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HERMITE – HADAMARD-TYPE INEQUALITIES FOR r-CONVEX FUNCTIONS USING RIEMANN – LIOUVILLE FRACTIONAL INTEGRALS*

НЕРІВНОСТІ ТИПУ ЕРМІТА – АДАМАРА ДЛЯ r-ОПУКЛИХ ФУНКЦІЙ ІЗ ВИКОРИСТАННЯМ ДРОБОВИХ ІНТЕГРАЛІВ РІМАНА – ЛІУВІЛЛЯ

By using two fundamental fractional integral identities, we derive some new Hermite – Hadamard-type inequalities for differentiable r-convex functions and twice-differentiable r-convex functions involving Riemann – Liouville fractional integrals.

Із використанням двох фундаментальних дробових інтегральних тотожностей отримано нові нерівності типу Ерміта – Адамара для диференційовних r-опуклих функцій та двічі диференційовних r-опуклих функцій, що містять дробові інтеграли Рімана – Ліувілля.

1. Introduction. It is well-known that one of the most fundamental and interesting inequalities for classical convex functions is that associated with the name of Hermite – Hadamard inequality which provides a lower and an upper estimations for the integral average of any convex functions defined on a compact interval, involving the midpoint and the endpoints of the domain. More precisely, if $f\colon [a,b] \to R$ is a convex function, then it is integrable in sense of Riemann and

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(t)dt \le \frac{f(a)+f(b)}{2}. \tag{1}$$

This above inequality (1) was firstly discovered by Hermite in 1881 in the journal Mathesis (see Mitrinović and Lacković [1]). But, this beautiful result was nowhere mentioned in the mathematical literature and was not widely known as Hermite's result (see Pečarić et al. [2]). For more recent results which generalize, improve, and extend this classical Hermite – Hadamard inequality, one can see [3 – 13] and references therein.

Meanwhile, fractional integrals and derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes. It draws a great application in nonlinear oscillations of earthquakes, many physical phenomena such as seepage flow in porous media and in fluid dynamic traffic model. For more recent development on fractional calculus, one can see the monographs [14 – 21].

Due to the widely application of Hermite – Hadamard-type inequalities and fractional integrals, many researchers turn to study Hermite – Hadamard-type inequalities involving fractional integrals not limited to integer integrals. Recently, more and more Hermite – Hadamard-type inequalities involving fractional integrals have been obtained for different classes of functions; see for convex functions [22] and nondecreasing functions [23], for m-convex functions and (s,m)-convex functions [24, 25], for functions satisfying s-e-condition [26] and the references therein.

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The notion of r-convexity undoubtedly plays a dominant role in almost all aspects of mathematical programming [27] and Hermite – Hadamard-type inequalities [28]. However, Hermite – Hadamard-type inequalities for r-convex functions involving fractional integrals have not been studied. Thus, the purpose of this paper is to establish Hermite – Hadamard-type inequalities for r-convex functions via Riemann – Liouville fractional integral by using two fundamental fractional integrals identity in Sarikaya et al. [22] and Wang et al. [25].

2. Preliminaries. In this section, we introduce notations, definitions, and preliminary facts.

In [28], Pearce et al. introduced the definition of r-convex function via power means.

Definition 2.1. *The function $f\colon [0,b^*] \to R$ is said to be r-convex, where $r \geq 0$ and $b^* > 0$, if for every $x, y \in [0,b^*]$ and $t \in [0,1]$, we have*

$$f(tx+(1-t)y) \leq \left[t(f(x))^r + (1-t)(f(y))^r\right]^{1/r}, \quad r \neq 0,$$

$$f(tx+(1-t)y) \leq \big(f(x)\big)^t \big(f(y)\big)^{(1-t)}, \quad r = 0.$$

***Remark* 2.1.** Clearly, a r-convex function must be a convex function, however, the inverse is false. For example, $f(x) = x^{1/2} (x > 0)$ is a 4-convex function, but it is not a convex function in anyway.

We also give some necessary definitions of fractional calculus which are used further in this paper. For more details, one can see Kilbas et al. [16].

Definition 2.2. *Let $f \in L[a,b]$. The symbols ${}_{RL}J^{\alpha}_{a^+}f$ and ${}_{RL}J^{\alpha}_{b^-}f$ denote the left-sided and right-sided Riemann – Liouville fractional integrals of the order $\alpha \in R^+$ are defined by*

$$({}_{RL}J^{\alpha}_{a^+}f)(x) = \frac{1}{\Gamma(\alpha)} \int\limits_a^x (x-t)^{\alpha-1} f(t)dt, \quad 0 \leq a < x \leq b,$$

and

$$({}_{RL}J^{\alpha}_{b^-}f)(x) = \frac{1}{\Gamma(\alpha)} \int\limits_x^b (t-x)^{\alpha-1} f(t)dt, \quad 0 \leq a \leq x < b,$$

respectively. Here $\Gamma(\cdot)$ is the Gamma function.

The following results will be used in the sequel.

Lemma 2.1 (see Lemma 4.1 [26]). *For $\alpha > 0$ and $k > 0$, we have*

$$I(\alpha,k) := \int\limits_0^1 t^{\alpha-1} k^t dt = k \sum_{i=1}^{\infty} (-1)^{i-1} \frac{(\ln\ k)^{i-1}}{(\alpha)_i} < +\infty,$$

where

$$(\alpha)_i = \alpha(\alpha+1)(\alpha+2)\ldots(\alpha+i-1).$$

Moreover, it holds

$$\left| I(\alpha,k) - k \sum_{i=1}^{m} \frac{(-\ln k)^{i-1}}{(\alpha)_i} \right| \leq \frac{|\ln k|}{\alpha\sqrt{2\pi(m-1)}} \left(\frac{|\ln k| e}{m-1}\right)^{m-1}.$$

Lemma 2.2. *For $\alpha > 0$ and $k > 0$, $z > 0$, we have*

$$J(\alpha, k) := \int_0^1 (1-t)^{\alpha-1} k^t dt = \sum_{i=1}^{\infty} \frac{(\ln\ k)^{i-1}}{(\alpha)_i} < +\infty,$$

$$H(\alpha, k, z) := \int_0^z t^{\alpha-1} k^t dt = z^\alpha k^z \sum_{i=1}^{\infty} \frac{(-z \ln\ k)^{i-1}}{(\alpha)_i} < +\infty.$$

Proof. By using Lemma 2.1, we obtain

$$J(\alpha, k) := \int_0^1 (1-t)^{\alpha-1} k^t dt = \int_0^1 t^{\alpha-1} k^{1-t} dt = kI(\alpha, k^{-1}) = \sum_{i=1}^{\infty} \frac{(\ln\ k)^{i-1}}{(\alpha)_i} < +\infty,$$

$$H(\alpha, k, z) := \int_0^z t^{\alpha-1} k^t dt = z^\alpha \int_0^1 T^{\alpha-1} (k^z)^T dT = z^\alpha I(\alpha, k^z) = z^\alpha k^z \sum_{i=1}^{\infty} \frac{(-z \ln\ k)^{i-1}}{(\alpha)_i} < +\infty,$$

which implies the desired results.

