)| \leq \frac{n! \sup_{z\in K^{(\tau)}} |P(z)|}{(\tau/2)^n}$$

for $n = 0, 1, 2, \ldots$. So,

$$\sup_{x\in K^{\tau/2}} |D^n P(x)| \leq A_\tau \sup_{x\in K^{(\tau)}} |P(x)| \tag{4}$$

for $n = 0, 1, 2$, where $A_\tau = 2\left(1 + (2/\tau)^2\right)$ is independent of $P(x)$. By using (3), (4), we have

$$\sup_{x\in\mathbb{R}} \left|x^\beta \Psi(x)\right| \leq (2\pi)^{-1/2} \sum_{\gamma\leq\beta} \left(\frac{\beta!}{\gamma!(\beta-\gamma)!} A_\tau \sup_{x\in K^{(\tau)}} |P(x)| \int\limits_{\zeta\in K^{\tau/2}} |D^\gamma \vartheta(\zeta)|\, d\zeta \right) \leq$$

$$\leq 4(2\pi)^{-1/2} A_\tau A_{\tau,K} \sup_{x\in K^{(\tau)}} |P(x)| \tag{5}$$

for all $\beta = 0, 1, 2$, where

$$A_{\tau,K} := \max_{\gamma\leq 2} \int\limits_{\zeta\in K^{\tau/2}} |D^\gamma \vartheta(\zeta)|\ d\zeta.$$

Then it follows from (5) that

$$\sup_{x\in\mathbb{R}} \left|\left(1 + x^2\right) \Psi(x)\right| \leq 8(2\pi)^{-1/2} A_\tau A_{\tau,K} \sup_{x\in K^{(\tau)}} |P(x)|. \tag{6}$$

From (2) and (6) we obtain (1).

Sufficiency. Assume (1) is true, we need to prove $\operatorname{Spec}(f) \subset K$. Indeed, assume the contrary that there exists $\varrho \in \operatorname{Spec}(f)$ and $\varrho \notin K$. We construct the polynomial $G(x) = t - (x - \varrho)^2$, where $t = \sup_{x\in K}(x - \varrho)^2$. Then applying (1) for $P(x) = G^m(x)$, we get, for all $m \in \mathbb{Z}_+$,

$$\|G^m(D)f\|_{L(\mathbb{X})} \leq C_\tau \|f\|_{L(\mathbb{X})} \sup_{x\in K^{(\tau)}} |G^m(x)|,$$

which gives

$$\limsup_{m\to\infty} \left(\|G^m(D)f\|_{L(\mathbb{X})}\right)^{1/m} \leq \sup_{x\in K^{(\tau)}} |G(x)|.$$

Letting $\tau \to 0$, we obtain

$$\limsup_{m\to\infty} \left(\|G^m(D)f\|_{L(\mathbb{X})}\right)^{1/m} \leq \sup_{x\in K} |G(x)|. \tag{7}$$

Then it follows from Lemma 2.3 that

$$|G(\varrho)| \leq \sup_{x\in K} |G(x)|$$

and then

$$t = |G(\varrho)| \leq \sup_{x \in K} \left(t - (x - \varrho)^2 \right).$$

This is a contradiction. So, $\mathrm{Spec}(f) \subset K$.

Theorem 2.1 is proved.

It follows from Lemma 2.3 that if $f \in L(\mathbb{X})$ and $\mathrm{Spec}(f) \subset K$, then for any $\tau > 0$ there exists a constant $C_{P,\tau,f} < \infty$ ($C_{P,\tau,f}$ depends on P, τ and f) such that

$$\|P^m(D)f\|_{L(\mathbb{X})} \leq C_{P,\tau,f} \|f\|_{L(\mathbb{X})} \sup_{x \in K^{(\tau)}} |P^m(x)| \quad \forall m \in \mathbb{N},$$

while by Theorem 2.1 we have the stronger result that for any $\tau > 0$ there exists a constant $C_\tau < \infty$ (independent of P, m, f) such that

$$\|P^m(D)f\|_{L(\mathbb{X})} \leq C_\tau \|f\|_{L(\mathbb{X})} \sup_{x \in K^{(\tau)}} |P^m(x)|.$$

2.2. Paley – Wiener type theorem for sets generated by polynomials. Let $P(x)$ be a polynomial with complex coefficients. We put, for $r > 0$,

$$Q(P)_r := \{x \in \mathbb{R} \colon |P(x)| \leq r\}$$

and $Q(P)_r$ is called the set generated by $P(x)$ with respect to r. Note that if $\deg(P) \geq 1$, then the $Q(P)_r$ is compact. Moreover, if $a, b \in \mathbb{R}$, $a \leq b$, $\alpha > 0$, then $[a, a+\alpha] \cup [b, b+\alpha]$ are sets generated by polynomials.

