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ON THE SEPARATION PROBLEM FOR A FAMILY OF BOREL AND BAIRE G -POWERS OF SHIFT MEASURES ON \mathbb{R}

ПРО ЗАДАЧУ ВІДОКРЕМЛЕННЯ ДЛЯ СІМ'Ї БОРЕЛІВСЬКИХ ТА БЕРІВСЬКИХ G -СТЕПЕНІВ МІР ЗСУВУ НА \mathbb{R}

The separation problem for a family of Borel and Baire G -powers of shift measures on \mathbb{R} is studied for an arbitrary infinite additive group G using the technique developed in [Kuipers L., Niederreiter H. Uniform distribution of sequences. – New York etc.: John Wiley & Sons, 1974], [Shiryayev A. N. Probability (in Russian). – Moscow: Nauka, 1980] and [Pantsulaia G. R. Invariant and quasiinvariant measures in infinite-dimensional topological vector spaces. – New York: Nova Sci. Publ., Inc., 2007]. It is proved that $T_n: \mathbb{R}^n \rightarrow \mathbb{R}$, $n \in \mathbb{N}$, defined by

$$T_n(x_1, \dots, x_n) = -F^{-1}(n^{-1} \#(\{x_1, \dots, x_n\} \cap (-\infty; 0]))$$

for $(x_1, \dots, x_n) \in \mathbb{R}^n$ is a consistent estimator of a useful signal θ in the one-dimensional linear stochastic model

$$\xi_k = \theta + \Delta_k, \quad k \in \mathbb{N},$$

where $\#(\cdot)$ is a counting measure, Δ_k , $k \in \mathbb{N}$, is a sequence of independent identically distributed random variables on \mathbb{R} with a strictly increasing continuous distribution function F , and the expectation of Δ_1 does not exist.

Вивчається задача відокремлення для сім'ї борелівських та берівських G -степенів мір зсуву на \mathbb{R} для довільної нескінченної адитивної групи G із використанням підходу, розвинутого в роботах [Kuipers L., Niederreiter H. Uniform distribution of sequences. – New York etc.: John Wiley & Sons, 1974], [Ширяев А. Н. Вероятность. – М.: Наука, 1980] та [Pantsulaia G. R. Invariant and quasiinvariant measures in infinite-dimensional topological vector spaces. – New York: Nova Sci. Publ., Inc., 2007]. Доведено, що $T_n: \mathbb{R}^n \rightarrow \mathbb{R}$, $n \in \mathbb{N}$, означений формулою

$$T_n(x_1, \dots, x_n) = -F^{-1}(n^{-1} \#(\{x_1, \dots, x_n\} \cap (-\infty; 0]))$$

при $(x_1, \dots, x_n) \in \mathbb{R}^n$, є консистентною оцінкою корисного сигналу θ в одновимірній лінійній стохастичній моделі

$$\xi_k = \theta + \Delta_k, \quad k \in \mathbb{N},$$

де $\#(\cdot)$ – зліченна міра, Δ_k , $k \in \mathbb{N}$, – послідовність незалежних однаково розподілених випадкових величин на \mathbb{R} із строго зростаючою неперервною функцією розподілу F , а сподівання величини Δ_1 не існує.

1. Introduction. In the general theory of statistical decisions there often arises a problem of transition from a weakly separated family of probability measures to the corresponding strongly separated family. In 1981, A. Skorokhod [1] proved that if the Continuum Hypothesis is true, then an arbitrary weakly separated family of probability measures, whose cardinality is not greater than the cardinality of the continuum, is strongly separable. The validity of the inverse relation was established in [3] (see also [4]). In particular, it was shown there that if an arbitrary weakly separated family of probability measures whose cardinality is less than or equal to the cardinality of the continuum is strongly separated, then the Continuum Hypothesis is true. Applying Martin's axiom, in 1984 Z. Zerakidze [7] proved that an arbitrary weakly separated family of Borel probability measures defined in a separable completely metrizable space (i.e., Polish space) is strongly separated if its cardinality is not greater than the cardinality of the continuum. In [3], this result is extended to all complete metric spaces whose topological weights are not measurable in a wider sense.

Below we give some definitions from the theory of stochastic processes.

Definition 1.1. Let (Ω, \mathcal{F}, p) be a probability space and G be an infinite additive group. A stochastic process $X = (X_g)_{g \in G}: \Omega \rightarrow \mathbb{R}^G$ is called a G -process on (Ω, \mathcal{F}, p) if a joint probability distribution

$$F_{(g_1, \dots, g_n)}^{(X)}(x_1, \dots, x_n) = p(\{\omega: X_{g_1}(\omega) \leq x_1, \dots, X_{g_n}(\omega) \leq x_n\})$$

with $(g_1, \dots, g_n) \in G^n$ and $(x_1, \dots, x_n) \in \mathbb{R}^n$, does not change when shifted in a group, i.e., the following equality

$$F_{(g_1, \dots, g_n)}^{(X)}(x_1, \dots, x_n) = F_{(g_1+h, \dots, g_n+h)}^{(X)}(x_1, \dots, x_n)$$

holds for an arbitrary $h \in G$.