Lemma 2.3. *For $\alpha > 0$ and $k > 0, 1 \geq z > 0$, we have*

$$R(\alpha, k, z) := \int_0^z (1-t)^{\alpha-1} k^t dt = \sum_{i=1}^{\infty} \frac{(\ln\ k)^{i-1}}{(\alpha)_i} \left(1 - k^z (1-z)^{\alpha+i-1}\right).$$

Proof. By using Lemma 2.1 and Lemma 2.2, we obtain

$$R(\alpha, k, z) = \int_0^1 (1-t)^{\alpha-1} k^t dt - \int_z^1 (1-t)^{\alpha-1} k^t dt =$$

$$= J(\alpha, k) - \int_0^{1-z} t^{\alpha-1} k^{1-t} dt = J(\alpha, k) - kH(\alpha, k^{-1}, 1-z) =$$

$$= \sum_{i=1}^{\infty} \frac{(\ln\ k)^{i-1}}{(\alpha)_i} - k(1-z)^\alpha k^{z-1} \sum_{i=1}^{\infty} \frac{(1-z)^{i-1} (\ln\ k)^{i-1}}{(\alpha)_i}.$$

Lemma 2.3 is proved.

Finally, we recall the following elementary inequalities.

Lemma 2.4. *For $A \geq 0$, $B \geq 0$, it holds*

$$(A+B)^\theta \leq 2^{\theta-1}(A^\theta + B^\theta) \quad \textit{when} \quad \theta \geq 1,$$

$$(A+B)^\theta \leq (A^\theta + B^\theta) \quad \textit{when} \quad 0 < \theta \leq 1.$$

3. Hermite – Hadamard-type inequalities for differentiable r-convex functions. In this section, we apply the most fundamental fractional integrals identity given by Sarikaya et al. [22] to present some new Hermite – Hadamard-type inequalities for differentiable r-convex functions.

Lemma 3.1 (see Lemma 2 [22]). *Let $f\colon [a,b] \to R$ be a differentiable mapping on (a,b) with $a < b$. If $f' \in L[a,b]$, then the following equality for fractional integrals holds:*

$$\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[{}_{RL}J^{\alpha}_{a+}f(b) + {}_{RL}J^{\alpha}_{b-}f(a)] =$$

$$= \frac{b-a}{2}\int\limits_0^1 [(1-t)^{\alpha} - t^{\alpha}]f'(ta+(1-t)b)dt. \tag{2}$$

By using the above lemma, we can obtain the main results in this section.

Theorem 3.1. *Let $f\colon [0,b^*] \to R$ be a differentiable mapping with $b^* > 0$. If $|f'|$ is measurable and r-convex on $[a,b]$ for some fixed $0 \le r < \infty$, $0 \le a < b$, then the following inequality for fractional integrals holds*:

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[{}_{RL}J^{\alpha}_{a+}f(b) + {}_{RL}J^{\alpha}_{b-}f(a)]\right| \le K_r,$$

where

$$K_r := 2^{1/r-2}\frac{1-2^{-\alpha}}{1+\alpha}(b-a)\left(|f'(a)|+|f'(b)|\right) \quad \textit{for} \quad 0 < r \le 1,$$

$$K_r := 2^{-1/r}\frac{1-2^{-\alpha}}{1+\alpha}(b-a)\left(|f'(a)|+|f'(b)|\right) \quad \textit{for} \quad r > 1,$$

$$K_0 := \frac{(b-a)|f'(b)|}{2}\sum_{i=1}^{\infty}\left[\frac{(\ln k)^{2i-1}}{(\alpha+1)_{2i}}(1-k) + \frac{(\ln k)^{2i-2}}{(\alpha+1)_{2i-1}}\left(k+1-\frac{\sqrt{k}}{2^{\alpha+2i-3}}\right)\right],$$

and $k = \dfrac{|f'(a)|}{|f'(b)|}$.

Proof. *Case* 1: $0 < r \le 1$. By Definition 2.1, Lemmas 2.4 and 3.1, we have

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[{}_{RL}J^{\alpha}_{a+}f(b) + {}_{RL}J^{\alpha}_{b-}f(a)]\right| \le$$

$$\le \frac{b-a}{2}\int\limits_0^1 |(1-t)^{\alpha} - t^{\alpha}||f'(ta+(1-t)b)|dt \le$$

$$\le \frac{b-a}{2}\int\limits_0^1 |(1-t)^{\alpha} - t^{\alpha}|\left[t|f'(a)|^r + (1-t)|f'(b)|^r\right]^{1/r}dt \le$$

$$\le \frac{b-a}{2}\int\limits_0^1 |(1-t)^{\alpha} - t^{\alpha}|2^{1/r-1}\left[t^{1/r}|f'(a)| + (1-t)^{1/r}|f'(b)|\right]dt =$$

$$= 2^{1/r-2}(b-a)\left(|f'(a)|\int\limits_0^1 |(1-t)^{\alpha} - t^{\alpha}|t^{1/r}dt + \right.$$

$$+\left|f'(b)\right|\int\limits_0^1 |(1-t)^\alpha - t^\alpha|(1-t)^{1/r}dt\right) =$$

$$= 2^{1/r-2}(b-a)\left(|f'(a)|+|f'(b)|\right)\int\limits_0^1 |(1-t)^\alpha - t^\alpha|t^{1/r}dt =$$

$$= 2^{1/r-2}(b-a)\left(|f'(a)|+|f'(b)|\right)\int\limits_0^{1/2} ((1-t)^\alpha - t^\alpha)(t^{1/r}+(1-t)^{1/r})dt \leq$$

$$\leq 2^{1/r-2}(b-a)\left(|f'(a)|+|f'(b)|\right)\int\limits_0^{1/2} ((1-t)^\alpha - t^\alpha)dt =$$