Theorem 2.2. *Let $f \in L(\mathbb{X}), r > 0$ and $P(x)$ be a polynomial. Then $\mathrm{Spec}(f) \subset Q(P)_r$ if and only if for any $\tau > 0$ there exists a constant $C_{\tau,r,P} < \infty$ independent of f such that*

$$\|P^m(D)f\|_{L(\mathbb{X})} \leq C_{\tau,r,P} \|f\|_{L(\mathbb{X})} (r+\tau)^m \tag{8}$$

for all $m \in \mathbb{Z}_+$.

Proof. *Necessity* is follows from Theorem 2.1.

Sufficiency. Assume the contrary that there exists $\sigma \in \mathrm{Spec}(f)$ and $\sigma \notin Q(P)_r$. Combining $\sigma \notin Q(P)_r$ and $Q(P)_r = \{x \in \mathbb{R} \colon |P(x)| \leq r\}$, we have

$$|P(\sigma)| > r.$$

By using (8), we obtain

$$\limsup_{m \to \infty} \left(\|P^m(D)f\|_{L(\mathbb{X})} \right)^{1/m} \leq r + \tau. \tag{9}$$

Applying Lemma 2.3, we have

$$\liminf_{m \to \infty} \left(\|P^m(D)f\|_{L(\mathbb{X})} \right)^{1/m} \geq |P(\sigma)|. \tag{10}$$

From (9) and (10), we get $|P(\sigma)| \leq r + \tau$. Letting $\tau \to 0$, we obtain $|P(\sigma)| \leq r$. This is a contradiction. So, $\mathrm{Spec}(f) \subset Q(P)_r$.

Theorem 2.2 is proved.

By Theorem 2.2 we get the following corollary.

***Corollary* 2.1.** *Let $r > 0$ and $f \in L(\mathbb{X})$. Then* $\operatorname{Spec}(f) \subset [-r, r]$ *if and only if for any $\tau > 0$ there exists a constant $C_\tau < \infty$ such that*

$$\|D^m f\|_{L(\mathbb{X})} \leq C_\tau (r + \tau)^m \|f\|_{L(\mathbb{X})}$$

for all $m \in \mathbb{Z}_+$.

In general, for $a, b \in \mathbb{R},\ a < b$, then $[a, b]$ is the set generated by the polynomial $P(x) = x - \dfrac{a+b}{2}$ with respect to $\dfrac{b-a}{2}$. Therefore, $\operatorname{Spec}(f) \subset [a, b]$ if and only if for any $\tau > 0$ there exists a constant $C_\tau < \infty$ such that

$$\left\| \left(x - \frac{a+b}{2} \right)^m (D) f \right\|_{L(\mathbb{X})} \leq C_\tau \left(\frac{b-a}{2} + \tau \right)^m \|f\|_{L(\mathbb{X})}$$

for all $m \in \mathbb{Z}_+$.

Moreover, for $a, b \in \mathbb{R},\ a < b,\ \alpha > 0$, then $[a, a+\alpha] \cup [b, b+\alpha]$ is the set generated by the polynomial $Q(x) = x^2 - (a + b + \alpha)x + ab + \dfrac{(a+b)\alpha}{2}$ with respect to $r = \dfrac{(b-a)\alpha}{2}$. Hence, $\operatorname{Spec}(f) \subset [a, a+\alpha] \cup [b, b+\alpha]$ if and only if for any $\tau > 0$ there exists a constant $C_\tau < \infty$ such that

$$\|Q^m(D) f\|_{L(\mathbb{X})} \leq C_\tau \left(\frac{(b-a)\alpha}{2} + \tau \right)^m \|f\|_{L(\mathbb{X})}$$

for all $m \in \mathbb{Z}_+$. Consequently, for $0 < a < b$, $\operatorname{Spec}(f) \subset [a, b] \cup [-b, -a]$ if and only if for any $\tau > 0$ there exists a constant $C_\tau < \infty$ such that

$$\left\| \left(x^2 - \frac{a^2+b^2}{2} \right)^m (D) f \right\|_{L(\mathbb{X})} \leq C_\tau \left(\frac{b^2-a^2}{2} + \tau \right)^m \|f\|_{L(\mathbb{X})}$$

for all $m \in \mathbb{Z}_+$.