Remark 1.1. For $G = \mathbb{R}^n$, $n > 1$, a G -process coincides with a homogenous field. If $G = \mathbb{R}$, then a G -process coincides with a stationary process.

Definition 1.2. Let p be a Borel probability measure on \mathbb{R} and $\theta \in \mathbb{R}$. Then a probability measure p_θ defined by

$$(\forall X)(X \in \mathcal{B}(\mathbb{R}) \rightarrow p_\theta(X) = p(X + \theta))$$

is called a θ -shift measure of p .

Definition 1.3. Let p be a Borel probability measure on \mathbb{R} and G be an infinite additive group. Suppose that $p_g = p$ for $g \in G$. Then the product measure $\prod_{g \in G} p_g$ is called the Baire G -power of p and is denoted by p^G . If p^G admits a Borel extension, then that extension is called the Borel G -power of p .

Remark 1.2. Note that the notions of Baire G -power and Borel G -power of p coincide when the group G is countable.

Example 1.1. Let p be a Borel probability measure on \mathbb{R} and G be an infinite additive group. Then the family of all coordinate projections $(Pr_g)_{g \in G}$ defined on a probability space $(\mathbb{R}^G, Ba(\mathbb{R}^G), p_\theta^G)$ is a G -process for every $\theta \in \mathbb{R}$, where p_θ^G is the G -power of a shift measure p_θ on \mathbb{R} .

The main aim of the present paper is to consider the separation problem for a family of G -powers of shift measures on \mathbb{R} , where G is an arbitrary additive group. Note that such measures generate G -processes on \mathbb{R}^G .

The attention is focused on two essentially different examples of strongly separated families of Borel and Baire G -powers of shift measures in \mathbb{R} for an arbitrary infinite additive group G .

Our tools of investigation are the techniques developed in [2, 5, 6].

The paper is organized as follows. Some auxiliary notions and facts from the theory of uniformly distributed sequences and the probability theory are considered in Section 2. Section 3 contains the formulations and proofs of the obtained results. In Section 4, the existence of some consistent estimators of a useful signal in the one-dimensional linear stochastic model is proved and some examples of the corresponding simulations with numerical computations are considered.

2. Some auxiliary notions and facts. We start this section with some standard notions and definitions from the probability theory.

Let I be an arbitrary nonempty set of parameters. Denote by (\mathbb{R}^I, τ) the vector space of all real-valued functions on I equipped with the Tychonoff topology τ . We denote by $B(\mathbb{R}^I)$ a σ -algebra of all Borel subsets of the space \mathbb{R}^I generated by the Tychonoff topology τ .

Let $(Pr_i)_{i \in I}$ be the family of all coordinate projections defined by

$$(\forall i)(\forall (x_j)_{j \in I})(i \in I \ \& \ (x_j)_{j \in I} \in \mathbb{R}^I \rightarrow Pr_i((x_j)_{j \in I}) = x_i).$$

A minimal σ -algebra of subsets of \mathbb{R}^I generated by the class of subsets

$$\{Pr_i^{-1}(X) : i \in I \ \& \ X \in B(\mathbb{R})\}$$

is denoted by $Ba(\mathbb{R}^I)$ and is called a Baire σ -algebra of subsets of \mathbb{R}^I .

Remark 2.1. Note that $Ba(\mathbb{R}^I) = B(\mathbb{R}^I)$ for $\text{card}(I) \leq \aleph_0$, where \aleph_0 denotes the cardinality of the set of all natural numbers. If $\text{card}(I) > \aleph_0$, then

$$Ba(\mathbb{R}^I) \subset B(\mathbb{R}^I) \ \& \ B(\mathbb{R}^I) \setminus Ba(\mathbb{R}^I) \neq \emptyset.$$

As usual, a measure defined on $B(\mathbb{R}^I)$ is called a Borel measure. Analogously, a measure defined on $Ba(\mathbb{R}^I)$ is called a Baire measure.

Definition 2.1. Let μ_1 be a Baire measure defined on \mathbb{R}^I . A Borel measure μ_2 defined on \mathbb{R}^I is called a Borel extension of μ_1 if

$$(\forall X)(X \in Ba(\mathbb{R}^I) \rightarrow \mu_2(X) = \mu_1(X)).$$

Example 2.1. Let I be an arbitrary nonempty parametric set and p_i be a Borel probability measure on $\mathbb{R}_i := \mathbb{R}$ for all $i \in I$. If $\text{card}(I) > \aleph_0$, then the probability product-measure $\prod_{i \in I} p_i$ is defined on the σ -algebra

$$\prod_{i \in I} B(\mathbb{R}_i) = Ba(\mathbb{R}^I).$$

Accordingly, this measure is an example of a Baire probability measure which is not defined on $B(\mathbb{R}^I)$.