$$= 2^{1/r-2}\frac{1-2^{-\alpha}}{1+\alpha}(b-a)\left(|f'(a)|+|f'(b)|\right).$$

Case 2: $1 < r$. Like in Case 1, we obtain

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[{}_{RL}J^\alpha_{a+}f(b) +_{RL} J^\alpha_{b-}f(a)]\right| \leq$$

$$\leq \frac{b-a}{2}\int\limits_0^1 |(1-t)^\alpha - t^\alpha|\left[t|f'(a)|^r + (1-t)|f'(b)|^r\right]^{1/r} dt \leq$$

$$\leq \frac{b-a}{2}\int\limits_0^1 |(1-t)^\alpha - t^\alpha|\left[t^{1/r}|f'(a)| + (1-t)^{1/r}|f'(b)|\right] dt =$$

$$= \frac{b-a}{2}\left(|f'(a)|\int\limits_0^1 |(1-t)^\alpha - t^\alpha|t^{1/r}dt+\right.$$

$$\left.+|f'(b)|\int\limits_0^1 |(1-t)^\alpha - t^\alpha|(1-t)^{1/r}dt\right) = \frac{b-a}{2}\left(|f'(a)|+|f'(b)|\right)\int\limits_0^1 |(1-t)^\alpha - t^\alpha|t^{1/r}dt =$$

$$= \frac{b-a}{2}\left(|f'(a)|+|f'(b)|\right)\int\limits_0^{1/2} ((1-t)^\alpha - t^\alpha)(t^{1/r}+(1-t)^{1/r})dt \leq$$

$$\leq 2^{-1/r}(b-a)\left(|f'(a)|+|f'(b)|\right)\int\limits_0^{1/2} ((1-t)^\alpha - t^\alpha)dt =$$

$$= 2^{-1/r}\frac{1-2^{-\alpha}}{1+\alpha}(b-a)\left(|f'(a)|+|f'(b)|\right).$$

Case 3: $r = 0$. By using Definition 2.1 and Lemmas 2.1, 2.3, 3.1, we have

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[{}_{RL}J^\alpha_{a^+}f(b) +_{RL} J^\alpha_{b^-}f(a)]\right| \leq$$

$$\leq \frac{b-a}{2}\int\limits_0^1 |(1-t)^\alpha - t^\alpha||f'(ta+(1-t)b)|dt \leq$$

$$\leq \frac{b-a}{2}\int\limits_0^1 |(1-t)^\alpha - t^\alpha||f'(a)|^t|f'(b)|^{1-t}dt = \frac{(b-a)|f'(b)|}{2}\int\limits_0^1 |(1-t)^\alpha - t^\alpha|k^t dt =$$

$$= \frac{(b-a)|f'(b)|}{2}\left(\int\limits_0^{1/2}(1-t)^\alpha k^t dt - \int\limits_0^{1/2} t^\alpha k^t dt + \int\limits_{1/2}^1 t^\alpha k^t dt - \int\limits_{1/2}^1 (1-t)^\alpha k^t dt\right) =$$

$$= \frac{(b-a)|f'(b)|}{2}\left(2\int\limits_0^{1/2}(1-t)^\alpha k^t dt - 2\int\limits_0^{1/2} t^\alpha k^t dt + \int\limits_0^1 t^\alpha k^t dt - \int\limits_0^1 (1-t)^\alpha k^t dt\right) =$$

$$= \frac{(b-a)|f'(b)|}{2}\sum_{i=1}^\infty\left[\frac{(\ln k)^{2i-1}}{(\alpha+1)_{2i}}(1-k) + \frac{(\ln k)^{2i-2}}{(\alpha+1)_{2i-1}}\left(k+1-\frac{\sqrt{k}}{2^{\alpha+2i-3}}\right)\right].$$

Theorem 3.1 is proved.

Theorem 3.2. *Let $f\colon [0,b^*] \to R$ be a differentiable mapping with $b^* > 0$. If $|f'|^q$, $q > 1$, is measurable and r-convex on $[a,b]$ for some fixed, $0 \leq r < \infty$, $0 \leq a < b$, then the following inequality for fractional integrals holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[{}_{RL}J^\alpha_{a^+}f(b) +_{RL} J^\alpha_{b^-}f(a)]\right| \leq K_r,$$

where

$$K_r := (b-a)2^{\frac{1-2r}{qr}}\left(\frac{1-2^{-p\alpha}}{p\alpha+1}\right)^{1/p}\left[\frac{r(|f'(a)|^q+|f'(b)|^q)}{r+1}\right]^{1/q} \quad \textit{for} \quad 0<r\leq 1,$$

$$K_r := \frac{b-a}{2^{1-1/p}}\left(\frac{1-2^{-p\alpha}}{p\alpha+1}\right)^{1/p}\left[\frac{r(|f'(a)|^q+|f'(b)|^q)}{r+1}\right]^{1/q} \quad \textit{for} \quad r>1,$$

$$K_0 := \frac{(b-a)}{2^{1-1/p}}\left(\frac{1-2^{-p\alpha}}{p\alpha+1}\right)^{1/p}\left(\frac{|f'(a)|^q-|f'(b)|^q}{q\ln|f'(a)| - q\ln|f'(b)|}\right)^{1/q} \quad \textit{when} \quad |f'(a)| \neq |f'(b)|,$$

$$K_0 := \frac{(b-a)}{2^{1-1/p}}\left(\frac{1-2^{-p\alpha}}{p\alpha+1}\right)^{1/p}|f'(a)| \quad \textit{when} \quad |f'(a)| = |f'(b)|,$$

and $\dfrac{1}{p}+\dfrac{1}{q} = 1$.

Proof. *Case* 1: $0 < r \leq 1$. By Definition 2.1, Lemmas 2.4, 3.1 and using Hölder inequality, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[{}_{RL}J^{\alpha}_{a^+}f(b)+_{RL}J^{\alpha}_{b^-}f(a)\right]\right|\leq$$

$$\leq\frac{b-a}{2}\int\limits_0^1|(1-t)^{\alpha}-t^{\alpha}||f'(ta+(1-t)b)|dt\leq$$

$$\leq\frac{b-a}{2}\left[\int\limits_0^1|(1-t)^{\alpha}-t^{\alpha}|^p dt\right]^{1/p}\left[\int\limits_0^1|f'(ta+(1-t)b)|^q dt\right]^{1/q}=$$

$$=\frac{b-a}{2}\left[2\int\limits_0^{1/2}((1-t)^{\alpha}-t^{\alpha})^p dt\right]^{1/p}\left[\int\limits_0^1|f'(ta+(1-t)b)|^q dt\right]^{1/q}\leq$$