3. Paley – Wiener type theorem for integral operators. ***3.1. Paley – Wiener type theorem for any compact K.*** We define $I_\lambda = (\lambda - D)^{-1}$, where $\lambda \in \mathbb{C} \backslash i\mathbb{R}$. The integral operator $P(I_\lambda)$ is obtained from $P(x)$ by substituting $x \to I_\lambda$. We have the following result for $P(I_\lambda)$.

Theorem 3.1. *Let K be a compact set in $\mathbb{R}$, $f \in L(\mathbb{X})$ and $\lambda \in \mathbb{C} \backslash i\mathbb{R}$. Then* $\operatorname{Spec}(f) \subset K$ *if and only if for any $\tau > 0$ there exists $C_{\tau,K} > 0$ independent of f such that*

$$\|P(I_\lambda) f\|_{L(\mathbb{X})} \leq C_{\tau,K} \sup_{x \in K^{(\tau)}} |P(1/(\lambda - ix))| \|f\|_{L(\mathbb{X})} \tag{11}$$

for all polynomials with complex coefficients $P(x)$.

Proof. *Necessity.* Assume that $\operatorname{Spec}(f) \subset K$. Now, we choose a function $\phi \in C^\infty(\mathbb{R})$ such that $\phi(x) = (2\pi)^{-1/2}$ if $x \in K^{\tau/4}$ and $\phi(x) = 0$ if $x \notin K^{\tau/2}$. Since $\lambda \in \mathbb{C} \backslash i\mathbb{R}$, there is a small enough positive number τ such that $\lambda - ix \neq 0$ for all $x \in K^{(\tau)}$, so, the following function is well defined:

$$\Phi = \mathcal{F}^{-1} \left(\phi(x) P(1/(\lambda - ix)) \right).$$

By using Lemma 2.2, we have $f = \breve{\phi} * f$, and then

$$(\lambda - D) \left((I_\lambda \breve{\phi}) * f \right) = \lambda \left((I_\lambda \breve{\phi}) * f \right) - D \left((I_\lambda \breve{\phi}) * f \right) =$$

$$= \lambda\Big(\big(I_\lambda \breve{\phi}\big) * f\Big) - \big(D I_\lambda \breve{\phi}\big) * f = \Big(\big(\lambda - D\big)\big(I_\lambda \breve{\phi}\big)\Big) * f = \breve{\phi} * f = f.$$

Hence, $I_\lambda f = \Big(I_\lambda \breve{\phi}\Big) * f$. Similarly, $I_\lambda^k f = \Big(I_\lambda^k \breve{\phi}\Big) * f \ \forall k \in \mathbb{Z}_+$. So,

$$P(I_\lambda) f = \Big(P(I_\lambda)\breve{\phi}\Big) * f. \tag{12}$$

From $I_\lambda^k \breve{\phi} = \mathcal{F}^{-1}\left(\phi(x)/(\lambda - ix)^k\right)$, we conclude $P(I_\lambda)\breve{\phi} = \mathcal{F}^{-1}\left(\phi(x) P(1/(\lambda - ix))\right) = \Phi$. Therefore, applying (12), we get $P(I_\lambda) f = \Phi * f$. Hence, it follows from Lemma 2.1 that

$$\|P(I_\lambda) f\|_{L(\mathbb{X})} \le \|f\|_{L(\mathbb{X})} \|\Phi\|_{L^1}. \tag{13}$$

For each $x \in K^{\tau/2}$, we consider $\gamma = \{z \in \mathbb{C} \colon |z - x| = \tau/2\}$ as a simple closed curve on $K^{(\tau)}$ oriented counterclockwise. Because $Q(x) := P(1/(\lambda - ix))$ is a holomorphic function on $K^{(\tau)}$ and by Cauchy's integral formula for derivatives, we obtain

$$D^n Q(x) = \frac{n!}{2\pi i} \int\limits_\gamma \frac{Q(z)dz}{(z - x)^{n+1}}, \quad n = 0, 1, 2, \ldots.$$

Consequently,

$$|D^n Q(x)| \le \frac{n! \sup_{z \in \gamma} |Q(z)|}{(\tau/2)^n}, \quad n = 0, 1, 2, \ldots.$$