Lemma 2.1 ([5, p. 67], Lemma 4.4). Let (E_1, τ_1) and (E_2, τ_2) be two topological spaces. Denote by $B(E_1)$ and $B(E_2)$ (respectively, by $B(E_1 \times E_2)$) the class of all Borel subsets generated by the topologies τ_1 and τ_2 (respectively, by $\tau_1 \times \tau_2$). If at least one of these topological spaces has a countable base, then the equality

$$B(E_1) \times B(E_2) = B(E_1 \times E_2)$$

holds.

Lemma 2.2 ([5, p. 70], Remark 4.5). Let $(p_i)_{i \in I}$ be a family of Borel probability measures on \mathbb{R} with strictly increasing continuous distribution functions. Then there exists only one Borel extension \mathcal{P}_I of the Baire product-measure $\prod_{i \in I} p_i$.

Corollary 2.1 ([5, p. 75], Corollary 4.2). The product of an arbitrary family $(p_i)_{i \in I}$ of nontrivial Gaussian Borel probability measures defined on \mathbb{R}^I has only one Borel extension.

Corollary 2.2 ([5, p. 75], Corollary 4.3). In the case of the space \mathbb{R}^I , for $\text{Card}(I) > \aleph_0$, Lemma 2.2 is a generalization of Anderson well-known theorem which gives only the construction of a Baire product-measure.

Definition 2.2 [2]. A sequence s_1, s_2, s_3, \dots of real numbers from the interval (a, b) is said to be equidistributed or uniformly distributed on the interval (a, b) if for any subinterval $[c, d]$ of (a, b) we have

$$\lim_{n \rightarrow \infty} \frac{\#\{s_1, s_2, s_3, \dots, s_n\} \cap [c, d]}{n} = \frac{d - c}{b - a},$$

where $\#$ is a counter measure.

Now let X be a compact Polish space and μ be a probability Borel measure on X . Let $\mathcal{R}(X)$ be a space of all bounded continuous functions on X .

Definition 2.3. A sequence s_1, s_2, s_3, \dots of elements of X is said to be μ -equidistributed or μ -uniformly distributed on X if for every $f \in \mathcal{R}(X)$ we have

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N f(s_n) = \int_X f d\mu.$$

Lemma 2.3 [2, p. 199–201]. Let $f \in \mathcal{R}(X)$, $\mu_\infty := \mu^\infty$ and S be a set of all μ -equidistributed sequences in X^∞ . Then

- (i) $\mu_\infty(S) = 1$;
- (ii) S is a set of the first category;
- (iii) S is everywhere dense in the Tychonoff topology.

Corollary 2.3. Let ℓ_1 be a Lebesgue measure on $(0, 1)$. Let D be a set of all ℓ_1 -equidistributed sequences in $(0, 1)^\infty$. Then

- (i) $\ell_1^\infty(D) = 1$;
- (ii) D is a set of the first category;
- (iii) D is everywhere dense in the Tychonoff topology.

Definition 2.4. Let μ be a probability Borel measure on \mathbb{R} such that its distribution function F is continuous. A sequence s_1, s_2, s_3, \dots of elements of \mathbb{R} is said to be μ -equidistributed or μ -uniformly distributed on \mathbb{R} if for every interval $[a, b]$ ($-\infty \leq a < b \leq +\infty$) we have

$$\lim_{n \rightarrow \infty} \frac{\#\{[a, b] \cap \{x_1, \dots, x_n\}\}}{n} = F(b) - F(a).$$

Lemma 2.4. Let $(x_k)_{k \in \mathbb{N}}$ be an ℓ_1 -equidistributed sequence in $(0, 1)$ and F be a strictly increasing continuous distribution function on \mathbb{R} . Let p be a Borel probability measure on \mathbb{R} defined by F . Then $(F^{-1}(x_k))_{k \in \mathbb{N}}$ is a p -equidistributed sequence on \mathbb{R} .

Proof. We have

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{\#\{[a, b] \cap \{F^{-1}(x_1), \dots, F^{-1}(x_n)\}\}}{n} = \\ & = \lim_{n \rightarrow \infty} \frac{\#\{[F(a), F(b)] \cap \{x_1, \dots, x_n\}\}}{n} = F(b) - F(a). \end{aligned}$$

Corollary 2.4. Let F be a strictly increasing continuous distribution function on \mathbb{R} and p be a Borel probability measure on \mathbb{R} defined by F . Then for a set of all p -equidistributed sequences $D_F \subset \mathbb{R}^\infty$ we have

- (i) $D_F = \{(F^{-1}(x_k))_{k \in \mathbb{N}} : (x_k)_{k \in \mathbb{N}} \in D\}$, where D is from Corollary 2.3;

- (ii) $p^\infty(D_F) = 1$;
- (iii) D_F is a set of the first category;
- (iv) D_F is everywhere dense in the Tychonoff topology.