$$\leq\frac{b-a}{2^{1-1/p}}\left[\int\limits_0^{1/2}((1-t)^{p\alpha}-t^{p\alpha})^p dt\right]^{1/p}\left[\int\limits_0^1|f'(ta+(1-t)b)|^q dt\right]^{1/q}=$$

$$=\frac{b-a}{2^{1-1/p}}\left(\frac{1-2^{-p\alpha}}{p\alpha+1}\right)^{1/p}\left[\int\limits_0^1|f'(ta+(1-t)b)|^q dt\right]^{1/q}\leq$$

$$\leq\frac{b-a}{2^{1-1/p}}\left(\frac{1-2^{-p\alpha}}{p\alpha+1}\right)^{1/p}\left[\int\limits_0^1\left[t|f'(a)|^{qr}+(1-t)|f'(b)|^{qr}\right]^{1/r}dt\right]^{1/q}\leq$$

$$\leq\frac{b-a}{2^{1-1/p}}\left(\frac{1-2^{-p\alpha}}{p\alpha+1}\right)^{1/p}\left[2^{1/r-1}\int\limits_0^1\left[t^{1/r}|f'(a)|^{q}+(1-t)^{1/r}|f'(b)|^{q}\right]dt\right]^{1/q}=$$

$$=(b-a)2^{\frac{1-2r}{qr}}\left(\frac{1-2^{-p\alpha}}{p\alpha+1}\right)^{1/p}\left[\frac{r(|f'(a)|^q+|f'(b)|^q)}{r+1}\right]^{1/q}.$$

Case 2: $1<r$. By Definition 2.1, Lemmas 2.4, 3.1 and using Hölder inequality, like above we obtain

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[{}_{RL}J^{\alpha}_{a^+}f(b)+_{RL}J^{\alpha}_{b^-}f(a)\right]\right|\leq$$

$$\leq\frac{b-a}{2^{1-1/p}}\left(\frac{1-2^{-p\alpha}}{p\alpha+1}\right)^{1/p}\left[\int\limits_0^1\left[t|f'(a)|^{qr}+(1-t)|f'(b)|^{qr}\right]^{1/r}dt\right]^{1/q}\leq$$

$$\leq\frac{b-a}{2^{1-1/p}}\left(\frac{1-2^{-p\alpha}}{p\alpha+1}\right)^{1/p}\left[\int\limits_0^1\left[t^{1/r}|f'(a)|^{q}+(1-t)^{1/r}|f'(b)|^{q}\right]dt\right]^{1/q}=$$

$$= \frac{b-a}{2^{1-1/p}} \left(\frac{1-2^{-p\alpha}}{p\alpha+1} \right)^{1/p} \left[\frac{r(|f'(a)|^q + |f'(b)|^q)}{r+1} \right]^{1/q}.$$

Case 3: $r = 0$. By Definition 2.1, Lemma 3.1 and using Hölder inequality, like above we have

$$\left| \frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [{}_{RL}J^\alpha_{a+} f(b) +_{RL} J^\alpha_{b-} f(a)] \right| \leq$$

$$\leq \frac{(b-a)}{2^{1-1/p}} \left(\frac{1-2^{-p\alpha}}{p\alpha+1} \right)^{1/p} \left[\int\limits_0^1 |f'(ta+(1-t)b)|^q dt \right]^{1/q} \leq$$

$$\leq \frac{(b-a)}{2^{1-1/p}} \left(\frac{1-2^{-p\alpha}}{p\alpha+1} \right)^{1/p} \left[\int\limits_0^1 |f'(a)|^{qt} |f'(b)|^{q(1-t)} dt \right]^{1/q}.$$

Since

$$\left(\frac{|f'(a)|^q - |f'(b)|^q}{q\ln|f'(a)| - q\ln|f'(b)|} \right)^{1/q} = \begin{cases} \left(\dfrac{|f'(a)|^q - |f'(b)|^q}{q\ln|f'(a)| - q\ln|f'(b)|} \right)^{1/q} & \text{when } |f'(a)| \neq |f'(b)|, \\ |f'(a)|^q & \text{when } |f'(a)| = |f'(b)|. \end{cases}$$

Theorem 3.2 is proved.

4. Hermite – Hadamard-type inequalities for twice-differentiable r-convex functions. In this section, we will first recall from our previous work an important fractional integrals identity including the second order derivative of a function. Then, we apply this new fractional integrals identity to present some new Hermite – Hadamard-type inequalities for twice-differentiable r-convex functions.

Lemma 4.1 (see Lemma 2.1, [25]). *Let $f\colon [a,b] \to R$ be twice-differentiable mapping on (a,b) with $a < b$. If $f'' \in L[a,b]$, then the following equality for fractional integrals holds*:

$$\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left[{}_{RL}J^\alpha_{a+} f(b) +_{RL} J^\alpha_{b-} f(a) \right] =$$

$$= \frac{(b-a)^2}{2} \int\limits_0^1 \frac{1-(1-t)^{\alpha+1} - t^{\alpha+1}}{\alpha+1} f''(ta+(1-t)b) dt. \tag{3}$$

Now we are ready to present the main results in this section.

Theorem 4.1. *Let $f\colon [0,b^*] \to R$ be a twice-differentiable mapping with $b^* > 0$. If $|f''|^q (q > 1)$ is measurable and r-convex on $[a,b]$ for some fixed $0 \leq r < \infty$, $0 \leq a < b$, then the following inequality for fractional integrals holds*:

$$\left| \frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [{}_{RL}J^\alpha_{a+} f(b) +_{RL} J^\alpha_{b-} f(a)] \right| \leq I_r,$$

where

$$I_r = \frac{2^{\frac{1-r-qr}{qr}} (b-a)^2}{\alpha+1} \left(1 - \frac{2}{p\alpha+p+1} \right)^{1/p} \times$$

$$\times \left(|f''(a)|^q + |f''(b)|^q\right)^{1/q} \left(\frac{r}{r+1}\right)^{1/q} \quad \textit{for} \quad 0 < r \le 1,$$

$$I_r = \frac{(b-a)^2}{2(\alpha+1)} \left(1 - \frac{2}{p\alpha+p+1}\right)^{1/p} \left(|f''(a)|^q + |f''(b)|^q\right)^{1/q} \left(\frac{r}{r+1}\right)^{1/q} \quad \textit{for} \quad 1 < r,$$

$$I_0 = \frac{(b-a)^2}{2(\alpha+1)} \left(1 - \frac{2}{p\alpha+p+1}\right)^{1/p} \times$$

$$\times \left(\frac{|f''(a)|^q - |f''(b)|^q}{q\ln|f''(a)| - q\ln|f''(b)|}\right)^{1/q} \quad \textit{when} \quad |f''(a)| \ne |f''(b)|,$$

$$I_0 = \frac{(b-a)^2|f''(a)|}{2(\alpha+1)} \left(1 - \frac{2}{p\alpha+p+1}\right)^{1/p} \quad \textit{when} \quad |f''(a)| = |f''(b)|,$$

and $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

Proof. *Case* 1: $0 < r \le 1$. By Definition 2.1, Lemmas 2.4, 4.1 and using Hölder inequality, we have

