UDC 517.9

J. Diblík * (Brno Univ. Technology, Czech Republic),
M. Fečkan** (Comenius Univ. and Math. Inst. Slovak Acad. Sci., Bratislava, Slovakia),
M. Pospíšil*** (Brno Univ. Technology, Czech Republic)

REPRESENTATION OF A SOLUTION OF THE CAUCHY PROBLEM FOR AN OSCILLATING SYSTEM WITH TWO DELAYS AND PERMUTABLE MATRICES

ЗОБРАЖЕННЯ РОЗВ'ЯЗКУ ЗАДАЧІ КОШІ ДЛЯ КОЛИВНОЇ СИСТЕМИ З ДВОМА ЗАПІЗНЮВАННЯМИ ТА ПЕРЕСТАВНИМИ МАТРИЦЯМИ

We represent a solution of a nonhomogeneous second-order differential equation with two delays using matrix functions under the assumption that the linear parts are given by permutable matrices.

Отримано зображення розв'язку неоднорідного диференціального рівняння другого порядку з двома запізнюваннями із використанням матричних функцій за припущення, що лінійні частини задано переставними матрицями.

1. Introduction. Representation of a solution of system of first-order differential equations with single delay using matrix polynomial derived in [9] led to many new results in theory of ordinary differential equations with delay, such as controllability, exponential stability, boundary-value problems, etc. [1 – 3, 10, 13], but also in theory of boundary-value problems in partial differential equations [4]. On the other side, solutions of differential equations with multiple fixed or variable delays and difference equations with single and more delays were represented in similar ways [7, 11, 12, 15]. Also their asymptotic properties and controllability were investigated [6, 11, 12, 14, 15].

In [8], any solution of a system of differential equations of second order with single delay is represented using matrix polynomials. This representation was used in control theory [5].

In the present paper, by the use of permutable matrices, we are able to construct matrix polynomials which solve a linear homogeneous system of differential equations of second order with two different delays and linear terms given by these matrices. Later, using these functions, we represent a solution of initial problem of the corresponding nonhomogeneous system. So, the assumption of permutability leads to extension of [8] to systems with two different delays. Without the assumption, the matrix functions would not solve the homogeneous equation.

Let us recall the result from [8].

Theorem 1.1. *Let* $\tau > 0$, $\varphi \in C^2([-\tau, 0], \mathbb{R}^n)$, B *be a nonsingular* $n \times n$ *matrix and* $f\colon [0, \infty) \to \mathbb{R}^n$ *be a given function. Solution* $x\colon [-\tau, \infty) \to \mathbb{R}^n$ *of equation*

$$\ddot{x}(t) = -B^2 x(t - \tau) + f(t) \tag{1.1}$$

satisfying initial condition

$$\begin{aligned} x(t) &= \varphi(t), \\ \dot{x}(t) &= \dot{\varphi}(t), \end{aligned} \qquad -\tau \le t \le 0, \tag{1.2}$$

* Supported by the Grant GAČR P201/11/0768 and by the project No. CZ.1.05/2.1.00/03.0097.
** Supported in part by the Grants VEGA-MS 1/0507/11, VEGA-SAV 2/7140/27 and APVV-0134-10.
*** Supported by the project No. CZ.1.07/2.3.00/30.0005 funded by European Regional Development Fund.

has the form

$$x(t) = \mathrm{Cos}_\tau Bt\,\varphi(-\tau) + B^{-1}\mathrm{Sin}_\tau Bt\,\dot{\varphi}(-\tau)+$$

$$+B^{-1}\int\limits_{-\tau}^{0}\mathrm{Sin}_\tau B(t-\tau-s)\,\ddot{\varphi}(s)ds + B^{-1}\int\limits_{0}^{t}\mathrm{Sin}_\tau B(t-\tau-s)\,f(s)ds$$

for $t \in [-\tau, \infty)$, *where*

$$\mathrm{Cos}_\tau Bt = \begin{cases} \Theta, & -\infty < t < -\tau, \\ E, & -\tau \le t < 0, \\ E - B^2\dfrac{t^2}{2!} + \ldots + (-1)^k B^{2k}\dfrac{(t-(k-1)\tau)^{2k}}{(2k)!}, & (k-1)\tau \le t < k\tau, \quad k \in \mathbb{N}, \end{cases} \tag{1.3}$$

$$\mathrm{Sin}_\tau Bt = \begin{cases} \Theta, & -\infty < t < -\tau, \\ B(t+\tau), & -\tau \le t < 0, \\ B(t+\tau) - B^3\dfrac{t^3}{3!} + \ldots + (-1)^k B^{2k+1}\dfrac{(t-(k-1)\tau)^{2k+1}}{(2k+1)!}, & \\ & (k-1)\tau \le t < k\tau, \quad k \in \mathbb{N}. \end{cases} \tag{1.4}$$

Here, we used the notation Θ and E for the $n\times n$ zero and identity matrix, respectively. Moreover, $\mathbb{N}$ denotes the set of all positive integers. The above stated matrix functions $\mathrm{Cos}_\tau Bt$, $\mathrm{Sin}_\tau Bt$ have the following properties (see [8])

$$\begin{aligned} &\frac{d}{dt}\mathrm{Cos}_\tau Bt = -B\mathrm{Sin}_\tau B(t-\tau), \\ &\frac{d^2}{dt^2}\mathrm{Cos}_\tau Bt = -B^2\mathrm{Cos}_\tau B(t-\tau), \\ &\frac{d}{dt}\mathrm{Sin}_\tau Bt = B\mathrm{Cos}_\tau Bt, \\ &\frac{d^2}{dt^2}\mathrm{Sin}_\tau Bt = -B^2\mathrm{Sin}_\tau B(t-\tau) \end{aligned} \tag{1.5}$$

for any $t \in \mathbb{R}$, considering one-sided derivatives at $-\tau$ and 0.