Let $(\mu_\theta)_{\theta \in \Theta}$ be a sequence of probability measures defined on a measurable space (E, S) . For $\theta \in \Theta$, we denote by $\bar{\mu}_\theta$ the completion of the measure μ_θ and by $\text{dom}(\bar{\mu}_\theta)$ the σ -algebra of all $\bar{\mu}_\theta$ -measurable subsets of E .

Definition 2.5. We say that the family $(\mu_\theta)_{\theta \in \Theta}$ is strongly separated if there exists a family $(Z_\theta)_{\theta \in \Theta}$ of elements of the σ -algebra $\cap_{\theta \in \Theta} \text{dom}(\bar{\mu}_\theta)$ such that

- (i) $\bar{\mu}_\theta(Z_\theta) = 1$ for $\theta \in \Theta$;
- (ii) $Z_{\theta_1} \cap Z_{\theta_2} = \emptyset$ for all different parameters θ_1 and θ_2 from Θ ;
- (iii) $\cup_{\theta \in \Theta} Z_\theta = E$.

Definition 2.6. Let $(\mu_\theta)_{\theta \in \Theta}$ be a family of pairwise singular probability measures on a measurable space (E, S) , where Θ is equipped with a σ -algebra $L(\Theta)$ that contains all singletons of Θ and $S_1 := \cap_{\theta \in \Theta} \text{dom}(\bar{\mu}_\theta)$. We say that a measurable mapping $\tilde{\theta}: E \rightarrow \Theta$ is a consistent estimator of the parameter θ if

$$(\forall \theta) \left(\theta \in \Theta \rightarrow \bar{\mu}_\theta(\{x: \tilde{\theta}(x) = \theta\}) = 1 \right).$$

Lemma 2.5. Let $(\mu_\theta)_{\theta \in \Theta}$ be a family of pairwise singular probability measures on a measurable space (E, S) , where Θ is equipped with σ -algebra $L(\Theta)$ that contains all singletons of Θ and $S_1 := \cap_{\theta \in \Theta} \text{dom}(\bar{\mu}_\theta)$. Then the following sentences are equivalent:

- (a) there is a consistent estimator $\tilde{\theta}: E \rightarrow \Theta$ of the parameter θ ;
- (b) the family of measures $(\mu_\theta)_{\theta \in \Theta}$ is strongly separated.

Proof. Let us show the validity of the implication (a) \rightarrow (b). The existence of a consistent estimator $\tilde{\theta}: E \rightarrow \Theta$ of the parameter θ implies that

$$(\forall \theta) \left(\theta \in \Theta \rightarrow \bar{\mu}_\theta(\{x: \tilde{\theta}(x) = \theta\}) = 1 \right).$$

Setting $Z_\theta = \{x: \tilde{\theta}(x) = \theta\}$ for $\theta \in \Theta$, we get:

- (i) $\bar{\mu}_\theta(Z_\theta) = \bar{\mu}_\theta(\{x: \tilde{\theta}(x) = \theta\}) = 1$ for $\theta \in \Theta$;
- (ii) $Z_{\theta_1} \cap Z_{\theta_2} = \emptyset$ for all different parameters θ_1 and θ_2 from Θ because

$$\{x: \tilde{\theta}(x) = \theta_1\} \cap \{x: \tilde{\theta}(x) = \theta_2\} = \emptyset;$$

- (iii) $\cup_{\theta \in \Theta} Z_\theta = \{x: \tilde{\theta}(x) \in \Theta\} = E$.

Let us show the validity of the implication (b) \rightarrow (a).

Since the family $(\mu_\theta)_{\theta \in \Theta}$ is strongly separated there exists a family $(Z_\theta)_{\theta \in \Theta}$ of elements of σ -algebra $S_1 := \cap_{\theta \in \Theta} \text{dom}(\bar{\mu}_\theta)$ such that:

- (i) $\bar{\mu}_\theta(Z_\theta) = 1$ for $\theta \in \Theta$;
- (ii) $Z_{\theta_1} \cap Z_{\theta_2} = \emptyset$ for all different parameters θ_1 and θ_2 from Θ ;
- (iii) $\cup_{\theta \in \Theta} Z_\theta = E$.

For $x \in E$, we put $\tilde{\theta}(x) = \theta$, where θ is a unique parameter from the set Θ for which $x \in Z_\theta$. The existence of such a unique parameter θ can be proved by using conditions (ii), (iii).

Now let $Y \in L(\Theta)$. Then $\{x: \tilde{\theta}(x) \in Y\} = \cup_{\theta \in Y} Z_\theta$. We have to show that $\{x: \tilde{\theta}(x) \in Y\} \in \text{dom}(\bar{\mu}_{\theta_0})$ for each $\theta_0 \in \Theta$.