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[{}_{RL}J^\alpha_{a^+} f(b) + {}_{RL}J^\alpha_{b^-} f(a)]\right| \le$$

$$\le \frac{(b-a)^2}{2} \int_0^1 \frac{1-(1-t)^{\alpha+1} - t^{\alpha+1}}{\alpha+1} |f''(ta+(1-t)b)| dt \le$$

$$\le \frac{(b-a)^2}{2(\alpha+1)} \left(\int_0^1 (1-(1-t)^{\alpha+1} - t^{\alpha+1})^p dt\right)^{1/p} \left(\int_0^1 |f''(ta+(1-t)b)|^q dt\right)^{1/q} \le$$

$$\le \frac{(b-a)^2}{2(\alpha+1)} \left(\int_0^1 (1-(1-t)^{p(\alpha+1)} - t^{p(\alpha+1)}) dt\right)^{1/p} \times$$

$$\times \left(\int_0^1 [t|f''(a)|^{qr} + (1-t)|f''(b)|^{qr}]^{1/r} dt\right)^{1/q} \le$$

$$\le \frac{(b-a)^2}{2(\alpha+1)} \left(1 - \frac{2}{p\alpha+p+1}\right)^{1/p} \left(2^{1/r-1} \int_0^1 [t^{1/r}|f''(a)|^q + (1-t)^{1/r}|f''(b)|^q] dt\right)^{1/q} =$$

$$= \frac{2^{\frac{1-r-qr}{qr}}(b-a)^2}{\alpha+1} \left(1 - \frac{2}{p\alpha+p+1}\right)^{1/p} \left(|f''(a)|^q + |f''(b)|^q\right)^{1/q} \left(\frac{r}{r+1}\right)^{1/q}.$$

Case 2: $1 < r$. By Definition 2.1, Lemmas 2.4, 4.1 and using Hölder inequality, like above we obtain

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[{}_{RL}J^{\alpha}_{a^+}f(b)+{}_{RL}\,J^{\alpha}_{b^-}f(a)\right]\right|\leq$$

$$\leq\frac{(b-a)^2}{2(\alpha+1)}\left(1-\frac{2}{p\alpha+p+1}\right)^{1/p}\left(\int\limits_0^1\left[t|f''(a)|^{qr}+(1-t)|f''(b)|^{qr}\right]^{1/r}dt\right)^{1/q}\leq$$

$$\leq\frac{(b-a)^2}{2(\alpha+1)}\left(1-\frac{2}{p\alpha+p+1}\right)^{1/p}\left(\int\limits_0^1\left[t^{1/r}|f''(a)|^{q}+(1-t)^{1/r}|f''(b)|^{q}\right]dt\right)^{1/q}=$$

$$=\frac{(b-a)^2}{2(\alpha+1)}\left(1-\frac{2}{p\alpha+p+1}\right)^{1/p}\left(|f''(a)|^q+|f''(b)|^q\right)^{1/q}\left(\frac{r}{r+1}\right)^{1/q}.$$

Case 3: $r=0$. By Definition 2.1, Lemma 4.1 and using Hölder inequality, like above we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[{}_{RL}J^{\alpha}_{a^+}f(b)+{}_{RL}\,J^{\alpha}_{b^-}f(a)\right]\right|\leq$$

$$\leq\frac{(b-a)^2}{2(\alpha+1)}\left(1-\frac{2}{p\alpha+p+1}\right)^{1/p}\left(\int\limits_0^1|f''(ta+(1-t)b)|^q dt\right)^{1/q}\leq$$

$$\leq\frac{(b-a)^2}{2(\alpha+1)}\left(1-\frac{2}{p\alpha+p+1}\right)^{1/p}\left(\int\limits_0^1|f''(a)|^{qt}|f''(b)|^{q(1-t)}dt\right)^{1/q}.$$

Since

$$\int\limits_0^1|f''(a)|^{qt}|f''(b)|^{q(1-t)}dt=\begin{cases}\dfrac{|f''(a)|^q-|f''(b)|^q}{q\ln|f''(a)|-q\ln|f''(b)|} & \text{when}\quad |f''(a)|\neq|f''(b)|,\\ |f''(a)|^q & \text{when}\quad |f''(a)|=|f''(b)|.\end{cases}$$

Theorem 4.1 is proved.

***Corollary* 4.1.** *Suppose that the assumptions of Theorem* 4.1 *hold. Moreover,* $|f''(x)|\leq M$ *on* $[a,b]$. *Then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[{}_{RL}J^{\alpha}_{a^+}f(b)+{}_{RL}\,J^{\alpha}_{b^-}f(a)\right]\right|\leq I'_r,$$

where

$$I'_r=\frac{2^{\frac{1-qr}{qr}}M(b-a)^2}{(\alpha+1)}\left(1-\frac{2}{p\alpha+p+1}\right)^{1/p}\left(\frac{r}{r+1}\right)^{1/q}\quad \textit{for}\quad 0<r\leq 1,$$

$$I'_2=\frac{2^{\frac{1-q}{q}}M(b-a)^2}{(\alpha+1)}\left(1-\frac{2}{p\alpha+p+1}\right)^{1/p}\left(\frac{r}{r+1}\right)^{1/q}\quad \textit{for}\quad 1<r,$$

$$I'_0=\frac{M(b-a)^2}{2(\alpha+1)}\left(1-\frac{2}{p\alpha+p+1}\right)^{1/p},$$

and $\dfrac{1}{p}+\dfrac{1}{q}=1.$

Next, we give another Hermit – Hadamard-type inequality for powers in terms of the second derivatives.