2. Systems with two delays. In this section, we derive the representation of a solution of equation (1.1) with two delays, using matrix functions analogical to (1.3) and (1.4). More precisely, we consider equation

$$\ddot{x}(t) = -B_1^2 x(t-\tau_1) - B_2^2 x(t-\tau_2) + f(t) \tag{2.1}$$

with $\tau_1, \tau_2 > 0$ and permutable matrices B_1, B_2, i.e., $B_1B_2 = B_2B_1$. But first, we slightly improve Theorem 1.1 to C^1-smooth initial function.

Proposition 2.1. *Let* $\tau > 0$, $\varphi \in C^1([-\tau, 0], \mathbb{R}^n)$, B *be a nonsingular* $n \times n$ *matrix and* $f\colon [0,\infty) \to \mathbb{R}^n$ *be a given function. Solution* $x\colon [-\tau,\infty) \to \mathbb{R}^n$ *of equation* (1.1) *satisfying initial condition* (1.2) *has the form*

$$x(t) = \begin{cases} \varphi(t), & -\tau \le t < 0, \\ \mathrm{Cos}_\tau B(t-\tau)\,\varphi(0) + B^{-1}\mathrm{Sin}_\tau B(t-\tau)\,\dot\varphi(0) - & \\ \quad -B\displaystyle\int_{-\tau}^{0} \mathrm{Sin}_\tau B(t-2\tau-s)\,\varphi(s)ds + & \\ \quad +B^{-1}\displaystyle\int_0^t \mathrm{Sin}_\tau B(t-\tau-s)\,f(s)ds, & 0 \le t. \end{cases} \tag{2.2}$$

Proof. Obviously, $x(t)$ satisfies the initial condition on $[-\tau, 0)$ and $x(0) = \varphi(0)$ due to (1.3), (1.4). Furthermore, if $0 \le t < \tau$, then

$$x(t) = \varphi(0) + t\dot\varphi(0) - B^2 \int_{-\tau}^{t-\tau} (t-\tau-s)\varphi(s)ds + \int_0^t (t-s)f(s)ds$$

since

$$\mathrm{Sin}_\tau B(t-2\tau-s) = \begin{cases} B(t-\tau-s), & s \in [-\tau, t-\tau], \\ \Theta, & s \in (t-\tau, 0], \end{cases}$$

for $0 \le t < \tau$ and $s \in [-\tau, 0]$. Accordingly,

$$\dot x(t) = \dot\varphi(0) - B^2 \int_{-\tau}^{t-\tau} \varphi(s)ds + \int_0^t f(s)ds,$$

$$\ddot x(t) = -B^2\varphi(t-\tau) + f(t). \tag{2.3}$$

Hence, one can see that $\lim_{t\to 0^+} \dot x(t) = \dot\varphi(0)$, i.e., $x \in C^1([-\tau,\infty), \mathbb{R}^n)$.

Next, using properties (1.5) for $t \ge \tau$,

$$\dot x(t) = -B\mathrm{Sin}_\tau B(t-2\tau)\,\varphi(0) + \mathrm{Cos}_\tau B(t-\tau)\,\dot\varphi(0) -$$

$$-B^2 \int_{-\tau}^{0} \mathrm{Cos}_\tau B(t-2\tau-s)\,\varphi(s)ds + \int_0^t \mathrm{Cos}_\tau B(t-\tau-s)\,f(s)ds,$$

$$\ddot x(t) = -B^2\mathrm{Cos}_\tau B(t-2\tau)\,\varphi(0) - B\mathrm{Sin}_\tau B(t-2\tau)\,\dot\varphi(0) +$$

$$+B^3 \int_{-\tau}^{0} \mathrm{Sin}_\tau B(t-3\tau-s)\,\varphi(s)ds - B\int_0^t \mathrm{Sin}_\tau B(t-2\tau-s)\,f(s)ds + f(t) =$$

$$= -B^2 x(t-\tau) + f(t). \tag{2.4}$$

In the last step, we applied the identity

$$\int\limits_0^t \mathrm{Sin}_\tau B(t-2\tau-s)\, f(s)ds = \int\limits_0^{t-\tau} \mathrm{Sin}_\tau B(t-2\tau-s)\, f(s)ds.$$

Taking the second derivatives at τ, we get

$$\lim_{t\to\tau^-} \ddot{x}(t) = -B^2\varphi(0) + f(\tau) = \lim_{t\to\tau^+} \ddot{x}(t)$$

by (2.3) and (2.4). Summarizing, $x(t)$ given by (2.2) solves equation (1.1) on $[0,\infty)$, satisfies condition (1.2) on $[-\tau,0]$ and

$$x \in C^1([-\tau,\infty),\mathbb{R}^n) \cap C^2([0,\infty),\mathbb{R}^n).$$

Proposition 2.1 is proved.