If $\theta_0 \in Y$, then

$$\{x: \tilde{\theta}(x) \in Y\} = \cup_{\theta \in Y} Z_\theta = Z_{\theta_0} \cup \cup_{\theta \in Y \setminus \theta_0} Z_\theta.$$

On the one hand, from the validity of the condition (b) it follows that

$$Z_{\theta_0} \in S_1 = \bigcap_{\theta \in \Theta} \text{dom}(\bar{\mu}_\theta) \subseteq \text{dom}(\bar{\mu}_{\theta_0}).$$

On the other hand, the validity of the condition

$$\cup_{\theta \in Y \setminus \theta_0} Z_\theta \subseteq (E \setminus Z_{\theta_0})$$

implies that

$$\bar{\mu}_{\theta_0}(\cup_{\theta \in Y \setminus \theta_0} Z_\theta) = 0.$$

The latter equality yields that

$$\cup_{\theta \in Y \setminus \theta_0} Z_\theta \in \text{dom}(\bar{\mu}_{\theta_0}).$$

Since $\text{dom}(\bar{\mu}_{\theta_0})$ is a σ -algebra, we deduce that

$$\{x: \tilde{\theta}(x) \in Y\} = Z_{\theta_0} \cup \cup_{\theta \in Y \setminus \theta_0} Z_\theta \in \text{dom}(\bar{\mu}_{\theta_0}).$$

If $\theta_0 \notin Y$, then

$$\{x: \tilde{\theta}(x) \in Y\} = \cup_{\theta \in Y} Z_\theta \subseteq (E \setminus Z_{\theta_0})$$

and we claim that $\bar{\mu}_{\theta_0}(\{x: \tilde{\theta}(x) \in Y\}) = 0$. The latter relation implies that

$$\{x: \tilde{\theta}(x) \in Y\} \in \text{dom}(\bar{\mu}_{\theta_0}).$$

Thus we have shown the validity of the condition

$$\{x: \tilde{\theta}(x) \in Y\} \in \text{dom}(\bar{\mu}_{\theta_0})$$

for an arbitrary $\theta_0 \in \Theta$. Hence

$$\{x: \tilde{\theta}(x) \in Y\} \in \bigcup_{\theta_0 \in \Theta} \text{dom}(\bar{\mu}_{\theta_0}) = S_1.$$

Since $L(\Theta)$ contains all singletons of Θ , we claim that

$$(\forall \theta) \left(\theta \in \Theta \rightarrow \bar{\mu}_\theta(\{x: \tilde{\theta}(x) = \theta\}) = \bar{\mu}_\theta(Z_\theta) = 1 \right).$$

Lemma 2.5 is proved.

Remark 2.2. Let E be a Polish space and $(\mu_\theta)_{\theta \in \Theta}$ be a family of Borel probability measures on E . Then the σ -algebra $S_1 = \bigcap_{\theta \in \Theta} \text{dom}(\bar{\mu}_\theta)$ contains a class of all universally measurable¹ subsets. Note that each universally measurable consistent estimator $\tilde{\theta}: E \rightarrow \Theta$ (if such an estimator exists) will be measurable also in the sense of Definition 2.6.

Definition 2.7. Let μ be a Borel measure on \mathbb{R} and $\theta \in \mathbb{R}$. Let I be a non-empty parameter set. A measure μ_θ^I defined by $\mu_\theta^I = \prod_{i \in I} \lambda_i$ with $\lambda_i = \mu_\theta$ for $i \in I$, where μ_θ denotes a θ -shift measure of μ (i.e., $\mu_\theta(X) = \mu(X + \theta)$ for $X \in \mathcal{B}(\mathbb{R})$), is called a Baire I -power of the θ -shift measure μ_θ on \mathbb{R} .

Definition 2.8. Let μ be a Borel measure on \mathbb{R} , $\theta \in \mathbb{R}$ and I be a non-empty parameter set. Assume that \mathcal{P}_I is a Borel extension of the Baire I -power of a θ -shift measure in \mathbb{R}^I . Then \mathcal{P}_I is called a Borel I -power of the θ -shift measure μ_θ on \mathbb{R} .

Lemma 2.6. Let F be a strictly increasing continuous distribution function on \mathbb{R} and let p be a Borel probability measure on \mathbb{R} defined by F . Then the family of Baire (equivalently, Borel) \mathbb{N} -powers $(p_\theta^{\mathbb{N}})_{\theta \in \mathbb{R}}$ of shift measures $(p_\theta)_{\theta \in \mathbb{R}}$, where \mathbb{N} denotes a set of all natural numbers, is strongly separated.

