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INVARIANT MEASURES FOR DISCRETE DYNAMICAL SYSTEMS AND ERGODIC PROPERTIES OF GENERALIZED BOOLE-TYPE TRANSFORMATIONS

ІНВАРІАНТНІ МІРИ ДЛЯ ДИСКРЕТНИХ ДИНАМІЧНИХ СИСТЕМ ТА ЕРГОДИЧНІ ВЛАСТИВОСТІ УЗАГАЛЬНЕНИХ ПЕРЕТВОРЕНЬ БУЛЕВОГО ТИПУ

Invariant ergodic measures for generalized Boole-type transformations are studied using an invariant quasimeasure generating function approach based on special solutions for the Frobenius – Perron operator. New two-dimensional Boole-type transformations are introduced, and their invariant measures and ergodicity properties are analyzed.

Вивчаються ергодичні міри для узагальнених перетворень булевого типу із використанням підходу твірних функцій інваріантних квазімір, що базується на спеціальних розв'язках для оператора Фробеніуса – Перрона. Запропоновано нові двовимірні перетворення булевого типу та досліджено їхні інваріантні міри та ергодичні властивості.

1. Invariant measures: introductory setting. It is well known that discrete dynamical systems on finite-dimensional manifolds play an important role [8, 9, 12, 21] in describing evolution properties of many processes in the applied sciences. Of particular interest are discrete dynamical systems on manifolds with invariant measures, often possessing additional properties such as ergodicity or mixing, which allow to explain such phenomenon as chaotic behavior and instability of the physical objects being studied. Therefore, methods of constructing invariant (with respect to a given discrete dynamical system) measures, such as those we develop in the sequel, are of crucial importance.

Suppose that a topological phase space M is endowed with a structure of a measurable space, that is a σ-algebra $\mathcal{A}(M)$ of subsets in M, on which there is a finite normalized measure $\mu\colon \mathcal{A}(M) \to \mathbb{R}_+$, $\mu(M) = 1$. As is well known [25], a measurable mapping $\varphi\colon M \to M$ of the measurable space $(M, \mathcal{A}(M))$ is called an *ergodic* discrete dynamical system if μ-almost everywhere (μ-a.e.) there exists an $x \in M$ limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(\varphi^k x) \tag{1.1}$$

for any bounded measurable function $f \in \mathcal{B}(M;\mathbb{R})$.

We now assume that the limit (1.1) exists μ-a.e., that is one can define a bounded measurable function $f_\varphi \in \mathcal{B}(M;\mathbb{R})$, where

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(\varphi^k x) := f_\varphi(x) \tag{1.2}$$

for all $x \in M$ the function (1.2) defines a finite measure $\mu_\varphi\colon \mathcal{A}(M) \to \mathbb{R}_+$ on M such that

$$\int\limits_M f_\varphi(x)d\mu(x) := \int\limits_M f(x)d\mu_\varphi(x). \tag{1.3}$$

Actually, the Lebesgue – Helley theorem on bounded convergence [22] implies that

$$\int\limits_M f_\varphi(x)d\mu(x) = \lim_{n\to\infty}\int\limits_M f(x)d\mu_{n,\varphi}(x), \tag{1.4}$$

where

$$\mu_{n,\varphi}(A) := \frac{1}{n}\sum_{k=0}^{n-1}\mu(\varphi^{-k}A), \tag{1.5}$$

is the Schur average, $n \in \mathbb{Z}_+$, $A \subset \mathcal{A}(M)$, and $\varphi^{-k}A := \{x \in M \colon \varphi^k x \in A\}$ for any $k \in \mathbb{Z}_+$. The limit on the right-hand side of (1.4) obviously exists for any bounded measurable function $f \in \mathcal{B}(M;\mathbb{R})$. Consequently, the equality

$$\mu_\varphi(A) := \lim_{n\to\infty}\mu_{n,\varphi}(A), \tag{1.6}$$

for any $A \subset \mathcal{A}(M)$ defines on the σ-algebra $\mathcal{A}(M)$ an additive non-negative bounded mapping $\mu_\varphi \colon \mathcal{A}(M) \to \mathbb{R}_+$. Besides, from the existence of a uniform approximation of arbitrary measurable bounded function by means of finite-valued measurable (simple) functions, one immediately infers the equality (1.3) for any $f \in \mathcal{B}(M;\mathbb{R})$. The requirement for countable additivity of the mapping $\mu_\varphi \colon \mathcal{A}(M) \to \mathbb{R}_+$ follows from the equivalent expression [23]

$$\lim_{k\to\infty}\sup_{n\in\mathbb{Z}_+}\mu_{n,\varphi}(A_k) = 0$$

for any monotonic sequence of sets $A_j \supset A_{j+1}$, $j \in \mathbb{Z}_+$, of $\mathcal{A}(M)$ with empty intersection.

The measure $\mu_\varphi \colon \mathcal{A}(M) \to \mathbb{R}_+$ defined by (1.6), has the following invariance property with respect to the dynamical system $\varphi \colon M \to M$:

$$\mu_\varphi(\varphi^{-1}A) = \mu_\varphi(A) \tag{1.7}$$

for any $A \in \mathcal{A}(M)$, which follows from simple identity

$$\mu_{n,\varphi}(\varphi^{-1}A) = \frac{n+1}{n}\mu_{n+1,\varphi}(A) - \frac{1}{n}\mu(A),$$

upon taking the limit as $n \to \infty$. It is easy to see that (1.7) is completely equivalent to the equality

$$\int\limits_M f(\varphi x)d\mu_\varphi(x) = \int\limits_M f(x)d\mu_\varphi(x)$$

for any $f \in \mathcal{B}(M;\mathbb{R})$. Moreover, if a σ-measurable set $A \in \mathcal{A}(M)$ is invariant with respect to the mapping $\varphi \colon M \to M$, that is $\varphi^{-1}(M) = M$, then evidently $\mu_\varphi(A) = \mu(A)$.