From now on, we assume the property of empty sum, i.e.,

$$\sum_{i\in\varnothing} f(i) = 0, \qquad \sum_{i\in\varnothing} F(i) = \Theta$$

for any function f and matrix function F, whether they are defined or not. Define functions $\mathcal{X}_\tau^{B^2}$, $\mathcal{Y}_\tau^{B^2} \colon \mathbb{R} \to L(\mathbb{R}^n)$ as

$$\mathcal{X}_\tau^{B^2}(t) := \sum_{\substack{i\ge 0\\ i\tau\le t}} (-1)^i B^{2i} \frac{(t-i\tau)^{2i}}{(2i)!},$$

$$\mathcal{Y}_\tau^{B^2}(t) := \sum_{\substack{i\ge 0\\ i\tau\le t}} (-1)^i B^{2i} \frac{(t-i\tau)^{2i+1}}{(2i+1)!}$$

for any $t\in\mathbb{R}$. Note that

$$\mathrm{Cos}_\tau B(t-\tau) = \mathcal{X}_\tau^{B^2}(t), \qquad \mathrm{Sin}_\tau B(t-\tau) = B\mathcal{Y}_\tau^{B^2}(t).$$

Function $\mathcal{X}_\tau^{B^2}$ inherits its properties from function $\mathrm{Cos}_\tau B(\cdot-\tau)$. As the next lemma explains, so does the function $\mathcal{Y}_\tau^{B^2}$ from $\mathrm{Sin}_\tau B(\cdot-\tau)$.

Lemma 2.1. *Let $\tau>0$ and B be $n\times n$ complex matrix. Function $\mathcal{Y}_\tau^{B^2}(t)$ solves equation*

$$\ddot{y}(t) = -B^2 y(t-\tau) \tag{2.5}$$

for $t\ge 0$ with initial condition

$$y(t)=\Theta, \quad -\tau\le t\le 0, \qquad \dot{y}(t) = \begin{cases} \Theta, & -\tau\le t<0,\\ E, & t=0.\end{cases}$$

Proof. The initial condition is immediately verified. If the matrix B is regular, then $\mathcal{Y}_\tau^{B^2}(t) = B^{-1}\mathrm{Sin}_\tau B(t-\tau)$ and equation (2.5) is fulfilled by (1.5) (clearly, this is valid also if B is complex). Also the other case can be proved easily:

Let $t \geq 0$. Then

$$\mathcal{Y}_\tau^{B^2}(t) = \sum_{\substack{i\geq 0\\ i\tau\leq t}} (-1)^i B^{2i}\frac{(t-i\tau)^{2i+1}}{(2i+1)!} = tE\chi_{[0,\infty)}(t) + \sum_{\substack{i\geq 1\\ i\tau\leq t}} (-1)^i B^{2i}\frac{(t-i\tau)^{2i+1}}{(2i+1)!},$$

where χ_M is a characteristic function of a set M given by

$$\chi_M(t) = \begin{cases} 1, & t\in M,\\ 0, & t\notin M. \end{cases}$$

Hence,

$$\ddot{\mathcal{Y}}_\tau^{B^2}(t) = \sum_{\substack{i\geq 1\\ i\tau\leq t}} (-1)^i B^{2i}\frac{(t-i\tau)^{2i-1}}{(2i-1)!} =$$

$$= -B^2 \sum_{\substack{i-1\geq 0\\ (i-1)\tau\leq t-\tau}} (-1)^i B^{2(i-1)}\frac{(t-\tau-(i-1)\tau)^{2(i-1)+1}}{(2(i-1)+1)!} = -B^2\mathcal{Y}_\tau^{B^2}(t-\tau). \qquad (2.6)$$

Lemma 2.1 is proved.

Define functions $\mathcal{X}_{\tau_1,\tau_2}^{B_1^2,B_2^2}, \mathcal{Y}_{\tau_1,\tau_2}^{B_1^2,B_2^2} : \mathbb{R}\to L(\mathbb{R}^n)$ as

$$\mathcal{X}_{\tau_1,\tau_2}^{B_1^2,B_2^2}(t) := \sum_{\substack{i,j\geq 0\\ i\tau_1+j\tau_2\leq t}} (-1)^{i+j}\binom{i+j}{i} B_1^{2i}B_2^{2j}\frac{(t-i\tau_1-j\tau_2)^{2(i+j)}}{(2(i+j))!},$$

$$\mathcal{Y}_{\tau_1,\tau_2}^{B_1^2,B_2^2}(t) := \sum_{\substack{i,j\geq 0\\ i\tau_1+j\tau_2\leq t}} (-1)^{i+j}\binom{i+j}{i} B_1^{2i}B_2^{2j}\frac{(t-i\tau_1-j\tau_2)^{2(i+j)+1}}{(2(i+j)+1)!} \qquad (2.7)$$

for any $t\in\mathbb{R}$. Some properties of functions $\mathcal{X}_{\tau_1,\tau_2}^{B_1^2,B_2^2}$ and $\mathcal{Y}_{\tau_1,\tau_2}^{B_1^2,B_2^2}$ are concluded in the next lemma.