Theorem 4.2. *Let* $f\colon [0,b^*]\to R$ *be a twice-differentiable mapping with* $b^*>0$. *If* $|f''|^q (q>1)$ *is measurable and r-convex on* $[a,b]$ *for some fixed* $0\le r<\infty$, $0\le a<b$, *then the following inequality for fractional integrals holds*:

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[{}_{RL}J^{\alpha}_{a+}f(b)+{}_{RL}J^{\alpha}_{b-}f(a)\right]\right|\le I_r,$$

where

$$I_r=\frac{2^{\frac{1-r-qr}{qr}}(b-a)^2}{(\alpha+1)}\Big(|f''(a)|^q+|f''(b)|^q\Big)^{1/q}\times$$

$$\times\left(\frac{r}{r+1}-\beta\left(\frac{r}{r+1},\alpha q+q+1\right)-\frac{r}{\alpha qr+qr+r+1}\right)^{1/q}\quad \textit{for}\quad 0<r\le 1,$$

$$I_r=\frac{(b-a)^2}{2(\alpha+1)}\Big(|f''(a)|^q+|f''(b)|^q\Big)^{1/q}\times$$

$$\times\left(\frac{r}{r+1}-\beta\left(\frac{r}{r+1},\alpha q+q+1\right)-\frac{r}{\alpha qr+qr+r+1}\right)^{1/q}\quad \textit{for}\quad 1<r,$$

$$I_0=\frac{(b-a)^2}{2(\alpha+1)}\left(\frac{|f''(a)|^q-|f''(b)|^q}{q\left(\ln|f''(a)|-\ln|f''(b)|\right)}-\right.$$

$$\left.-\sum_{i=1}^{\infty}\frac{(\ln|f''(a)|-\ln|f''(b)|)^{i-1}}{(\alpha q+q+1)_i}\left[q^{i-1}|f''(b)|^q+(-q)^{i-1}|f''(a)|^q\right]\right)^{1/q}$$

$$\textit{when}\quad |f''(a)|\ne|f''(b)|,$$

$$I_0=\frac{(b-a)^2|f''(a)|}{2(\alpha+1)}\left(1-\frac{2}{\alpha q+q+1}\right)^{1/q}\quad \textit{when}\quad |f''(a)|=|f''(b)|,$$

and $\dfrac{1}{p}+\dfrac{1}{q}=1.$

Proof. *Case* 1: $0<r\le 1$. By Definition 2.1, Lemmas 2.4, 4.1 and using Hölder inequality, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[{}_{RL}J^{\alpha}_{a+}f(b)+{}_{RL}J^{\alpha}_{b-}f(a)\right]\right|\le$$

$$\le\frac{(b-a)^2}{2}\int_0^1\frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}|f''(ta+(1-t)b)|dt\le$$

$$\le\frac{(b-a)^2}{2(\alpha+1)}\left(\int_0^1 1dt\right)^{1/p}\left(\int_0^1(1-(1-t)^{\alpha+1}-t^{\alpha+1})^q|f''(ta+(1-t)b)|^q dt\right)^{1/q}\le$$

$$\leq \frac{(b-a)^2}{2(\alpha+1)}\left(\int_0^1 (1-(1-t)^{(\alpha+1)q}-t^{(\alpha+1)q})[t|f''(a)|^{qr}+(1-t)|f''(b)|^{qr}]^{1/r}dt\right)^{1/q} \leq$$

$$\leq \frac{2^{\frac{1-r-qr}{qr}}(b-a)^2}{\alpha+1}\left(\int_0^1 (1-(1-t)^{(\alpha+1)q}-t^{(\alpha+1)q})\big[t^{1/r}|f''(a)|^{q}+(1-t)^{1/r}|f''(b)|^{q}\big]dt\right)^{1/q} =$$

$$= \frac{2^{\frac{1-r-qr}{qr}}(b-a)^2}{(\alpha+1)}\Big(|f''(a)|^q+|f''(b)|^q\Big)^{1/q}\times$$

$$\times\left(\frac{r}{r+1}-\beta\left(\frac{r}{r+1},\alpha q+q+1\right)-\frac{r}{\alpha qr+qr+r+1}\right)^{1/q}.$$

Case 2: $1<r$. By Definition 2.1, Lemmas 2.4, 4.1 and using Hölder inequality, like above we obtain

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[{}_{RL}J^\alpha_{a^+}f(b)+{}_{RL}J^\alpha_{b^-}f(a)]\right| \leq$$

$$\leq \frac{(b-a)^2}{2(\alpha+1)}\left(\int_0^1 (1-(1-t)^{(\alpha+1)q}-t^{(\alpha+1)q})[t|f''(a)|^{qr}+(1-t)|f''(b)|^{qr}]^{1/r}dt\right)^{1/q} \leq$$

$$\leq \frac{(b-a)^2}{2(\alpha+1)}\left(\int_0^1 (1-(1-t)^{(\alpha+1)q}-t^{(\alpha+1)q})[t^{1/r}|f''(a)|^{q}+(1-t)^{1/r}|f''(b)|^{q}]dt\right)^{1/q} =$$

$$= \frac{(b-a)^2}{2(\alpha+1)}\Big(|f''(a)|^q+|f''(b)|^q\Big)^{1/q}\left(\frac{r}{r+1}-\beta\left(\frac{r}{r+1},\alpha q+q+1\right)-\frac{r}{\alpha qr+qr+r+1}\right)^{1/q}.$$

Case 3: $r=0$. According to Lemmas 2.1, 2.2, 4.1, using Hölder inequality, like above we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[{}_{RL}J^\alpha_{a^+}f(b)+{}_{RL}J^\alpha_{b^-}f(a)]\right| \leq$$

$$\leq \frac{(b-a)^2}{2(\alpha+1)}\left(\int_0^1 (1-(1-t)^{\alpha+1}-t^{\alpha+1})^q|f''(ta+(1-t)b)|^q dt\right)^{1/q} \leq$$

$$\leq \frac{(b-a)^2}{2(\alpha+1)}\left(\int_0^1 (1-(1-t)^{(\alpha+1)q}-t^{(\alpha+1)q})|f''(a)|^{qt}|f''(b)|^{q(1-t)}dt\right)^{1/q}.$$

Since

$$\int_0^1 (1-(1-t)^{(\alpha+1)q}-t^{(\alpha+1)q})|f''(a)|^{qt}|f''(b)|^{q(1-t)}dt =$$

$$= \begin{cases} \dfrac{|f''(a)|^q - |f''(b)|^q}{q\,(\ln|f''(a)| - \ln|f''(b)|)} - \displaystyle\sum_{i=1}^{\infty} \dfrac{(\ln|f''(a)| - \ln|f''(b)|)^{i-1}}{(\alpha q + q + 1)_i} \times \\ \quad \times \Big[q^{i-1}|f''(b)|^q + (-q)^{i-1}|f''(a)|^q \Big] & \text{when} \quad |f''(a)| \neq |f''(b)|, \\ |f''(a)|^q & \text{when} \quad |f''(a)| = |f''(b)|. \end{cases}$$

Theorem 4.2 is proved.