Therefore, since $\gamma \subset K^{(\tau)}$,

$$\sup_{x \in K^{\tau/2}} |D^n Q(x)| \le \frac{n! \sup_{z \in K^{(\tau)}} |Q(z)|}{(\tau/2)^n}, \quad n = 0, 1, 2, \ldots.$$

Then it follows from

$$\sqrt{2\pi} \sup_{x \in \mathbb{R}} \left|\left(1 + x^2\right) \Phi(x)\right| \le$$

$$\le \int\limits_{x \in K^{\tau/2}} \left(\left|D^2\left(\phi(x) P(1/(\lambda - ix))\right)\right| + |\phi(x) P(1/(\lambda - ix))|\right) dx$$

that

$$\sup_{x \in \mathbb{R}} \left|\left(1 + x^2\right) \Phi(x)\right| \le C(\tau, K) \sup_{x \in K^{(\tau)}} |P(1/(\lambda - ix))|, \tag{14}$$

where $C(\tau, K)$ does not depend on $P(x)$. Further, we have

$$\|\Phi\|_{L^1} \le \pi C(\tau, K) \sup_{x \in \mathbb{R}} \left|\left(1 + x^2\right) \Phi(x)\right|. \tag{15}$$

Combining (13)–(15), we obtain

$$\|P(I_\lambda) f\|_{L(\mathbb{X})} \le \pi C(\tau, K) \sup_{x \in K^\tau} |P(1/(\lambda - ix))| \|f\|_{L(\mathbb{X})}.$$

Sufficiency. Assume the contrary that there exists $\sigma \in \operatorname{Spec}(f)$ and $\sigma \notin K$. Let $t = \sup_{x\in K}(\sigma - x)^2$, we define the following polynomial:

$$Q(x) = t + (x - \lambda + i\sigma)^2.$$

Clearly,

$$\sup_{x\in K} |Q(\lambda - ix)| = \sup_{x\in K} \left|t - (\sigma - x)^2\right| < t = |Q(\lambda - i\sigma)|.$$

Put

$$H = \left\{ z = a + bi \colon |a| < \mathrm{Re}\lambda,\ |b| < 2(|\sigma| + \sup_{x\in K} |x|) \right\},$$

$$H_1 = \left\{ z = a + bi \colon |a| < \frac{\mathrm{Re}\,\lambda}{2},\ |b| < |\sigma| + \sup_{x\in K} |x| \right\}$$

and $R(x) = Q\left(\dfrac{1}{x}\right)$. Then it follows from $\lambda \in \mathbb{C}\backslash i\mathbb{R}$ that $R(x)$ is a holomorphic function on H and

$$|R(1/(\lambda - i\sigma))| > \sup_{x\in K} |R(1/(\lambda - ix))|. \tag{16}$$

Because $R(x)$ is a holomorphic function in the complex domain H, there exists a sequence of polynomials $\{P_n\}$ such that P_n converges uniformly to $R(x)$ on H_1. Combining this with (16), we can choose an integer j_0 such that

$$|P_{j_0}(1/(\lambda - i\sigma))| > \sup_{x\in K} |P_{j_0}(1/(\lambda - ix))|. \tag{17}$$

For a small enough positive number τ we have $P_{j_0}(1/(\lambda - ix)) \neq 0$ for all $x \in (\sigma - \tau, \sigma + \tau)$. From the definition of Beurling spectrum, there exists $\varphi \in C_0^\infty(\mathbb{R})$, $\operatorname{supp}\varphi \subset (\sigma - \tau, \sigma + \tau)$ such that $\check{\varphi} * f \not\equiv 0$. Put

$$\varphi_m = \mathcal{F}^{-1}\left(\varphi(x)/P_{j_0}^m(1/(\lambda - ix))\right).$$

Then φ_m is well defined, $\varphi_m \in \mathcal{S}(\mathbb{R})$ and $P_{j_0}^m(I_\lambda)\varphi_m = \check{\varphi}$. Clearly, $\varphi_m * \left(I_\lambda^k f\right) = \left(I_\lambda^k \varphi_m\right) * f$ for all $k \in \mathbb{Z}_+$. That gives $\varphi_m * \left(P_{j_0}^m(I_\lambda) f\right) = \left(P_{j_0}^m(I_\lambda)\varphi_m\right) * f$, and then $\varphi_m * P_{j_0}^m(I_\lambda) f = \check{\varphi} * f$. So, by Lemma 2.1, we get

$$0 < \|\check{\varphi} * f\|_{L(\mathbb{X})} = \left\|\varphi_m * P_{j_0}^m(I_\lambda) f\right\|_{L(\mathbb{X})} \le \left\|P_{j_0}^m(I_\lambda) f\right\|_{L(\mathbb{X})} \|\varphi_m\|_{L^1}.$$