Lemma 2.2. *Let $\tau_1,\tau_2>0$, B_1, B_2 be permutable matrices. Then the following holds for any $t\in\mathbb{R}$:*

(1) *if $B_1=\Theta$, then $\mathcal{X}_{\tau_1,\tau_2}^{B_1^2,B_2^2}(t) = \mathcal{X}_{\tau_2}^{B_2^2}(t)$,*

(2) *if $B_2=\Theta$, then $\mathcal{X}_{\tau_1,\tau_2}^{B_1^2,B_2^2}(t) = \mathcal{X}_{\tau_1}^{B_1^2}(t)$,*

(3) *if $\tau:=\tau_1=\tau_2$, then $\mathcal{X}_{\tau_1,\tau_2}^{B_1^2,B_2^2}(t) = \mathcal{X}_{\tau}^{B_1^2+B_2^2}(t)$,*

(4) $\mathcal{X}_{\tau_1,\tau_2}^{B_1^2,B_2^2}(t) = \mathcal{X}_{\tau_2,\tau_1}^{B_2^2,B_1^2}(t)$,

(5) *taking the one-sided derivatives at 0, τ_1, τ_2*

$$\ddot{\mathcal{X}}_{\tau_1,\tau_2}^{B_1^2,B_2^2}(t) = -B_1^2\mathcal{X}_{\tau_1,\tau_2}^{B_1^2,B_2^2}(t-\tau_1) - B_2^2\mathcal{X}_{\tau_1,\tau_2}^{B_1^2,B_2^2}(t-\tau_2), \qquad (2.8)$$

(6) *considering the one-sided derivatives at* 0 (*they both equal* Θ)

$$\dot{\mathcal{Y}}_{\tau_1,\tau_2}^{B_1^2,B_2^2}(t) = \mathcal{X}_{\tau_1,\tau_2}^{B_1^2,B_2^2}(t).$$

Statements (1)–(5) *hold with* $\mathcal{Y}$ *instead of* $\mathcal{X}$.

Proof. (1) and (2) are obtained easily from definition of $\mathcal{X}_{\tau_1,\tau_2}^{B_1^2,B_2^2}$, because $\Theta^{2i} = E$ if $i = 0$ and $\Theta^{2i} = \Theta$ whenever $i > 0$. Since

$$\sum_{\substack{i,j\geq 0\\(i+j)\tau\leq t}} f(i,j) = \sum_{\substack{l\geq 0\\l\tau\leq t}} \sum_{\substack{i,j\geq 0\\i+j=l}} f(i,j)$$

for any (matrix) function f, for (3) we get

$$\mathcal{X}_{\tau_1,\tau_2}^{B_1^2,B_2^2}(t) = \sum_{\substack{l\geq 0\\l\tau\leq t}} (-1)^l (B_1^2 + B_2^2)^l \frac{(t - l\tau)^{2l}}{(2l)!} = \mathcal{X}_{\tau}^{B_1^2+B_2^2}(t).$$

Property (4) is trivial.

Next, if $\tau := \tau_1 = \tau_2$, then

$$\ddot{\mathcal{X}}_{\tau_1,\tau_2}^{B_1^2,B_2^2}(t) = \ddot{\mathcal{X}}_{\tau}^{B_1^2+B_2^2}(t) = -(B_1^2 + B_2^2)\mathcal{X}_{\tau}^{B_1^2+B_2^2}(t - \tau) =$$

$$= -B_1^2\mathcal{X}_{\tau_1,\tau_2}^{B_1^2,B_2^2}(t - \tau_1) - B_2^2\mathcal{X}_{\tau_1,\tau_2}^{B_1^2,B_2^2}(t - \tau_2)$$

for any $t \in \mathbb{R}$, by (3) and the property of $\mathrm{Cos}_\tau \sqrt{B_1^2 + B_2^2}t$ (see (1.5)). Note that $\mathrm{Cos}_\tau Bt$ solves equation (2.5) even for complex matrix B.

Without any loss of generality, we assume that $\tau_1 < \tau_2$ and consider two cases: $t < \tau_2$ and $t \geq \tau_2$.

If $t < \tau_2$, then $t - \tau_2 < 0$ and $\mathcal{X}_{\tau_1,\tau_2}^{B_1^2,B_2^2}(t) = \mathcal{X}_{\tau_1}^{B_1^2}(t)$. Thus (2.8) is verified for $t < \tau_2$ by the property of $\mathrm{Cos}_{\tau_1} B_1(t - \tau_1)$.