Therefore, the existence of the φ-invariant measure $\mu_\varphi \colon \mathcal{A}(M) \to \mathbb{R}_+$, coinciding with the measure $\mu \colon \mathcal{A}(M) \to \mathbb{R}_+$ on the σ-algebra $\mathcal{I}(M) \subset \mathcal{A}(M)$ of invariant (with respect to the dynamical system $\varphi \colon M \to M$) sets, is a necessary condition of the convergence μ-a.e. on M of the mean values (1.1) as $n \to \infty$ for any $f \in \mathcal{B}(M;\mathbb{R})$. That the converse is also true follows from a theorem of Birkhoff [22]: if the mapping $\varphi \colon M \to M$ conserves a finite measure $\mu_\varphi \colon \mathcal{A}(M) \to \mathbb{R}_+$, the mean values (1.1) are convergent μ_φ-a.e. on M, and the convergence set is invariant. Thus, if the reduction of the measure $\mu\colon \mathcal{A}(M) \to \mathbb{R}_+$ upon the invariant σ-algebra $\mathcal{I}(M) \subset \mathcal{A}(M)$ is absolutely continuous with respect to that of the measure $\mu_\varphi\colon \mathcal{A}(M) \to \mathbb{R}_+$, the convergence holds μ-a.e. on M.

2. An invariant measure generating construction. Assume we are given a discrete dynamical system $\varphi\colon M \to M$ and a sequence of associated measures $\mu_{n,\varphi}\colon \mathcal{A}(M) \to \mathbb{R}_+$, $n \in \mathbb{Z}_+$, defined by (1.5). Then one can define measure generating functions (m.g.f.) $\mu_{n,\varphi}(\lambda; A)$, $n \in \mathbb{Z}_+$, where for any $A \in \mathcal{A}(M)$, $\lambda \in \mathbb{C}$,

$$\mu_{n,\varphi}(\lambda; A) := \sum_{k=0}^{n-1} \lambda^k\ \mu(\varphi^{-k}A). \tag{2.1}$$

Define now the following measure generating function:

$$\mu_\varphi(\lambda; A) := \lim_{n\to\infty} \sum_{k=0}^{n-1} \lambda^k\ \mu(\varphi^{-k}A), \tag{2.2}$$

where $A \in \mathcal{A}(M)$, and $|\lambda| < 1$ to insure the finiteness of the expression (2.2). It is easy now to prove the following result.

Lemma 2.1. *The m.g.f.* (2.2) *satisfies the functional equation*

$$\mu_\varphi(\lambda; A) = \lambda\mu_\varphi(\lambda; \varphi^{-1}A) + \mu(A) \tag{2.3}$$

for any $A \in A(M)$ *and* $|\lambda| < 1$.

Proof. From (2.3) one finds by iteration directly that

$$\mu_\varphi(\lambda; A) - \sum_{k=0}^{n-1} \lambda^k\ \mu(\varphi^{-k}A) = \lambda^{n+1}\mu_\varphi(\lambda; \varphi^{-k-1}A) \tag{2.4}$$

for any $n \in \mathbb{Z}_+$, $A \in A(M)$ and $|\lambda| < 1$. Taking the limit in (2.4) as $n \to \infty$, one arrives at the determining expression (2.2) that completes the proof.

***Corollary* 2.1.** *Assume we are given a mapping* $\mu^{(s)} := \mu - s\ \mu\circ\varphi^{-1}$ *on* $\mathcal{A}(M)$, *where* $|s| < 1$. *Then the equality*

$$\mu^{(s)}_\varphi(s; A) = \mu(A) \tag{2.5}$$

holds for all $A \in \mathcal{A}(M)$, $|s| < 1$.

Proof. This follows from a straightforward substitution of the mapping $\mu^{(s)}\colon \mathcal{A}(M) \to \mathbb{R}$ for $|s| < 1$ into (2.3).

***Example* 2.1.** *The induced functional expansion.*

Let $M = [0, 1] \subset \mathbb{R}$ and $\varphi\colon M \to M$ is the “baker” transformation, that is

$$\varphi(x) := \begin{cases} 2x, & \text{if } x \in [0, 1/2), \\ 2(1-x), & \text{if } x \in [1/2, 1]. \end{cases} \tag{2.6}$$

Take now a mapping $f\colon M \to M$, given as

$$f(x) := 2x - x^2 \tag{2.7}$$

for any $x \in M$ and construct the convolution of (2.5) with the function (2.7) at the parametric measure $\mu(A;x) := \int_A d\vartheta_x(y)$, $A \in \mathcal{A}(M)$, $x \in M$, where $\vartheta_x \colon M \to \mathbb{R}$, $x \in M$, is the standard Heavyside function with the support supp $\vartheta_x = \{y \in M \colon y - x \geq 0\}$. Then the decomposition

$$f(x) = (2-4s)\sum_{n\in\mathbb{Z}_+} s^n\varphi^n(x) + (4s-1)\sum_{n\in\mathbb{Z}_+} s^n\varphi^n(x)\varphi^n(x)$$

holds [26] for any $x \in M$. In the cases $s = 1/2$ and $s = 1/4$, one readily obtains for any $x \in M$ the decompositions

$$\sum_{n\in\mathbb{Z}_+} (1/2)^n\varphi^n(x)\varphi^n(x) = 2x - x^2 = \sum_{n\in\mathbb{Z}_+} (1/4)^n\varphi^n(x),$$

which are useful for some applied set-theoretical considerations. Note here also that a similar expansion given by

$$\sum_{n\in\mathbb{Z}_+} (1/2)^n\varphi^n(x) := \xi(x)$$

for any $x \in M$, yields the well-known Weierstrass function $\xi \colon [0,1] \to [0,1]$, which is continuous but nowhere differentiable [24] on $M = [0,1] \subset \mathbb{R}$.

3. Representation of invariant measures. Assume now that the limit (1.6) exists owing to (1.7) being measure preserving on $\mathcal{A}(M)$. Then the following important Tauberian type [11] result holds.

Theorem 3.1. *Let the measure generating function* $\mu_\varphi \colon \mathbb{C}\times\, \mathcal{A}(M) \to \mathbb{C}$, *corresponding to a discrete dynamical system* $\varphi \colon M \to M$, *exist and satisfy the invariance condition* (1.7). *Then the limit expression*

$$\lim_{\lambda\uparrow 1 \ (\mathrm{Im}\lambda=0)} (1-\lambda)\mu_\varphi(\lambda;A) = \mu_\varphi(A) \tag{3.1}$$

holds for any $A \in \mathcal{A}(M)$. *Moreover, the converse is also true.*

Proof. Since all coefficients of the series (2.1) are bounded, that is are of $O(1)$, then it follows from a well-known Tauberian theorem of [11] Hardy that

$$\lim_{\lambda\uparrow 1 \ (\mathrm{Im}\lambda=0)} (1-\lambda)\mu_\varphi(\lambda;A) = \lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\mu(\varphi^{-k}A) := \mu_\varphi(A) \tag{3.2}$$

for any $A \in \mathcal{A}(M)$, which completes the proof.