Proof. For $\theta \in \mathbb{R}$, we denote by D_θ the set of all p_θ -equidistributed sequences in $\mathbb{R}^{\mathbb{N}}$. Let us show that $D_{\theta_1} \cap D_{\theta_2} = \emptyset$ for $-\infty < \theta_1 < \theta_2 < +\infty$. For $(x_k)_{k \in \mathbb{N}} \in D_{\theta_1}$, we have

$$\lim_{n \rightarrow \infty} \frac{\#\left((-\infty, 0] \cap \{x_1, \dots, x_n\}\right)}{n} = F_{\theta_1}(0) = F(\theta_1).$$

Analogously, for all $(x_k)_{k \in \mathbb{N}} \in D_{\theta_2}$, we have

$$\lim_{n \rightarrow \infty} \frac{\#\left((-\infty, 0] \cap \{x_1, \dots, x_n\}\right)}{n} = F_{\theta_2}(0) = F(\theta_2).$$

Since F is a strictly increasing continuous distribution function on \mathbb{R} , we deduce that $F(\theta_1) < F(\theta_2)$. The latter relation implies that $D_{\theta_1} \cap D_{\theta_2} = \emptyset$.

For $\theta \in \mathbb{R}$, let Y_θ be a F_σ -subset of X_θ such that $p_\theta^{\mathbb{N}}(Y_\theta) = 1$.

For $\theta \in \mathbb{R} \setminus \{0\}$, we set $Z_\theta = Y_\theta$ and

$$Z_0 = Y_0 \cup \left(\mathbb{R}^{\mathbb{N}} \setminus \bigcup_{\theta \in \mathbb{R}} Z_\theta \right).$$

Let us show that $Z_\theta \in S$ for $\theta \in \mathbb{R}$. It is clear that $Z_\theta \in \mathcal{B}(\mathbb{R}^{\mathbb{N}}) \subseteq S$ for $\theta \in \mathbb{R} \setminus \{0\}$.

We have

$$Z_0 = Y_0 \cup \left(\mathbb{R}^{\mathbb{N}} \setminus \bigcup_{\theta \in \mathbb{R} \setminus \{0\}} Z_\theta \right).$$

On the one hand, $Y_0 \in \mathcal{B}(\mathbb{R}^{\mathbb{N}}) \subset S$ since Y_0 is F_σ -set. On the other hand, we have

$$\mathbb{R}^{\mathbb{N}} \setminus \bigcup_{\theta \in \mathbb{R} \setminus \{0\}} Z_\theta \subset \mathbb{R}^{\mathbb{N}} \setminus Z_\theta$$

¹Let $(\mu_i)_{i \in I}$ be a class of all Borel probability measures defined on a Polish space E . A σ -algebra $U(E)$ defined by $U(E) = \bigcap_{i \in I} \text{dom}(\bar{\mu}_i)$, where $\bar{\mu}_i$ denotes a usual completion of μ_i for $i \in I$, is called a class of all universally measurable subsets of E .

for each $\theta \in \mathbb{R}$. Hence

$$\mathbb{R}^{\mathbb{N}} \setminus \bigcup_{\theta \in \mathbb{R} \setminus \{0\}} Z_{\theta} \in \text{dom}(\bar{\mu}_{\theta})$$

for $\theta \in \mathbb{R}$. Finally, we get

$$\mathbb{R}^{\mathbb{N}} \setminus \bigcup_{\theta \in \mathbb{R} \setminus \{0\}} Z_{\theta} \in \bigcap_{\theta \in \mathbb{R}} \text{dom}(\bar{\mu}_{\theta}) = S.$$

Since S is a σ -algebra we claim that

$$Z_0 = Y_0 \cup \left(\mathbb{R}^{\mathbb{N}} \setminus \bigcup_{\theta \in \mathbb{R}} Z_{\theta} \right) \in S,$$

because $Y_0 \in S$ and $(\mathbb{R}^{\mathbb{N}} \setminus \bigcup_{\theta \in \mathbb{R}} Z_{\theta}) \in S$.

Now it is not difficult to verify that

- (i) $\bar{p}_{\theta}^{\mathbb{N}}(Z_{\theta}) = 1$;
- (ii) $Z_{\theta_1} \cap Z_{\theta_2} = \emptyset$;
- (iii) $\bigcup_{\theta \in \mathbb{R}} X_{\theta} = \mathbb{R}^{\mathbb{N}}$.

Lemma 2.6 is proved.