Now, let $t \geq \tau_2$. We decompose

$$\mathcal{X}_{\tau_1,\tau_2}^{B_1^2,B_2^2}(t) = E\chi_{[0,\infty)}(t) + S_1(t) + S_2(t) + S_3(t),$$

where

$$S_1(t) = \sum_{\substack{i\geq 1\\i\tau_1\leq t}} (-1)^i B_1^{2i} \frac{(t - i\tau_1)^{2i}}{(2i)!},$$

$$S_2(t) = \sum_{\substack{j\geq 1\\j\tau_2\leq t}} (-1)^j B_2^{2j} \frac{(t - j\tau_2)^{2j}}{(2j)!},$$

$$S_3(t) = \sum_{\substack{i,j\geq 1\\i\tau_1+j\tau_2\leq t}} (-1)^{i+j} \binom{i+j}{i} B_1^{2i} B_2^{2j} \frac{(t - i\tau_1 - j\tau_2)^{2(i+j)}}{(2(i+j))!}.$$

Similarly to (2.6), we obtain

$$S_1''(t) = \sum_{\substack{i\geq 1\\ i\tau_1\leq t}} (-1)^i B_1^{2i} \frac{(t-i\tau_1)^{2(i-1)}}{(2(i-1))!} =$$

$$= -B_1^2 \sum_{\substack{i\geq 0\\ i\tau_1\leq t-\tau_1}} (-1)^i B_1^{2i} \frac{(t-\tau_1-i\tau_1)^{2i}}{(2i)!} = -B_1^2 - B_1^2 S_1(t-\tau_1),$$

$$S_2''(t) = \sum_{\substack{j\geq 1\\ j\tau_2\leq t}} (-1)^j B_2^{2j} \frac{(t-j\tau_2)^{2(j-1)}}{(2(j-1))!} =$$

$$= -B_2^2 \sum_{\substack{j\geq 0\\ j\tau_2\leq t-\tau_2}} (-1)^j B_2^{2j} \frac{(t-\tau_2-j\tau_2)^{2j}}{(2j)!} = -B_2^2 - B_2^2 S_2(t-\tau_2).$$

Using the property of binomial numbers

$$\binom{i+j}{i} = \binom{i-1+j}{i-1} + \binom{i+j-1}{j-1},$$

for $i, j \geq 1$ we derive

$$S_3''(t) = \sum_{\substack{i,j\geq 1\\ i\tau_1+j\tau_2\leq t}} (-1)^{i+j} \binom{i+j}{i} B_1^{2i} B_2^{2j} \frac{(t-i\tau_1-j\tau_2)^{2(i+j-1)}}{(2(i+j-1))!} =$$

$$= -B_1^2 \sum_{\substack{i-1\geq 0, j\geq 1\\ (i-1)\tau_1+j\tau_2\leq t-\tau_1}} (-1)^{i-1+j} \binom{i-1+j}{i-1} \times$$

$$\times B_1^{2(i-1)} B_2^{2j} \frac{(t-\tau_1-(i-1)\tau_1-j\tau_2)^{2(i-1+j)}}{(2(i-1+j))!} -$$

$$-B_2^2 \sum_{\substack{i\geq 1, j-1\geq 0\\ i\tau_1+(j-1)\tau_2\leq t-\tau_2}} (-1)^{i+j-1} \binom{i+j-1}{j-1} \times$$

$$\times B_1^{2i} B_2^{2(j-1)} \frac{(t-\tau_2-i\tau_1-(j-1)\tau_2)^{2(i+j-1)}}{(2(i+j-1))!}.$$

Rewrite $i-1\to i$ in the first sum and $j-1\to j$ in the second one:

$$S_3''(t) = -B_1^2 \sum_{\substack{i\geq 0, j\geq 1\\ i\tau_1+j\tau_2\leq t-\tau_1}} (-1)^{i+j} \binom{i+j}{i} B_1^{2i} B_2^{2j} \frac{(t-\tau_1-i\tau_1-j\tau_2)^{2(i+j)}}{(2(i+j))!} -$$

$$-B_2^2 \sum_{\substack{i\geq 1, j\geq 0\\ i\tau_1+j\tau_2\leq t-\tau_2}} (-1)^{i+j}\binom{i+j}{j} B_1^{2i}B_2^{2j}\frac{(t-\tau_2-i\tau_1-j\tau_2)^{2(i+j)}}{(2(i+j))!}.$$

Now, we split the first sum to $i=0$ and $i\geq 1$, and the second sum to $j=0$ and $j\geq 1$:

$$S_3''(t) = -B_1^2 \sum_{\substack{j\geq 1\\ j\tau_2\leq t-\tau_1}} (-1)^j B_2^{2j}\frac{(t-\tau_1-j\tau_2)^{2j}}{(2j)!} - B_1^2 S_3(t-\tau_1)-$$

$$-B_2^2 \sum_{\substack{i\geq 1\\ i\tau_1\leq t-\tau_2}} (-1)^i B_1^{2i}\frac{(t-\tau_2-i\tau_1)^{2i}}{(2i)!} - B_2^2 S_3(t-\tau_2) =$$

$$= -B_1^2 S_2(t-\tau_1) - B_1^2 S_3(t-\tau_1) - B_2^2 S_1(t-\tau_2) - B_2^2 S_3(t-\tau_2).$$

In conclusion (adding formulae for $S_1''(t)$, $S_2''(t)$ and $S_3''(t)$), we get exactly formula (2.8) for $t\geq\tau_2$.