We can now use the above theorem to produce an invariant measure $\mu_\varphi \colon \mathcal{A}(M) \to \mathbb{R}_+$ on M by means of the measure generating function $\mu_\varphi \colon \mathbb{C}\times\, \mathcal{A}(M) \to \mathbb{C}$ defined by (2.1) for a given discrete dynamical system $\varphi \colon M \to M$. Also, observe that the series (2.1) generates an analytic function when $|\lambda| < 1$ such that for any $\lambda \in (-1,1)$ and $A \in \mathcal{A}(M)$,

$$\mathrm{Im}\,\mu_\varphi(\lambda;A) = 0. \tag{3.3}$$

Now, using classical analytic function theory [16, 17], one can readily verify the following result.

Theorem 3.2. *Let a measure generating function* $\mu_\varphi \colon \mathbb{C}\times\mathcal{A}(M) \to \mathbb{C}$ *satisfy the condition* (3.3). *Then the following representation holds*:

$$\mu_\varphi(\lambda; A) = \int_0^{2\pi} \frac{(1-\lambda^2) d\sigma_\varphi(s; A)}{1 - 2\lambda \cos s + \lambda^2} \tag{3.4}$$

for any $A \in \mathcal{A}(M)$, *where* $\sigma_\varphi(\circ; A)\colon [0, 2\pi] \to \mathbb{R}_+$ *is a function of bounded variation*

$$0 \le \sigma_\varphi(s; A) \le \mu(A) \tag{3.5}$$

for any $s \in [0, 2\pi]$ *and* $A \in \mathcal{A}(M)$.

This theorem appears to be exceptionally interesting for applications since it reduces the problem of detecting the invariant measure $\mu_\varphi\colon \mathcal{A}(M) \to \mathbb{R}_+$ defined by (1.6) to a calculation of the following complex analytical limit:

$$\mu_\varphi(A) = \lim_{\lambda \uparrow 1 \ (\mathrm{Im}\lambda = 0)} \int_0^{2\pi} \frac{2(1-\lambda)^2 d\sigma_\varphi(s; A)}{1 - 2\lambda \cos s + \lambda^2}, \tag{3.6}$$

where $A \in \mathcal{A}(M)$ and $\sigma_\varphi\colon [0, 2\pi] \times \mathcal{A}(M) \to \mathbb{R}_+$ — some Stieltjes measure on $[0, 2\pi]$, generated by a given *a priori* dynamical system $\varphi\colon M \to M$ and a measure $\mu\colon \mathcal{A}(M) \to \mathbb{R}_+$.

***Example* 3.1.** *The Gauss mapping.*

Consider the case of the Gauss mapping $\varphi\colon M \to M$, where $M = [0, 1]$ and for any $x \in (0, 1]$, $\varphi(x) := \{1/x\}$, $\varphi(0) = 0$ (here "$\{\cdot\}$" means taking the fractional part of a number $x \in [0, 1]$). One can show by means of simple but somewhat cumbersome calculations that it is indeed ergodic [22] and possesses the following invariant measure on M:

$$\mu_\varphi(A) = \frac{1}{\ln 2} \int_A \frac{dx}{1+x},$$

which obviously yields the well-known Gauss measure $\mu_\varphi\colon \ \mathcal{A}(M) \to \mathbb{R}_+$ on $M = (0, 1]$. As a result, the following limit for arbitrary $f \in L^1(0, 1)$ obtains:

$$\lim_{n\to\infty} \sum_{k=0}^{n-1} f(\varphi^n x) \overset{\text{a.e.}}{=} \frac{1}{\ln 2} \int_0^1 \frac{f(x) dx}{1+x}.$$

The analytical expression (3.6) obtained above for the invariant measure $\mu_\varphi\colon \ A(M) \to \mathbb{R}_+$, generated by a discrete dynamical system $\varphi\colon M \to M$, should be quite useful for concrete calculations. In particular, it follows directly from (3.4) that the Stieltjes measure $\sigma_\varphi(\circ; A)\colon [0, 2\pi] \to [0, \mu(A)]$, $A \in \mathcal{A}(M)$, generates for any $s \in [0, 2\pi]$ a new positive definite measure on $A \in \mathcal{A}(M)$ as

$$\sigma_\varphi(s)(A) = \sigma_\varphi(s; A),$$

which can be regarded as smearing the measure $\mu\colon \mathcal{A}(M) \to \mathbb{R}_+$ along the unit circle $\mathbb{S}^1$ in the complex plane $\mathbb{C}$.

An important still open problem, which is closely linked with the expression (3.6), is the following inverse measure evaluation question: How can one retrieve the dynamical system $\varphi\colon M \to M$ which generated the above smeared Stieltjes measure $\sigma_\varphi\colon [0, 2\pi] \times \mathcal{A}(M) \to \mathbb{R}_+$ via the expression (3.4)?

4. New generalizations of the Boole transformation and their ergodicity. In this section we will study invariant measures and ergodicity properties of both the one-dimensional generalized Boole transformation

$$y \to \varphi(y) := \alpha y + a - \sum_{j=1}^{N} \frac{\beta_j}{y - b_j} \in \mathbb{R}, \tag{4.1}$$

where a and $b_j \in \mathbb{R}$ are real and $\alpha, \beta_j \in \mathbb{R}_+$ are positive parameters, $1 \leq j \leq N$, and naturally generalized two-dimensional Boole-type transformations

$$\begin{gathered} (x, y) \to (x - 1/x, y - 1/y) \in \mathbb{R}^2, \\ (x, y) \to (x - 1/y, y - 1/x) \in \mathbb{R}^2, \end{gathered} \tag{4.2}$$