Lemma 2.7 (The strong law of large numbers). *Let (Ω, \mathcal{F}, p) be a probability space and X_1, X_2, \dots be an infinite sequence of independent and identically distributed random variables on (Ω, \mathcal{F}, P) with finite expectation value $m \in \mathbb{R}$, where $m = E(X_1) = E(X_2) = \dots$. Then*

$$p \left(\left\{ \omega : \omega \in \Omega \ \& \ \lim_{n \rightarrow \infty} \frac{\sum_{k=1}^n X_k(\omega)}{n} = m \right\} \right) = 1.$$

Lemma 2.8. *Let F be a distribution function defined on the real axis \mathbb{R} , such that the integral $\int_{\mathbb{R}} x dF(x)$ is finite. Suppose that p is a Borel probability measure on \mathbb{R} defined by F . Then the family of Baire (equivalently, Borel) \mathbb{N} -powers $(p_{\theta}^{\mathbb{N}})_{\theta \in \mathbb{R}}$ of shift measures $(p_{\theta})_{\theta \in \mathbb{R}}$ is strongly separable.*

Proof. For $\theta \in \mathbb{R}$, we define D_{θ} with

$$D_{\theta} = \left\{ (x_i)_{i \in \mathbb{N}} : (x_i)_{i \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}} \ \& \ \lim_{n \rightarrow \infty} \frac{\sum_{k=1}^n x_k}{n} = \theta + m \right\} = 1,$$

where the k -th projection Pr_k is defined by $Pr_k((x_i)_{i \in \mathbb{N}}) = x_k$ for $(x_i)_{i \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$.

By Lemma 2.7 we conclude that $p_{\theta}^{\infty}(D_{\theta}) = 1$ for $\theta \in \mathbb{R}$.

It is obvious that $D_{\theta_1} \cap D_{\theta_2} = \emptyset$ for different $\theta_1, \theta_2 \in \mathbb{R}$.

The application of the argument used in the proof of Lemma 2.6 ends the proof of Lemma 2.8.

Lemma 2.9. *Let G be an infinite additive group. Let $(p_k)_{k \in \mathbb{N}}$ be a sequence of probability Borel measures defined on \mathbb{R} and let $(\alpha_k)_{k \in \mathbb{N}}$ be a sequence of positive real numbers such that $\sum_{k \in \mathbb{N}} \alpha_k = 1$. Let p_k^G be a Borel (or a Baire) G -power of the measure p_k for $k \in \mathbb{N}$ and $\mu = \sum_{k \in \mathbb{N}} \alpha_k p_k^G$. Then the family of all coordinate projections $X = (Pr_g)_{g \in G}$ defined on a probability space $(\mathbb{R}^G, B(\mathbb{R}^G), \mu)$ (or $(\mathbb{R}^G, Ba(\mathbb{R}^G), \mu)$) is a G -process.*

Proof. For $n \in \mathbb{N}$, $(g_1, \dots, g_n) \in G^n$, $(x_1, \dots, x_n) \in \mathbb{R}^n$ and $h \in G$, we have

$$\begin{aligned}
 & F_{(g_1, \dots, g_n)}^{(X)}(x_1, \dots, x_n) = \\
 & = \mu \left(\left\{ (\omega_g)_{g \in G} : (\omega_g)_{g \in G} \in \mathbb{R}^G \ \& \ (\omega_{g_1}, \dots, \omega_{g_n}) \in \prod_{k=1}^n (-\infty, x_k] \right\} \right) = \\
 & = \left(\sum_{k \in \mathbb{N}} \alpha_k p_k^G \right) \left(\left\{ (\omega_g)_{g \in G} : (\omega_g)_{g \in G} \in \mathbb{R}^G \ \& \ (\omega_{g_1}, \dots, \omega_{g_n}) \in \prod_{k=1}^n (-\infty, x_k] \right\} \right) = \\
 & = \sum_{k \in \mathbb{N}} \alpha_k p_k^G \left(\left\{ (\omega_g)_{g \in G} : (\omega_g)_{g \in G} \in \mathbb{R}^G \ \& \ (\omega_{g_1}, \dots, \omega_{g_n}) \in \prod_{k=1}^n (-\infty, x_k] \right\} \right) = \\
 & = \sum_{k \in \mathbb{N}} \alpha_k p_k^G \left(\left\{ (\omega_g)_{g \in G} : (\omega_g)_{g \in G} \in \mathbb{R}^G \ \& \ (\omega_{g_1+h}, \dots, \omega_{g_n+h}) \in \prod_{k=1}^n (-\infty, x_k] \right\} \right) = \\
 & = \left(\sum_{k \in \mathbb{N}} \alpha_k p_k^G \right) \left(\left\{ (\omega_g)_{g \in G} : (\omega_g)_{g \in G} \in \mathbb{R}^G \ \& \ (\omega_{g_1+h}, \dots, \omega_{g_n+h}) \in \prod_{k=1}^n (-\infty, x_k] \right\} \right) = \\
 & = F_{(g_1+h, \dots, g_n+h)}^{(X)}(x_1, \dots, x_n).
 \end{aligned}$$

Lemma 2.9 is proved.