For $\mathcal{Y}_{\tau_1,\tau_2}^{B_1^2,B_2^2}$, statements (1)–(4) are proved as for $\mathcal{X}_{\tau_1,\tau_2}^{B_1^2,B_2^2}$. Next, if $\tau:=\tau_1=\tau_2$, we apply (3) and Lemma 2.1 with $B=\sqrt{B_1^2+B_2^2}$ to get

$$\ddot{\mathcal{Y}}_{\tau_1,\tau_2}^{B_1^2,B_2^2}(t) = \ddot{\mathcal{Y}}_{\tau}^{B_1^2+B_2^2}(t) = -(B_1^2+B_2^2)\mathcal{Y}_{\tau}^{B_1^2+B_2^2}(t-\tau) =$$

$$= -B_1^2\mathcal{Y}_{\tau_1,\tau_2}^{B_1^2,B_2^2}(t-\tau_1) - B_2^2\mathcal{Y}_{\tau_1,\tau_2}^{B_1^2,B_2^2}(t-\tau_2).$$

For $\tau_1<\tau_2$, $t<\tau_2$, we have $\mathcal{Y}_{\tau_1,\tau_2}^{B_1^2,B_2^2}(t)=\mathcal{Y}_{\tau_1}^{B_1^2}(t)$ and (2.8) is verified with the aid of Lemma 2.1.

If $t\geq\tau_2$, we split the sum

$$\mathcal{Y}_{\tau_1,\tau_2}^{B_1^2,B_2^2}(t) = tE\chi_{[0,\infty)}(t) + S_1(t) + S_2(t) + S_3(t)$$

with

$$S_1(t) = \sum_{\substack{i\geq 1\\ i\tau_1\leq t}} (-1)^i B_1^{2i}\frac{(t-i\tau_1)^{2i+1}}{(2i+1)!},$$

$$S_2(t) = \sum_{\substack{j\geq 1\\ j\tau_2\leq t}} (-1)^j B_2^{2j}\frac{(t-j\tau_2)^{2j+1}}{(2j+1)!},$$

$$S_3(t) = \sum_{\substack{i,j\geq 1\\ i\tau_1+j\tau_2\leq t}} (-1)^{i+j}\binom{i+j}{i} B_1^{2i}B_2^{2j}\frac{(t-i\tau_1-j\tau_2)^{2(i+j)+1}}{(2(i+j)+1)!}.$$

The rest proceeds analogically to $\mathcal{X}_{\tau_1,\tau_2}^{B_1^2,B_2^2}(t)$.

Property (6) follows immediately from definition (2.7).

Lemma 2.2 is proved.

***Remark* 2.1.** Using statement (5) of the previous lemma, statements (1)–(4) can be proved easily by the uniqueness of a solution of the corresponding initial value problem. For instance in the statement (2), both $\mathcal{X}_{\tau_1,\tau_2}^{B_1^2,B_2^2}(t)$ and $\mathcal{X}_{\tau_1}^{B_1^2}(t)$ are matrix solutions of equation

$$\ddot{x}(t) = -B_1^2 x(t-\tau_1)$$

with initial condition

$$x(t) = \begin{cases} \Theta, & -\tau_1 \le t < 0, \\ E, & t = 0, \end{cases}$$

$$\dot{x}(t) = \Theta, \qquad -\tau_1 \le t \le 0.$$

Definition 2.1. *Let $\tau_1, \tau_2 > 0$, $\tau := \max\{\tau_1, \tau_2\}$, $\varphi \in C^1([-\tau, 0], \mathbb{R}^n)$, B_1, B_2 be $n \times n$ matrices and $f\colon [0,\infty) \to \mathbb{R}^n$ be a given function. Function $x\colon [-\tau,\infty) \to \mathbb{R}^n$ is a solution of equation* (2.1) *and initial condition* (1.2)*, if $x \in C^1([-\tau,\infty),\mathbb{R}^n) \cap C^2([0,\infty),\mathbb{R}^n)$ (taken the second right-hand derivative at* 0)*, satisfies equation* (2.1) *on $[0,\infty)$ and condition* (1.2) *on $[-\tau, 0]$.*

We are ready to state and prove our main result.

3. Main result.

Theorem 3.1. *Let $\tau_1, \tau_2 > 0$, $\tau := \max\{\tau_1, \tau_2\}$, $\varphi \in C^1([-\tau, 0], \mathbb{R}^n)$, B_1, B_2 be $n \times n$ permutable matrices and $f\colon [0,\infty) \to \mathbb{R}^n$ be a given function. Solution $x(t)$ of equation* (2.1) *satisfying initial condition* (1.2) *has the form*

$$x(t) = \begin{cases} \varphi(t), & -\tau \le t < 0, \\ \mathcal{X}(t)\varphi(0) + \mathcal{Y}(t)\dot{\varphi}(0) - B_1^2 \int\limits_{-\tau_1}^{0} \mathcal{Y}(t-\tau_1-s)\varphi(s)ds - \\ \quad -B_2^2 \int\limits_{-\tau_2}^{0} \mathcal{Y}(t-\tau_2-s)\varphi(s)ds + \int\limits_{0}^{t} \mathcal{Y}(t-s)f(s)ds, & 0 \le t, \end{cases} \tag{3.1}$$

where $\mathcal{X} = \mathcal{X}_{\tau_1,\tau_2}^{B_1^2,B_2^2}$, $\mathcal{Y} = \mathcal{Y}_{\tau_1,\tau_2}^{B_1^2,B_2^2}$.

Proof. We consider only the case $\tau_1 \ne \tau_2$, since if $\tau_1 = \tau_2$, one can use (3) in Lemma 2.2 to show that this theorem coincides with Proposition 2.1.