defined whenever $xy \neq 0$. They generalize the classical Boole transformation [10] $y \to \varphi(y) := := y - 1/y \in \mathbb{R}$, defined for $y \neq 0$, which was proved to be ergodic [6] with respect to the invariant standard infinite Lebesgue measure on $\mathbb{R}$. In the case $\alpha = 1$, $a = 0$, the analogous ergodicity result was proved in [1 – 3] making use of the specially devised inner function approach. The related spectral properties were in part studied in [3]. In spite of these results, the case $\alpha \neq 1$ still persists as a challenge. In fact, the only related result [4] concerns the following special case of (4.1): $y \to \varphi(y) := \alpha y - \beta/y \in \mathbb{R}$ for $0 < \alpha < 1$ and arbitrary $\beta \in \mathbb{R}_+$, where the corresponding invariant measure appeared to be finite absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$ and equal to

$$d\mu(x) := \frac{\sqrt{\beta(1-\alpha)}dx}{\pi[x^2(1-\alpha)+\beta]}, \tag{4.3}$$

where $x \in \mathbb{R}$. The ergodicity for the invariant measure (4.3) now can be easily proved. It should be recalled here that for a general nonsingular mapping $\varphi \colon \mathbb{R} \to \mathbb{R}$, the problem of constructing invariant ergodic measures is analyzed [4, 13] by studying the spectral properties of the adjoint Frobenius – Perron operator $\widehat{T}_\varphi \colon L_2(\mathbb{R}; \mathbb{R}) \to L_2(\mathbb{R}; \mathbb{R})$, where

$$\widehat{T}_\varphi \rho(x) := \sum_{y \in \{\varphi^{-1}(x)\}} \rho(y) J_\varphi^{-1}(y) \tag{4.4}$$

for any $\rho \in L_2(\mathbb{R}; \mathbb{R}_+)$ and $J_\varphi^{-1}(y) := \left| \frac{d\varphi(y)}{dy} \right|$, $y \in \mathbb{R}$. Then if $\widehat{T}_\varphi \rho = \rho$, $\rho \in L_2(\mathbb{R}; \mathbb{R}_+)$, the expression $d\mu(x) := \rho(x)dx$, $x \in \mathbb{R}$, will be an invariant (in general infinite) measure with respect to the mapping $\varphi \colon \mathbb{R} \to \mathbb{R}$.

Another way of finding a general algorithm for obtaining such an invariant measure was devised in [18, 19] using the generating measure function method.

Below we study some other special cases of the generalized Boole transformation (4.1), for which we derive the corresponding invariant measures and prove the related ergodicity and spectral properties.

4.1. Invariant measures and ergodic transformations. We will start with analyzing the following Boole-type surjective transformation:

$$\mathbb{R} \ni y \to \varphi(y) := \alpha y + a - \frac{\beta}{y - b} \in \mathbb{R} \tag{4.5}$$

for any $a, b \in \mathbb{R}$ and $2\beta := \gamma^2 \in \mathbb{R}_+$. The transformation (4.5) for $\alpha = 1/2$ and $b = 2a \in \mathbb{R}$ is measure preserving with respect to a measure like (4.3). Namely, the following lemma holds.

Lemma 4.1. *The Boole-type mapping* (4.5) *is measure preserving with respect to the measure*

$$d\mu(x) := \frac{|\gamma| dx}{\pi[(x-2a)^2+\gamma^2]}, \tag{4.6}$$

where $x \in \mathbb{R}$ and $\gamma^2 := 2\beta \in \mathbb{R}_+$.

Proof. A proof follows easily from the fact that the function

$$\rho(x) := \frac{\gamma}{\pi[(x-2a)^2+\gamma^2]} \tag{4.7}$$

satisfies for all $x \in \mathbb{R}\backslash\{2a\}$ the determining condition (4.4):

$$\widehat{T}_{\varphi}\rho(x) := \sum_{I} \rho(y_{\pm})|y'_{\pm}(x)|, \tag{4.8}$$

where $\varphi(y_{\pm}(x)) := x$ for any $x \in \mathbb{R}$. The relationship (4.8) is manifestly equivalent to the invariance condition

$$\sum_{\pm} d\mu(y_{\pm}(x)) = d\mu(x) := \mu(dx) \tag{4.9}$$

for any infinitesimal subset $dx \subset \mathbb{R}$.

Lemma 4.1 is proved.

The question about the ergodicity of the mapping (4.5) is solved here easily by the following theorem.

Theorem 4.1. *The measure* (4.7) *is ergodic with respect to the transformation* (4.5) *at $\alpha = 1/2$ and $b = 2a \in \mathbb{R}$ as such one is equivalent to the canonical ergodic mapping $\mathbb{R}/\mathbb{Z} \ni s :\to \psi(s) := 2s$ mod $\mathbb{Z} \in \mathbb{R}/\mathbb{Z}$ with respect to the standard Lebesgue measure on $\mathbb{R}/\mathbb{Z}$.*

Proof. Define $\mathbb{R}/\mathbb{Z} \ni s :\to \xi(s) = y \in \mathbb{R}$, where

$$\xi(s) := \gamma \cot \pi s + 2a. \tag{4.10}$$

Then the transformation (4.5) for $\alpha = 1/2$, $b = 2a \in \mathbb{R}$ and $\gamma^2 := 2\beta \in \mathbb{R}_+$ yields under the mapping (4.10)

$$\varphi(y) = \varphi(\xi(s)) = \frac{\gamma}{2}\cot \pi s + 2a - \frac{\gamma}{2}\tan \pi s = \frac{\gamma(\cos^2 \pi s - \sin^2 \pi s)}{2 \sin \pi s \cos \pi s} + 2a =$$

$$= \gamma \frac{\cos 2\pi s}{\sin 2\pi s} + 2a = \gamma \cot 2\pi s + 2a := \xi(2s) \tag{4.11}$$

for any $s \in \mathbb{R}/\mathbb{Z}$. The result (4.10) means that the transformation (4.5) is conjugated [3, 13] with the transformation

$$\mathbb{R}/\mathbb{Z} \ni s :\to \psi(s) = 2s \text{ mod } \mathbb{Z} \in \mathbb{R}/\mathbb{Z}; \tag{4.12}$$

that is, the following diagram is commutative:

$$\begin{array}{ccc} \mathbb{R}/\mathbb{Z} & \stackrel{\psi}{\to} & \mathbb{R}/\mathbb{Z} \\ \xi \downarrow & & \downarrow \xi \\ \mathbb{R} & \stackrel{\varphi}{\to} & \mathbb{R}, \end{array} \tag{4.13}$$