Theorem 4.3. *Let $f\colon [0,b^*] \to R$ be a twice-differentiable mapping with $b^* > 0$. If $|f''|^q$, $q > 1$, is measurable and r-convex on $[a,b]$ for some fixed $0 \le r < \infty$, $0 \le a < b$, then the following inequality for fractional integrals holds*:

$$\left| \frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [{}_{RL}J^\alpha_{a+} f(b) + {}_{RL}J^\alpha_{b-} f(a)] \right| \le I_r,$$

where

$$I_r = 2^{\frac{n-r-nqr}{qnr}} \frac{\alpha^{\frac{1}{p}+\frac{1}{qm}} (b-a)^2}{(\alpha+1)(\alpha+2)^{\frac{1}{p}+\frac{1}{qm}}} \Big(|f''(a)|^{qn} + |f''(b)|^{qn} \Big)^{\frac{1}{qn}} \times$$

$$\times \left(\frac{r}{n+r} - \beta\left(\frac{n+r}{r}, \alpha+2 \right) - \frac{r}{\alpha r + 2r + n} \right)^{\frac{1}{qn}} \quad \textit{for} \quad 0 < r \le n,$$

$$I_r = \frac{\alpha^{\frac{1}{p}+\frac{1}{qm}} (b-a)^2}{2(\alpha+1)(\alpha+2)^{\frac{1}{p}+\frac{1}{qm}}} \Big(|f''(a)|^{qn} + |f''(b)|^{qn} \Big)^{\frac{1}{qn}} \times$$

$$\times \left(\frac{r}{n+r} - \beta\left(\frac{n+r}{r}, \alpha+2 \right) - \frac{r}{\alpha r + 2r + n} \right)^{\frac{1}{qn}} \quad \textit{for} \quad n < r,$$

$$I_0 = \frac{\alpha^{\frac{1}{p}+\frac{1}{qm}} (b-a)^2}{2(\alpha+1)(\alpha+2)^{\frac{1}{p}+\frac{1}{qm}}} \left(\frac{|f''(a)|^{nq} - |f''(b)|^{nq}}{nq[\ln|f''(a)| - \ln|f''(b)|]} - \right.$$

$$\left. - \sum_{i=1}^{\infty} \frac{(nq)^{i-1} \Big(\ln|f''(a)| - \ln|f''(b)| \Big)^{i-1} \Big[|f''(b)|^{nq} + |f''(a)|^{nq}(-1)^{i-1} \Big]}{(\alpha+2)_i} \right)^{\frac{1}{qn}}$$

$$\textit{when} \quad |f''(a)| \neq |f''(b)|,$$

$$I_0 = \frac{\alpha^{\frac{1}{p}+\frac{1}{qm}+\frac{1}{qn}} (b-a)^2 |f''(a)|}{2(\alpha+1)(\alpha+2)^{\frac{1}{p}+\frac{1}{qm}+\frac{1}{qn}}} \quad \textit{when} \quad |f''(a)| = |f''(b)|,$$

and $\dfrac{1}{p} + \dfrac{1}{q} = 1$, $\dfrac{1}{m} + \dfrac{1}{n} = 1$, $n > 1$.

Proof. *Case* 1: $0 < r \le n$. By Lemmas 2.4, 4.1 and using Hölder inequality, we have

$$\left| \frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [{}_{RL}J^\alpha_{a+} f(b) + {}_{RL}J^\alpha_{b-} f(a)] \right| \le$$

$$\leq \frac{(b-a)^2}{2}\int\limits_0^1 \frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}|f''(ta+(1-t)b)|dt \leq$$

$$\leq \frac{(b-a)^2}{2(\alpha+1)}\left(\int\limits_0^1 (1-(1-t)^{\alpha+1}-t^{\alpha+1})dt\right)^{1/p}\times$$

$$\times\left(\int\limits_0^1 (1-(1-t)^{\alpha+1}-t^{\alpha+1})|f''(ta+(1-t)b)|^q dt\right)^{1/q} \leq$$

$$\leq \frac{\alpha^{1/p}(b-a)^2}{2(\alpha+1)(\alpha+2)^{1/p}}\left(\int\limits_0^1 (1-(1-t)^{(\alpha+1)}-t^{(\alpha+1)})[t|f''(a)|^{qr}+(1-t)|f''(b)|^{qr}]^{1/r}dt\right)^{1/q} \leq$$

$$\leq \frac{\alpha^{1/p}(b-a)^2}{2(\alpha+1)(\alpha+2)^{1/p}}\left(\int\limits_0^1 (1-(1-t)^{(\alpha+1)}-t^{(\alpha+1)})dt\right)^{\frac{1}{qm}}\times$$

$$\times\left(\int\limits_0^1 (1-(1-t)^{(\alpha+1)}-t^{(\alpha+1)})\big[t|f''(a)|^{qr}+(1-t)|f''(b)|^{qr}\big]^{\frac{n}{r}}dt\right)^{\frac{1}{qn}} \leq$$

$$\leq \frac{\alpha^{\frac{1}{p}+\frac{1}{qm}}(b-a)^2}{2(\alpha+1)(\alpha+2)^{\frac{1}{p}+\frac{1}{qm}}}\times$$

$$\times\left(2^{\frac{n}{r}-1}\int\limits_0^1 (1-(1-t)^{(\alpha+1)}-t^{(\alpha+1)})\big[t^{n/r}|f''(a)|^{qn}+(1-t)^{n/r}|f''(b)|^{qn}\big]dt\right)^{\frac{1}{qn}} =$$

$$= 2^{\frac{n-r-nqr}{qnr}}\frac{\alpha^{\frac{1}{p}+\frac{1}{qm}}(b-a)^2}{(\alpha+1)(\alpha+2)^{\frac{1}{p}+\frac{1}{qm}}}\left(|f''(a)|^{qn}+|f''(b)|^{qn}\right)^{\frac{1}{qn}}\times$$

$$\times\left(\frac{r}{n+r}-\beta\left(\frac{n+r}{r},\alpha+2\right)-\frac{r}{\alpha r+2r+n}\right)^{\frac{1}{qn}}.$$