Obviously, $x(t)$ satisfies the initial condition on $[-\tau, 0)$ and $x(0) = \varphi(0)$. For the derivative, it holds $\lim_{t\to 0^-} \dot{x}(t) = \dot{\varphi}(0)$. Moreover, if $0 \le t < \min\{\tau_1, \tau_2\}$, then

$$x(t) = \varphi(0) + t\dot{\varphi}(0) - B_1^2 \int\limits_{-\tau_1}^{t-\tau_1} (t-\tau_1-s)\varphi(s)ds -$$

$$-B_2^2 \int\limits_{-\tau_2}^{t-\tau_2} (t-\tau_2-s)\varphi(s)ds + \int\limits_{0}^{t} (t-s)f(s)ds \tag{3.2}$$

since

$$\mathcal{Y}(t-\tau_i-s)=\begin{cases}(t-\tau_i-s)E, & s\in[-\tau_i,t-\tau_i],\\ \Theta, & s\in(t-\tau_i,0],\end{cases}$$

for $0\le t<\min\{\tau_1,\tau_2\}$, $s\in[-\tau_i,0]$ and $i=1,2$. Thus

$$\dot{x}(t)=\dot{\varphi}(0)-B_1^2\int\limits_{-\tau_1}^{t-\tau_1}\varphi(s)ds-B_2^2\int\limits_{-\tau_2}^{t-\tau_2}\varphi(s)ds+\int\limits_0^t f(s)ds \qquad (3.3)$$

and $\lim_{t\to0^+}\dot{x}(t)=\dot{\varphi}(0)$. Clearly,

$$x\in C^1((0,\infty),\mathbb{R}^n)\cap C^2([0,\infty)\backslash\{\tau_1,\tau_2\},\mathbb{R}^n).$$

We show, that although $\mathcal{X}(t)$ is not C^2 at τ_1, τ_2, function $x(t)$ is C^2 at these points, and, therefore, in $[0,\infty)$. At once, we show that $x(t)$ is a solution of equation (2.1).

For instance, let $\tau_1<\tau_2$. Assume that $0\le t<\tau_1$. Then equalities (3.2), (3.3) are valid and we obtain

$$\ddot{x}(t)=-B_1^2\varphi(t-\tau_1)-B_2^2\varphi(t-\tau_2)+f(t).$$

Now, let $\tau_1\le t<\tau_2$. Then

$$x(t)=\mathcal{X}(t)\varphi(0)+\mathcal{Y}(t)\dot{\varphi}(0)-B_1^2\int\limits_{-\tau_1}^{0}\mathcal{Y}(t-\tau_1-s)\varphi(s)ds-$$

$$-B_2^2\int\limits_{-\tau_2}^{t-\tau_2}\mathcal{Y}(t-\tau_2-s)\varphi(s)ds+\int\limits_0^t\mathcal{Y}(t-s)f(s)ds$$

since $\mathcal{Y}(t-\tau_2-s)=\Theta$ if $s\in(t-\tau_2,0]$. By (6) of Lemma 2.2,

$$\dot{x}(t)=\dot{\mathcal{X}}(t)\varphi(0)+\dot{\mathcal{Y}}(t)\dot{\varphi}(0)-B_1^2\int\limits_{-\tau_1}^{0}\dot{\mathcal{Y}}(t-\tau_1-s)\varphi(s)ds-$$

$$-B_2^2\int\limits_{-\tau_2}^{t-\tau_2}\mathcal{X}(t-\tau_2-s)\varphi(s)ds+\int\limits_0^t\dot{\mathcal{Y}}(t-s)f(s)ds.$$

With the aid of (5) of Lemma 2.2 and since $\mathcal{X}(t)=\mathcal{Y}(t)=\Theta$ for $t<0$, we derive

$$\ddot{x}(t)=-B_1^2\mathcal{X}(t-\tau_1)\varphi(0)-B_1^2\mathcal{Y}(t-\tau_1)\dot{\varphi}(0)+$$

$$+B_1^4\int\limits_{-\tau_1}^{0}\mathcal{Y}(t-2\tau_1-s)\varphi(s)ds-B_2^2\varphi(t-\tau_2)+$$

$$+B_1^2B_2^2 \int\limits_{-\tau_2}^{t-\tau_1-\tau_2} \mathcal{Y}(t-\tau_1-\tau_2-s)\varphi(s)ds + f(t)-$$

$$-B_1^2 \int\limits_{0}^{t-\tau_1} \mathcal{Y}(t-\tau_1-s)f(s)ds =$$

$$= -B_1^2 x(t-\tau_1) - B_2^2\varphi(t-\tau_2) + f(t).$$

Finally, if $\tau_2 \le t$, we have

$$x(t) = \mathcal{X}(t)\varphi(0) + \mathcal{Y}(t)\dot{\varphi}(0) - B_1^2 \int\limits_{-\tau_1}^{0} \mathcal{Y}(t-\tau_1-s)\varphi(s)ds-$$

$$-B_2^2 \int\limits_{-\tau_2}^{0} \mathcal{Y}(t-\tau_2-s)\varphi(s)ds + \int\limits_{0}^{t} \mathcal{Y}(t-s)f(s)ds.$$