that is $\xi \circ \psi = \varphi \circ \xi$, where $\xi \colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ is the conjugate map defined by (4.10). It is easy now to check that the measure (4.6) under the conjugation (4.13) transforms into the standard normalized Lebesgue measure on $\mathbb{R}/\mathbb{Z}$:

$$d\mu(x)|_{x=\gamma \cot \pi s + 2a} = \frac{ds\gamma^2 \left| d(\cot \pi s)/ds \right|}{(\gamma^2 \cot^2 \pi s + \gamma^2)} = \frac{\sin^2 \pi s \cdot (\sin \pi s)^{-2}\, ds}{\cos^2 \pi s + \sin^2 \pi s} = ds, \tag{4.14}$$

where $s \in \mathbb{R}/\mathbb{Z}$. The infinitesimal measures ds on $\mathbb{R}/\mathbb{Z}$ and the infinitesimal measure (4.6) on $\mathbb{R}$ are normalized, so they are both probability measures. Now it suffices to make use of the fact that the measure ds on $\mathbb{R}/\mathbb{Z}$ on the interval $[0,1] \simeq \mathbb{R}/\mathbb{Z}$ is ergodic [4, 13] in order to obtain the desired result.

4.2. Ergodic measures: the inner function approach. Assume that there exists a function $\rho_\omega \in H_2(\mathbb{C}_+;\mathbb{C})$, holomorphic in parameter $\omega \in \mathbb{C}_+$, satisfying the following identity:

$$\widehat{T}_\varphi \rho_\omega = \rho_{\tilde{\varphi}(\omega)} \tag{4.15}$$

for any $\omega \in \mathbb{C}_+$ for some induced transformation $\mathbb{C}_+ \ni \omega \to \tilde{\varphi}(\omega) \in \mathbb{C}_+$. If we now take $\omega := \bar{\omega} \in \mathbb{C}_+$ as a fixed point of the mapping $\tilde{\varphi} \colon \mathbb{C}_+ \to \mathbb{C}_+$, then it follows directly from (4.15) that $\widehat{T}_\varphi \rho_{\bar{\omega}} = \rho_{\bar{\omega}}$, which means

$$d\mu(x) := \operatorname{Im} \rho_{\bar{\omega}}(x) dx \tag{4.16}$$

for $x \in \mathbb{R}$ is an invariant measure for the transformation $\varphi \colon \mathbb{R} \to \mathbb{R}$. There is no general rule for constructing such functions $\rho_\omega \in H_2(\mathbb{C}_+;\mathbb{C})$, analytic in $\omega \in \mathbb{C}_+$, and the related induced mappings $\tilde{\varphi} \colon \mathbb{C}_+ \to \mathbb{C}_+$. Nevertheless, for solving this problem one can adapt some natural ideas related to the exact functional form of the determining Frobenius – Perron operator $\widehat{T}_\varphi \colon L_2(\mathbb{R};\mathbb{R}) \to L_2(\mathbb{R};\mathbb{R})$. To explain this, let us consider the following Boole-type transformation:

$$\mathbb{R} \ni \varphi(y) := \alpha y + a - \frac{\beta}{y-b} \in \mathbb{R}, \tag{4.17}$$

where $a, b \in \mathbb{R}$ and $\beta \in \mathbb{R}_+$. It is easy to see that the Frobenius – Perron operator action on any $\rho_\omega \in H_2(\mathbb{C}_+;\mathbb{C})$ can be represented as follows:

$$\begin{aligned} \widehat{T}_\varphi \rho_\omega &:= \rho_\omega(y_+) y_+' + \rho_\omega(y_-) y_-' = \\ &= \frac{(\omega - y_+)\rho_\omega(y_+)(\omega - y_-)y_-'}{(\omega - y_+)(\omega - y_-)} + \frac{\rho_\omega(y_-)(\omega - y_+)(\omega - y_-)y_-'}{(\omega - y_+)(\omega - y_-)} = \\ &= \frac{k(\omega - y_-)y_+' + k(\omega - y_+)y_-'}{(\omega - y_+)(\omega - y_-)} = \frac{-k[(\omega - y_+)(\omega - y_-)]'}{(\omega - y_+)(\omega - y_-)} = \\ &= -k\frac{d}{dx} \ln \left[(\omega - y_+)(\omega - y_-) \right], \end{aligned} \tag{4.18}$$

where

$$\rho_\omega(x) = \frac{k}{\omega - x} \tag{4.19}$$

for all $\omega \in \mathbb{C}_+ \backslash \{x\}$, $x \in \mathbb{R}$, and some parameter $k \in \mathbb{R}$. As a result of (4.19), one can take

$$\rho_\omega(y_+)(\omega - y_+) = k = \rho_\omega(y_-)(\omega - y_-), \tag{4.20}$$

for all $x \in \mathbb{R}$ and $\omega \in \mathbb{C}_+$. Since the root functions y_+ and $y_- \colon \mathbb{R} \to \mathbb{R}$ satisfy, by definition, the same equation

$$\omega(y_\pm(x)) = x, \tag{4.21}$$

for all $x \in \mathbb{R}$, the following identity for all $\omega \in \mathbb{C}_+$ easily follows from (4.21) owing to the general form of (4.17):

$$\alpha(\omega - y_+)(\omega - y_-) = [\varphi(\omega) - x](\omega - b), \tag{4.22}$$

where

$$y_+(x) + y_-(x) = b + \frac{x-a}{2}, \qquad y_+(x)y_-(x) = \frac{bx - ab - \beta}{2}. \tag{4.23}$$

Whence, taking into account the expression (4.18), one computes that

$$\widehat{T}_\varphi \rho_\omega = -k\frac{d}{dx}\ln([\varphi(\omega) - x](\omega - b)) = \frac{k(\omega - b)}{[\varphi(\omega) - x](\omega - b)} = \frac{k}{\varphi(\omega) - x} = \rho_{\varphi(\omega)}, \tag{4.24}$$

for all $x \in \mathbb{R}$ and $\omega \in \mathbb{C}_+$. Therefore, the induced mapping $\tilde{\varphi} \colon \mathbb{C}_+ \to \mathbb{C}_+$ is exactly the transformation $\varphi \colon \mathbb{C}_+ \to \mathbb{C}_+$, extended naturally from $\mathbb{R}$ to the complex plane $\mathbb{C}_+$.