Consequently,

$$\liminf_{m\to\infty} \left\|P_{j_0}^m(I_\lambda) f\right\|_{L(\mathbb{X})}^{1/m} \ge 1/\limsup_{m\to\infty} \|\varphi_m\|_{L^1}^{1/m}. \tag{18}$$

From

$$\sup_{x\in\mathbb{R}} \left|(1 + x^2)\varphi_m(x)\right| \le$$

$$\leq \frac{1}{\sqrt{2\pi}} \int\limits_{\sigma-\tau}^{\sigma+\tau} \left(\left| D^2 \left(\varphi(x)/P_{j_0}^m \left(1/(\lambda - ix)\right)\right) \right| + \left| \varphi(x)/P_{j_0}^m \left(1/(\lambda - ix)\right) \right| \right) \, dx$$

we can deduce that

$$\sup_{x\in\mathbb{R}} \left| \left(1+x^2\right) \varphi_m(x) \right| \leq C_1 m^2 \sup_{x\in(\sigma-\tau,\sigma+\tau)} \left| 1/P_{j_0}^{m+2}(1/(\lambda - ix)) \right| \tag{19}$$

for some C_1 independent of m. Then it follows from $\|\varphi_m\|_{L^1} \leq \pi \sup_{x\in\mathbb{R}} \left|\left(1+x^2\right)\varphi_m(x)\right|$ that

$$\limsup_{m\to\infty} \|\varphi_m\|_{L^1}^{1/m} \leq \sup_{x\in(\sigma-\tau,\sigma+\tau)} |1/P_{j_0}(1/(\lambda - ix))| \, . \tag{20}$$

Relations (18) and (20) imply

$$\liminf_{m\to\infty} \left\| P_{j_0}^m(I_\lambda) f \right\|_{L(\mathbb{X})}^{1/m} \geq \inf_{x\in(\sigma-\tau,\sigma+\tau)} |P_{j_0}(1/(\lambda - ix))|.$$

Letting $\tau \to 0$, we get

$$\liminf_{m\to\infty} \left\| P_{j_0}^m(I_\lambda) f \right\|_{L(\mathbb{X})}^{1/m} \geq |P_{j_0}(1/(\lambda - i\sigma))|. \tag{21}$$

Combining this with (17), we have

$$\liminf_{m\to\infty} \left\| P_{j_0}^m(I_\lambda) f \right\|_{L(\mathbb{X})}^{1/m} > \sup_{x\in K} |P_{j_0}(1/(\lambda - ix))|.$$

This is contrary to (11).

Theorem 3.1 is proved.

3.2. Paley–Wiener type theorem for sets generated by polynomial type. Let $P(x)$ be a polynomial with complex coefficients, $r > 0$ and $\lambda \in \mathbb{C}\backslash i\mathbb{R}$. We put

$$(P)_{r,\lambda} := \{x \in \mathbb{R} \colon |P(1/(\lambda - ix))| \leq r\}$$

and $(P)_{r,\lambda}$ is called the set generated by the polynomial of type $(P(x), r, \lambda)$. Note that if $|P(0)| > r$, then $(P)_{r,\lambda}$ is compact, and if $(P)_{r,\lambda}$ is compact, then $|P(0)| \geq r$. However, $|P(0)| = r$ does not guarantee compactness of $(P)_{r,\lambda}$ or not. For example, we put $P(x) = 1 + x^4$, $P_1(x) = 1 + x^2$ and $\lambda = r = 1$. Then $(P)_{1,1}$ is compact and $(P_1)_{1,1}$ is not compact. Indeed,

$$\left| P\left(\frac{1}{\lambda - ix}\right) \right| = \left| 1 + \left(\frac{1}{1 - ix}\right)^4 \right| =$$

$$= \left| 1 + \left(\frac{1 + ix}{1 + x^2}\right)^4 \right| = \left| 1 + \frac{x^4 - 6x^2 + 1}{(1 + x^2)^4} + \frac{i\left(4x - 4x^3\right)}{(1 + x^2)^4} \right|,$$

$$\left| P_1\left(\frac{1}{\lambda - ix}\right) \right| = \left| 1 + \left(\frac{1}{1 - ix}\right)^2 \right| =$$