So, using

$$\left(\int\limits_{0}^{t} \mathcal{Y}(t-s)f(s)ds\right)'' = \left(\int\limits_{0}^{t} \mathcal{X}(t-s)f(s)ds\right)' = f(t) + \int\limits_{0}^{t} \ddot{\mathcal{Y}}(t-s)f(s)ds,$$

we get directly the second derivative

$$\ddot{x}(t) = \ddot{\mathcal{X}}(t)\varphi(0) + \ddot{\mathcal{Y}}(t)\dot{\varphi}(0) - B_1^2 \int\limits_{-\tau_1}^{0} \ddot{\mathcal{Y}}(t-\tau_1-s)\varphi(s)ds-$$

$$-B_2^2 \int\limits_{-\tau_2}^{0} \ddot{\mathcal{Y}}(t-\tau_2-s)\varphi(s)ds + \int\limits_{0}^{t} \ddot{\mathcal{Y}}(t-s)f(s)ds + f(t)$$

and after applying (5) of Lemma 2.2, relation (2.1) results. Hence, one can see, that function $x(t)$ given by (3.1) really solves equation (2.1), satisfies initial condition (1.2) and, moreover, that $x \in$ $\in C^2((0,\infty),\mathbb{R}^n)$. Indeed, taking limits at τ_1, τ_2 in the computed second derivatives, one gets

$$\lim_{t\to\tau_1^-} \ddot{x}(t) = -B_1^2\varphi(0) - B_2^2\varphi(\tau_1-\tau_2) + f(\tau_1) = \lim_{t\to\tau_1^+} \ddot{x}(t),$$

$$\lim_{t\to\tau_2^-} \ddot{x}(t) = -B_1^2 x(\tau_2-\tau_1) - B_2^2\varphi(0) + f(\tau_2) = \lim_{t\to\tau_2^+} \ddot{x}(t).$$

Cases $\tau_1 = \tau_2$ and $\tau_1 > \tau_2$ can be proved analogically.

Theorem 3.1 is proved.

1. *Boichuk A., Diblík J., Khusainov D., Růžičková M.* Boundary value problems for delay differential systems // Adv. Different. Equat. – 2010. – Article ID 593834. – 20 p.
2. *Boichuk A., Diblík J., Khusainov D., Růžičková M.* Fredholm's boundary-value problems for differential systems with a single delay // Nonlinear Anal.-Theor. – 2010. – **72**. – P. 2251 – 2258.
3. *Boichuk A., Diblík J., Khusainov D., Růžičková M.* Boundary-value problems for weakly nonlinear delay differential systems // Abstr. Appl. Anal. – 2011. – Article ID 631412. – 19 p.
4. *Diblík J., Khusainov D., Kukharenko O., Svoboda Z.* Solution of the first boundary-value problem for a system of autonomous second-order linear PDEs of parabolic type with a single delay // Abstr. Appl. Anal. – 2012. – Article ID 219040. – 27 p.
5. *Diblík J., Khusainov D. Ya., Lukáčová J., Růžičková M.* Control of oscillating systems with a single delay // Adv. Different. Equat. – 2010. – Article ID 108218. – 15 p.
6. *Diblík J., Khusainov D. Ya., Růžičková M.* Controllability of linear discrete systems with constant coefficients and pure delay // SIAM J. Contr. Optimiz. – 2008. – **47**. – P. 1140 – 1149.
7. *Khusainov D. Ya., Diblík J.* Representation of solutions of linear discrete systems with constant coefficients and pure delay // Adv. Different. Equat. – 2006. – Article ID 80825. – P. 1 – 13.
8. *Khusainov D. Ya., Diblík J., Růžičková M., Lukáčová J.* Representation of a solution of the Cauchy problem for an oscillating system with pure delay // Nonlinear Oscillations. – 2008. – **11**, № 2. – P. 276 – 285.
9. *Khusainov D. Ya., Shuklin G. V.* Linear autonomous time-delay system with permutation matrices solving // Stud. Univ. Žilina. Math. Ser. – 2003. – **17**. – P. 101 – 108.
10. *Khusainov D. Ya., Shuklin G. V.* Relative controllability in systems with pure delay // Int. Appl. Mech. – 2005. – **41**. – P. 210 – 221.
11. *Medved' M., Pospíšil M.* Sufficient conditions for the asymptotic stability of nonlinear multidelay differential equations with linear parts defined by pairwise permutable matrices // Nonlinear Anal.-Theor. – 2012. – **75**. – P. 3348 – 3363.
12. *Medved' M., Pospíšil M.* Representation and stability of solutions of systems of difference equations with multiple delays and linear parts defined by pairwise permutable matrices // Commun. Appl. Anal. – 2012 (to appear).
13. *Medved' M., Pospíšil M., Škripková L.* Stability and the nonexistence of blowing-up solutions of nonlinear delay systems with linear parts defined by permutable matrices // Nonlinear Anal.-Theor. – 2011. – **74**. – P. 3903 – 3911.
14. *Medved' M., L. Škripková L.* Sufficient conditions for the exponential stability of delay difference equations with linear parts defined by permutable matrices // Electron. J. Qual. Theory Different. Equat. – 2012. – **22**. – P. 1 – 13.
15. *Pospíšil M.* Representation and stability of solutions of systems of functional differential equations with multiple delays // Electron. J. Qual. Theory Different. Equat. – 2012. – **4**. – P. 1 – 30.

Received 17.12.12