Now let $\tilde{\omega} \in \mathbb{C}_+$ be a fixed point of the induced mapping $\varphi \colon \mathbb{C}_+ \to \mathbb{C}_+$, that is $\varphi(\bar{\omega}) = \bar{\omega} \in \mathbb{C}_+$. Then from (4.24), one finds that $\widehat{T}_\varphi \rho_{\bar{\omega}} = \rho_{\bar{\omega}}$, or the corresponding invariant quasi-measure on $\mathbb{R}$ has the form

$$d\mu(x) := \mathrm{Im}\frac{kdx}{\bar{\omega} - x} \tag{4.25}$$

for all $x \in \mathbb{R}$ and a suitable parameter $k \in \mathbb{C}$. As $\mathrm{Im}\rho_{\bar{\omega}} \in L_2(\mathbb{R}; \mathbb{R}_+)$ at any $\bar{\omega} \in \mathbb{C}_+ \backslash \mathbb{R}$ and some $k \in \mathbb{C}$, the invariant quasi-measure (4.25) transforms into an actual invariant measure. These results can be formulated as follows:

Theorem 4.2. *The quasi-measure* (4.25) *is invariant with respect to the transformation* (4.17) *for any* $\alpha \in \mathbb{R}_+ \backslash \{1\}$; *for* $\alpha = 1$ *at the condition* $a \neq 0$, $\mathrm{Im}\, k \neq 0$, *it is reduced upon the set* $\mathbb{R}/\pi\mathbb{Z}$, *being equivalent to the standard Gauss measure.*

Proof. The desired infinitesimal quasi-measure $d\mu(x)$ exist if there is at least one fixed point of the equation $\varphi(\omega) = \omega$ for $\omega \in \mathbb{C}_+$. If $\alpha \neq 1$, this equation is equivalent to

$$(\alpha - 1)\omega^2 - \omega[(\alpha - 1)b - a] - (ab + \beta) = 0, \tag{4.26}$$

which always has a solution $\bar{\omega} \in \mathbb{C}_+$, for which $\varphi(\bar{\omega}) = \bar{\omega}$. When $\alpha = 1$, the unique solution $\bar{\omega} = (ab + \beta)/a \in \mathbb{R}$ exists only if $a \neq 0$ and $\mathrm{Im}k \neq 0$, at which the quasi-measure (4.25) becomes degenerate and reduces to the standard Gauss measure [1, 13] on $\mathbb{R}/\pi\mathbb{Z}$.

Theorem 4.2 is proved.

Theorem 4.2 states only that the quasi-measure (4.25) is invariant with respect to the transformation (4.17), so its ergodicity still needs to be proved separately using only the additional property that the corresponding invariant measure is unique. Below we will proceed to study the general case of the transformation (4.1), searching for a suitable invariant quasi-measure that is actually a measure for some $\bar{\omega} \in \mathbb{C}_+ \backslash \mathbb{R}$, $k \in \mathbb{C}$.

4.3. Invariant measures: the general case. Consider the equation

$$\varphi(y) = x, \tag{4.27}$$

where $x, y \in \mathbb{R}$ and the mapping $\varphi\colon \mathbb{C}_+ \to \mathbb{C}_+$ is given by expression (4.1) for a fixed integer $N \in \mathbb{Z}_+ \backslash \{0, 1\}$. The equation (4.27) can be rewritten as

$$\alpha \prod_{j=1}^{N+1} (y - y_j) = [\varphi(y) - x] \prod_{j=1}^{N} (y - b_j) \tag{4.28}$$

for all $x, y \in \mathbb{R}$ and some functions $y_j\colon \mathbb{R} \to \mathbb{R}$, $1 \le j \le N + 1$. Then the relationship (4.28) is naturally extended on the complex plane $\mathbb{C}_+$ as

$$\alpha \prod_{j=1}^{N+1} (\omega - y_j) = [\varphi(\omega) - x] \prod_{j=1}^{N} (\omega - b_j) \tag{4.29}$$

for any $\omega \in \mathbb{C}_+$.

Consider now the relationship (4.15) in the manner of Section 3; namely

$$\widehat{T}_\varphi \rho_\omega = \sum_{j=1}^{N} \varphi(\omega)(y_j) y_j' = \sum_{j=1}^{N+1} \frac{\varphi(\omega)(y_j)(\omega - y_j \prod_{k \neq j}^{N+1} (\omega - y_k) y_j'}{\prod_{k=1}^{N+1} (\omega - y_k)} =$$

$$= \sum_{j=1}^{N+1} \frac{\varphi(\omega)(y_j)(\omega - y_j \prod_{k \neq j}^{N+1} (\omega - y_k) y_j'}{\prod_{k=1}^{N+1} (\omega - y_k)} = \sum_{j=1}^{N+1} \frac{k \prod_{k \neq j}^{N+1} (\omega - y_k) y_j'}{\prod_{k=1}^{N+1} (\omega - y_k)} =$$

$$= -k \frac{\frac{d}{dx} \prod_{k \neq j}^{N+1} (\omega - y_k)}{\prod_{k=1}^{N} (\omega - y_k)} = -k \frac{d}{dx} \pi \prod_{k=1}^{N+1} (\omega - y_k), \tag{4.30}$$

where we have put, as before,

$$\rho_\omega(y_j)(\omega - y_j) = k, \tag{4.31}$$

for all $j = 1, \ldots, N + 1$, $\omega \in \mathbb{C}_+$, and some parameters $k \in \mathbb{C}$. This clearly means that

$$\rho_\omega(y) = \frac{k}{\omega - y} \tag{4.32}$$

for any $y \in \mathbb{R}$ and $\omega \in \mathbb{C}_+$.