$$= \left| 1 + \left(\frac{1 + ix}{1 + x^2}\right)^2 \right| = \left| 1 - \frac{x^2 - 1}{(1 + x^2)^2} + \frac{2xi}{(1 + x^2)^2} \right|,$$

and then

$$\left|P\left(\frac{1}{\lambda - ix}\right)\right| \geq \left(1 + \frac{x^4 - 6x^2 + 1}{(1+x^2)^4}\right)^{1/2} > 1,$$

$$\left|P_1\left(\frac{1}{\lambda - ix}\right)\right| = \left(1 - \frac{2x^2 - 3}{(1+x^2)^2}\right)^{1/2} < 1$$

for all $x \in (-\infty, -6) \cup (6, +\infty)$.

Therefore, $(P)_{1,1}$ is compact but $(P_1)_{1,1}$ is not compact.

Moreover, if $a, b \in \mathbb{R},\ a < b$, then $[a, b]$ is a set generated by polynomial type. To see this we put $c = (a+b)/2,\ d = (b-a)/2,\ \lambda = 1 + ic$ and we choose two numbers $\kappa,\ r \in [1, +\infty)$ satisfying $(2\kappa - 1)/\left(\kappa^2 - r^2\right) = 1 + d^2$. Put $P(x) = \kappa - x$. Clearly,

$$|P(1/(\lambda - ix))| = \left|\kappa - \frac{1}{\lambda - ix}\right| = \left|\kappa - \frac{1}{1 + i(c-x)}\right| = \left|\kappa - \frac{1 - i(c-x)}{1 + (c-x)^2}\right| =$$

$$= \left(\left(\kappa - \frac{1}{1+(c-x)^2}\right)^2 + \left(\frac{(c-x)}{1+(c-x)^2}\right)^2\right)^{1/2} = \left(\kappa^2 - \frac{2\kappa - 1}{1+(c-x)^2}\right)^{1/2}.$$

Hence,

$$\{x \in \mathbb{R}\colon\ |P(1/(\lambda - ix))| \leq r\} = \left\{x \in \mathbb{R}\colon \kappa^2 - \frac{2\kappa - 1}{1+(c-x)^2} \leq r^2\right\} =$$

$$= \left\{x \in \mathbb{R}\colon 1 + (c-x)^2 \leq (2\kappa - 1)/\left(\kappa^2 - r^2\right)\right\} =$$

$$= \left\{x \in \mathbb{R}\colon 1 + (c-x)^2 \leq 1 + d^2\right\}.$$

Consequently, $(P)_{r,\lambda} = [a, b]$.

Theorem 3.2. *Let $f \in L(\mathbb{X})$, $r > 0$, $\lambda \in \mathbb{C}\backslash i\mathbb{R}$, $P(x)$ be a polynomial and $(P)_{r,\lambda}$ be compact. Then* $\operatorname{Spec}(f) \subset (P)_{r,\lambda}$ *if and only if for any $\tau > 0$ there exists a constant $C_{\tau,r,\lambda,P} < \infty$ independent of f such that*

$$\|P^m(I_\lambda)f\|_{L(\mathbb{X})} \leq C_{\tau,r,\lambda,P}\|f\|_{L(\mathbb{X})}(r+\tau)^m \tag{22}$$

for all $m \in \mathbb{Z}_+$.

Proof. *Necessity* is follows from Theorem 3.1.

Sufficiency. Assume the contrary that there exists $\sigma \in \operatorname{Spec}(f)$ and $\sigma \notin (P)_{r,\lambda}$. Hence, $|P(1/(\lambda - i\sigma))| > r$. According to (22), we obtain

$$\limsup_{m\to\infty} \left(\|P^m(I_\lambda)f\|_{L(\mathbb{X})}\right)^{1/m} \leq r + \tau. \tag{23}$$

Applying the proof of inequality (21), we have

$$\liminf_{m\to\infty} \left(\|P^m(I_\lambda)f\|_{L(\mathbb{X})}\right)^{1/m} \geq |P(1/(\lambda - i\sigma))|.$$

Combining this with (23), we deduce $|P(1/(\lambda - i\sigma))| \leq r + \tau$. Letting $\tau \to 0$, we obtain $|P(1/(\lambda - i\sigma))| \leq r$. This is a contradiction. So, $\operatorname{Spec}(f) \subset (P)_{r,\lambda}$.

Theorem 3.2 is proved.

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