Case 2: For $1<n<r$, like in Case 1, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\big[{}_{RL}J^\alpha_{a^+}f(b)+{}_{RL}J^\alpha_{b^-}f(a)\big]\right| \leq \frac{\alpha^{\frac{1}{p}+\frac{1}{qm}}(b-a)^2}{2(\alpha+1)(\alpha+2)^{\frac{1}{p}+\frac{1}{qm}}}\times$$

$$\times\left(\int\limits_0^1 (1-(1-t)^{(\alpha+1)}-t^{(\alpha+1)})\big[t|f''(a)|^{qr}+(1-t)|f''(b)|^{qr}\big]^{\frac{n}{r}}dt\right)^{\frac{1}{qn}} \leq$$

$$\leq \frac{\alpha^{\frac{1}{p}+\frac{1}{qm}}(b-a)^2}{2(\alpha+1)(\alpha+2)^{\frac{1}{p}+\frac{1}{qm}}}\times$$

$$\times\left(\int\limits_0^1 (1-(1-t)^{(\alpha+1)}-t^{(\alpha+1)})\big[t^{n/r}|f''(a)|^{qn}+(1-t)^{n/r}|f''(b)|^{qn}\big]dt\right)^{\frac{1}{qn}} \leq$$

$$\leq \frac{\alpha^{\frac{1}{p}+\frac{1}{qm}}(b-a)^2}{2(\alpha+1)(\alpha+2)^{\frac{1}{p}+\frac{1}{qm}}}\times$$

$$\times\left(\int\limits_0^1 (1-(1-t)^{(\alpha+1)}-t^{(\alpha+1)})[t^{\frac{n}{r}}|f''(a)|^{qn}+(1-t)^{\frac{n}{r}}|f''(b)|^{qn}]dt\right)^{\frac{1}{qn}} \leq$$

$$\leq \frac{\alpha^{\frac{1}{p}+\frac{1}{qm}}(b-a)^2}{2(\alpha+1)(\alpha+2)^{\frac{1}{p}+\frac{1}{qm}}}\left(|f''(a)|^{qn}+|f''(b)|^{qn}\right)^{\frac{1}{qn}}\times$$

$$\times\left(\frac{r}{n+r}-\beta\left(\frac{n+r}{r},\alpha+2\right)-\frac{r}{\alpha r+2r+n}\right)^{\frac{1}{qn}}.$$

Case 3: $r=0$. Denote

$$k=\left(\frac{|f''(a)|}{|f''(a)|}\right)^{nq}.$$

If $|f''(a)|\neq|f''(b)|$, we obtain

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[{}_{RL}J^{\alpha}_{a+}f(b)+{}_{RL}\,J^{\alpha}_{b-}f(a)]\right| \leq$$

$$\leq \frac{(b-a)^2}{2}\int\limits_0^1 \frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}|f''(ta+(1-t)b)|dt \leq$$

$$\leq \frac{(b-a)^2}{2(\alpha+1)}\left(\int\limits_0^1 (1-(1-t)^{\alpha+1}-t^{\alpha+1})dt\right)^{1/p}\times$$

$$\times\left(\int\limits_0^1 (1-(1-t)^{\alpha+1}-t^{\alpha+1})|f''(ta+(1-t)b)|^q dt\right)^{1/q} \leq$$

$$\leq \frac{\alpha^{1/p}(b-a)^2}{2(\alpha+1)(\alpha+2)^{1/p}}\left(\int\limits_0^1 (1-(1-t)^{(\alpha+1)}-t^{(\alpha+1)})[|f''(a)|^{qt}|f''(b)|^{q(1-t)}]dt\right)^{1/q} \leq$$

$$\leq \frac{\alpha^{1/p}(b-a)^2}{2(\alpha+1)(\alpha+2)^{1/p}} \left(\int_0^1 (1-(1-t)^{(\alpha+1)} - t^{(\alpha+1)})dt \right)^{\frac{1}{qm}} \times$$

$$\times \left(\int_0^1 (1-(1-t)^{(\alpha+1)} - t^{(\alpha+1)})[|f''(a)|^{qt}|f''(b)|^{q(1-t)}]^n dt \right)^{\frac{1}{qn}} =$$

$$= \frac{\alpha^{\frac{1}{p}+\frac{1}{qm}}(b-a)^2}{2(\alpha+1)(\alpha+2)^{\frac{1}{p}+\frac{1}{qm}}} |f''(b)| \left(\int_0^1 (1-(1-t)^{(\alpha+1)} - t^{(\alpha+1)})k^t dt \right)^{\frac{1}{qn}} =$$

$$= \frac{\alpha^{\frac{1}{p}+\frac{1}{qm}}(b-a)^2}{2(\alpha+1)(\alpha+2)^{\frac{1}{p}+\frac{1}{qm}}} |f''(b)| \left(\frac{k-1}{\ln k} - \sum_{i=1}^{\infty} \frac{(\ln k)^{i-1}}{(\alpha+2)_i} - k \sum_{i=1}^{\infty} (-1)^{i-1} \frac{(\ln k)^{i-1}}{(\alpha+2)_i} \right)^{\frac{1}{qn}} =$$

$$= \frac{\alpha^{\frac{1}{p}+\frac{1}{qm}}(b-a)^2}{2(\alpha+1)(\alpha+2)^{\frac{1}{p}+\frac{1}{qm}}} \left[\frac{|f''(a)|^{nq} - |f''(b)|^{nq}}{nq\big[\ln|f''(a)| - \ln|f''(b)|\big]} - \right.$$

$$\left. - \sum_{i=1}^{\infty} \frac{(nq)^{i-1}\Big(\ln|f''(a)| - \ln|f''(b)|\Big)^{i-1}\Big(|f''(b)|^{nq} + |f''(a)|^{nq}(-1)^{i-1}\Big)}{(\alpha+2)_i} \right]^{\frac{1}{qn}}.$$

If $|f''(a)| = |f''(b)|$, we have

$$\left| \frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [{}_{RL}J^\alpha_{a^+} f(b) + {}_{RL}J^\alpha_{b^-} f(a)] \right| \leq$$

$$\leq \frac{\alpha^{\frac{1}{p}+\frac{1}{qm}}(b-a)^2}{2(\alpha+1)(\alpha+2)^{\frac{1}{p}+\frac{1}{qm}}} \left(\int_0^1 (1-(1-t)^{(\alpha+1)} - t^{(\alpha+1)})|f''(a)|^{qn} dt \right)^{\frac{1}{qn}} =$$

$$= \frac{\alpha^{\frac{1}{p}+\frac{1}{qm}+\frac{1}{qn}}(b-a)^2|f''(a)|}{2(\alpha+1)(\alpha+2)^{\frac{1}{p}+\frac{1}{qm}+\frac{1}{qn}}}.$$

Theorem 4.3 is proved.

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