Upon substituting the expression (4.29) into (4.30), one readily finds that

$$\widehat{T}_\varphi \rho_\omega(x) = \frac{k}{\varphi(\omega) - x} = \rho_{\varphi(\omega)}(x), \tag{4.33}$$

for all $x \in \mathbb{R}$ and any $\omega \in \mathbb{C}_+$. Thus, the invariant quasi-measure for the discrete dynamical system (4.1) is given by the same expression (4.25) when $\bar{\omega} \in \mathbb{C}_+$ is a fixed point of the mapping $\varphi\colon \mathbb{C}_+ \to \mathbb{C}_+$. This means that

$$\alpha \bar{\omega} + a - \sum_{j=1}^{N} \frac{\beta_j}{\bar{\omega} - b_j} = \bar{\omega}, \tag{4.34}$$

or, equivalently,

$$\alpha\bar{\omega}\prod_{j=1}^{N}(\bar{\omega}-b_j)+a\prod_{j=1}^{N}(\bar{\omega}-b_j)-\sum_{j=1}^{N}\beta_j\prod_{k\neq j}^{N}(\bar{\omega}-b_k)=\bar{\omega}\prod_{j=1}^{N}(\bar{\omega}-b_j), \tag{4.35}$$

for some $\bar{\omega}\in\mathbb{C}_+$. Assume now that $\alpha\neq 1$; then it is easy to see that the algebraic equation (4.35) possesses exactly $N+1\in\mathbb{Z}_+$ roots, which can be used to constructing the invariant quasi-measure (4.25). When $\alpha=1$, the condition becomes

$$a\prod_{j=1}^{N}(\bar{\omega}-b_j)=\sum_{j=1}^{N}\beta_j\prod_{k\neq j}^{N}(\bar{\omega}-b_k), \tag{4.36}$$

which always possesses roots for arbitrary $a\in\mathbb{R}$ if $N\geq 2$. This leads directly to the following characterization for $N\geq 2$:

Theorem 4.3. *The expression* (4.25) *for some $k\in\mathbb{C}$ determines, in general, the infinitesimal invariant quasi-measure for the generalized Boole transformation* (4.1) *for all $N\geq 2$ with arbitrary parameters $a, b_j\in\mathbb{R}$ and $\alpha,\beta_j\in\mathbb{R}_+$, $1\leq j\leq N+1$.*

It is an important now to find in the set of invariant quasi-measures (4.25) that we obtained, those that are positive and ergodic with respect to the transformation (4.1) for $N\geq 2$. For positivity, the determining equation (4.35) must possess at least one pair of complex conjugate roots. A thorough analysis of the roots of equation (4.35) leads to the following result, which is analogous to that proved in [4].

Theorem 4.4. *The generalized Boole transformation* (4.1) *for any $N\geq 1$ is necessarily ergodic with respect to the measure* (4.25) *for some $\bar{\omega}\in\mathbb{C}_+\backslash\mathbb{R}$ and $k\in\mathbb{C}$ iff $\alpha=1$ and $a=0$. If $\alpha=1$ and $a\neq 0$, the transformation* (4.1) *is not ergodic since it is totally dissipative, that is the wandering set $\mathcal{D}(\varphi):=\bigcup\mathcal{W}_\varphi=\mathbb{R}$, where $\mathcal{W}_\varphi\subset\mathbb{R}$ are such subsets such that $\varphi^{-n}(\mathcal{W}_\varphi)$, $n\in\mathbb{Z}$, are disjoint.*

Proof (sketch). It is easy to see that for $N\geq 2$, $\alpha=1$ and $a=0$ the determining algebraic equation (4.35) always possesses exactly $N-1$ real roots $\bar{\omega}_j\in\mathbb{R}$, $j=1,\ldots,N-1$. Therefore, the invariant quasi-measure expression (4.25) is degenerate for all of the $\bar{\omega}_j\in\mathbb{R}$, which leads directly to the conclusion that the corresponding invariant measure $d\mu(x)=dx$, $x\in\mathbb{R}$, is the standard Lebesgue measure on $\mathbb{R}$. Its ergodicity with respect that transformation (4.1) then follows from the fact that the corresponding dissipative set $\mathcal{D}(\varphi)=\varnothing$ and the unique invariant set subalgebra $I(\varphi)=\{\varnothing,\mathbb{R}\}$.

Results similar to those above can also be obtained for the most generalized Boole-type transformation

$$\mathbb{R}\ni y\to\varphi(y):=\alpha y+a+\int_{\mathbb{R}}\frac{d\nu(s)}{s-y}\in\mathbb{R}, \tag{4.37}$$

where $a\in\mathbb{R}$, $\alpha\in\mathbb{R}_+$ and the measure ν on $\mathbb{R}$ has the compact support $\operatorname{supp}\,\nu\subset\mathbb{R}$ such that the natural conditions [1, 14]

$$\int_{\mathbb{R}}\frac{d\nu(s)}{1+s^2}=a, \qquad \int_{\mathbb{R}}d\nu(s)<\infty, \tag{4.38}$$

hold. Concerning the extension of the transformation (4.37) on the upper part $\mathbb{C}_+$ of the complex plane $\mathbb{C}$ so that that $\operatorname{Im}\varphi(\omega)\geq 0$ for all $\omega\in\mathbb{C}_+$, the following representation:

$$\varphi(\omega)=\alpha\omega+a+\int_{\mathbb{R}}\frac{1+s\omega}{s-\omega}d\sigma(s), \tag{4.39}$$

holds [1, 5], where the measure $d\sigma$ on $\mathbb{R}$ is closely related to the measure $d\nu$.

The general properties of the mapping (4.39) were in part studied in [4] in the framework of the theory of inner functions. The invariant measures corresponding to (4.37) and their ergodic properties can be also treated effectively by making use of the analytical and spectral properties of the associated Frobenius – Perron transfer operator (4.4).

5. Two-dimensional generalizations of the Boole transformation. Consider the two-dimensional Boole-type transformations $\varphi_2, \psi_{\sigma(2)} \colon \mathbb{R}^2 \backslash \{0, 0\} \to \mathbb{R}^2$

$$\varphi_2(x, y) := (x - 1/x, y - 1/y) \tag{5.1}$$

and

$$\psi_{\sigma(2)}\ (x, y) := (x - 1/y, y - 1/x). \tag{5.2}$$

It is easy to see that the infinitesimal (product) measure

$$d\mu(x, y) := dxdy, \tag{5.3}$$

is invariant with respect to the first mapping (5.1) since it is the product of two measures, each of which is invariant with respect to the corresponding classical Boole transformation. Therefore, the generalized Boole-type transformation (5.1) is ergodic too. In the case of the generalized two-dimensional transformation (5.2), the invariance property of the measure (5.3) is a direct consequence of the following result.

Lemma 5.1. *The mapping* (5.2) *satisfies the infinitesimal invariance property*

$$\mu(\psi_{\sigma(2)}^{-1}([u, u + du] \times [v, v + dv]) = dudv = \mu([u, u + du] \times [v, v + dv])$$

with respect to the product measure defined in (5.3) *for all infinitesimal rectangles in the image of* $\psi_{\sigma(2)}$.

Proof. Representing the map (5.2) in the form

$$\psi_{\sigma(2)}\ (x, y) := (u, v) = (x - 1/y, y - 1/x),$$

and "inverting", we obtain the solutions

$$\begin{aligned} x_\pm = x_\pm(u, v) &:= (1/2)\left[u \pm \sqrt{u^2 + (4u/v)}\right] := (1/2)\left[u \pm X(u, v)\right], \\ y_\pm = y_\pm(u, v) &:= v + (1/x_\pm(u, v)) = (1/2)\left[v \pm \sqrt{v^2 + (4v/u)}\right] := (1/2)\left[v \pm Y(u, v)\right], \end{aligned} \tag{5.4}$$

where mappings $X \colon \mathbb{R}^2 \to \mathbb{R}$ and $Y \colon \mathbb{R}^2 \to \mathbb{R}$ have the obvious definitions $X(u, v) = \sqrt{u^2 + (4u/v)}$ and $Y(u, v) = \sqrt{v^2 + (4v/u)}$, respectively. In order to make effective use of these formulas, it is convenient to define the mappings $\widehat{\psi}_+, \widehat{\psi}_- \colon \mathbb{R}^2 \to \mathbb{R}^2$ as

$$\widehat{\psi}_+(u, v) := (x_+(u, v), y_+(u, v)), \quad \widehat{\psi}_-(u, v) := (x_-(u, v), y_-(u, v)) \tag{5.5}$$

for $(u, v) \in \mathbb{R}^2$. It follows directly from (5.4) that for all points $(u, v) \in \mathbb{R}^2$ in the image of $\psi_{\sigma(2)} \colon \mathbb{R}^2 \to \mathbb{R}^2$, except for those of a subset of (product) measure zero, the preimage $\psi_{\sigma(2)}^{-1}\ (u, v\}) \subset$ $\subset \mathbb{R}^2$ is comprised of two points in the domain of the map. For such points, if we choose an infinitesimal rectangle $dR := [u, u + du] \times [v, v + dv]$ (sufficiently small by definition), the union

$$\psi_{\sigma(2)}^{-1}(dR) = \widehat{\psi}_+(dR) \sqcup \widehat{\psi}_-(dR)$$

is disjoint, which implies that

$$\mu\left(\psi_{\sigma(2)}^{-1}(dR)\right) = \mu\left(\widehat{\psi}_+(dR)\right) + \mu\left(\widehat{\psi}_-(dR)\right),$$

which is owing to (5.5) infinitesimally equivalent to the following equation involving the product measure:

$$\mu\left(\psi_{\sigma(2)}^{-1}(dR)\right) = dx_+dy_+ + dx_-dy_-.$$

It is useful to observe from (5.4) that we have the following algebraic relationships:

$$x_+ + x_- = u, \qquad y_+ + y_- = v, \qquad x_+/y_+ = u/v,$$

$$x_+x_- = u/v, \qquad y_+y_- = v/u, \qquad x_-/y_- = u/v.$$

Next, we perform an infinitesimal measure computation to verify the invariance making extensive use of the fact that for any real-valued Lebesgue measurable function $\varphi\colon \mathbb{R}^2 \to \mathbb{R}$, $d(\varphi d\varphi) = d\varphi d\varphi = 0$:

$$\mu\left(\psi_{\sigma(2)}^{-1}(dR)\right) = dx_+dy_+ + dx_-dy_- = d\left(x_+dy_+ + x_-dy_-\right) =$$

$$= d\left[\left(u/vy_+\right)dy_+ + \left(u/vy_-\right)dy_-\right] = (1/2)\,d\,(u/v)\,d\left[y_+^2 + y_-^2\right] = (1/2)d\,(u/v)\,d\left(v^2 - 2v/u\right) =$$

$$= d\,(u/v)\left[vdv - d(v/u)\right] = \left(du/v - udv/v^2\right)vdv - d(u/v)d(v/u) =$$

$$= dudv - (u/v)dvdv + (v^2/u^2)d(u/v)d(u/v) = dudv.$$

Lemma 5.1 is proved.

Consequently, it follows by a simple modification of the proof of the main theorem in [6] that $d\mu(x,y), (x,y) \in \mathbb{R}^2$, is the unique absolutely continuous invariant measure for the Boole-type transformation (5.2). This, in particular, implies that the map (5.2) is ergodic with respect to the infinitesimal measure $d\mu(x,y), (x,y) \in \mathbb{R}^2$, and so we have the following result.

Proposition 5.1. *The generalized two-dimensional Boole-type transformations* (5.1) *and* (5.2) *are ergodic with respect to the standard infinitesimal measure* $d\mu(x,y) = dxdy$ *for* $(x,y) \in \mathbb{R}^2$. *In particular, the following equalities*:

$$\int\limits_{\mathbb{R}^2} f(\varphi_2(x,y))dxdy = \int\limits_{\mathbb{R}^2} f(x,y)dxdy = \int\limits_{\mathbb{R}^2} f(\psi_{\sigma(2)}(x,y))dxdy$$

hold for any integrable function $f \in L_1(\mathbb{R}^2;\mathbb{R})$.

The above result strongly suggests the validity of the following conjecture.

Conjecture 5.1. *Let* $\sigma \in \Sigma_n$ *be any element* (*permutation*) *of the symmetric group* $\Sigma_n, n \in \mathbb{Z}_+$. *Then the following generalized Boole-type transformation* $\psi_\sigma\colon \mathbb{R}^n \backslash \{0,0,\ldots,0\} \to \mathbb{R}^n$, *where*

$$\psi_\sigma(x_1,x_2,\ldots,x_n) := (x_1 - 1/x_{\sigma(1)}, x_2 - 1/x_{\sigma(2)}, x_3 - 1/x_{\sigma(3)}, \ldots, x_n - 1/x_{\sigma(n)}),$$

is ergodic with respect to the standard infinitesimal measure $d\mu(x_1,x_2,\ldots,x_n) := dx_1dx_2\ldots dx_n$, $(x_1,x_2,\ldots,x_n) \in \mathbb{R}^n$.

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