3. Formulations and proofs of the main results.

Theorem 3.1. *Let F be a strictly increasing continuous distribution function on \mathbb{R} and p be a Borel probability measure on \mathbb{R} defined by F . Let G be an infinite additive group. Then the family of Borel G -powers $(p_\theta^G)_{\theta \in \mathbb{R}}$ of Borel shift measures $(p_\theta)_{\theta \in \mathbb{R}}$ is strongly separated and the family of all coordinate projections $X = (Pr_g)_{g \in G}$ defined on a probability space $(\mathbb{R}^G, \mathcal{B}(\mathbb{R}^G), p_\theta^G)$ is a G -process for every $\theta \in \mathbb{R}$.*

Proof. By Lemma 2.2, there is a unique Borel extension of the Baire measure p_θ^G . We preserve the notation p_θ^G for its Borel extension. Let G_0 be a subset of G with $\text{card}(G_0) = \aleph_0$. By Lemma 2.1 we can establish that $p_\theta^G = p_\theta^{G_0} \times p_\theta^{G \setminus G_0}$. By Lemma 2.6, $(p_\theta^{G_0})_{\theta \in \mathbb{R}}$ is strongly separated, which implies that there exists a family $(Z_\theta)_{\theta \in \mathbb{R}}$ of elements of the σ -algebra $\cap_{\theta \in \mathbb{R}} \overline{\text{dom}(p_\theta^{G_0})}$ such that:

- (i) $\overline{p_\theta^{G_0}}(Z_\theta) = 1$ for $\theta \in \mathbb{R}$;
- (ii) $Z_{\theta_1} \cap Z_{\theta_2} = \emptyset$ for all different parameters θ_1 and θ_2 from \mathbb{R} ;
- (iii) $\cup_{\theta \in \mathbb{R}} Z_\theta = \mathbb{R}^{G_0}$.

Now setting $D_\theta = Z_\theta \times \mathbb{R}^{G \setminus G_0}$, we get that $(D_\theta)_{\theta \in \mathbb{R}}$ is a family of elements of the σ -algebra $S = \cap_{\theta \in \mathbb{R}} \overline{\text{dom}(p_\theta^G)}$ such that the following three conditions are fulfilled:

- (i*) $\overline{p_\theta^G}(D_\theta) = 1$ for $\theta \in \mathbb{R}$;
- (ii*) $D_{\theta_1} \cap D_{\theta_2} = \emptyset$ for all different parameters θ_1 and θ_2 from \mathbb{R} ;
- (iii*) $\cup_{\theta \in \mathbb{R}} D_\theta = \mathbb{R}^G$.

Theorem 3.1 is proved.

Remark 3.1. Let us equip $\Theta = \mathbb{R}$ with a usual metric. Under the conditions of Theorem 3.1, by Lemmas 2.5 and 2.6 we deduce that the mapping $\tilde{\theta}: \mathbb{R}^G \rightarrow \Theta$ defined by $\tilde{\theta}(x) = \theta$ for $x \in D_\theta$ is a consistent estimator of the parameter θ .

Theorem 3.2. Let F be a distribution function on \mathbb{R} such that the integral $\int_{\mathbb{R}} x dF(x)$ is finite. Suppose that p is a Borel probability measure on \mathbb{R} defined by F . Let G be an infinite additive group. Then the family of Baire G -powers $(p_\theta^G)_{\theta \in \mathbb{R}}$ of Borel shift measures $(p_\theta)_{\theta \in \mathbb{R}}$ on \mathbb{R} is strongly separated and the family of all coordinate projections $X = (Pr_g)_{g \in G}$ defined on a probability space $(\mathbb{R}^G, Ba(\mathbb{R}^G), p_\theta^G)$ is a G -process for every $\theta \in \mathbb{R}$.

Remark 3.2. The proof of Theorem 3.2 can be obtained by using Lemmas 2.1 and 2.8. Let us equip $\Theta = \mathbb{R}$ with a usual metric. Under the conditions of Theorem 3.2, by Lemma 2.5 we claim that there exists a consistent estimator of the parameter θ .

Example 3.1. Let p be a Gaussian Borel measure on \mathbb{R} . Then by Theorem 3.1 (or by Theorem 3.2) we deduce that the family of Borel (or Baire) G -powers $(p_\theta^G)_{\theta \in \mathbb{R}}$ of shift measures $(p_\theta)_{\theta \in \mathbb{R}}$ on \mathbb{R} is strong separated for an arbitrary additive group G , and that the family of all coordinate projections $X = (Pr_g)_{g \in G}$ defined on a probability space $(\mathbb{R}^G, \mathcal{B}(\mathbb{R}^G), p_\theta^G)$ (or $(\mathbb{R}^G, Ba(\mathbb{R}^G), p_\theta^G)$) is a G -process for every $\theta \in \mathbb{R}$.

Example 3.2. Let p be a Poisson Borel probability measure on \mathbb{R} . Then by Theorem 3.2 we deduce that the family of Baire G -powers $(p_\theta^G)_{\theta \in \mathbb{R}}$ of shift measures $(p_\theta)_{\theta \in \mathbb{R}}$ on \mathbb{R} is strongly separated for an arbitrary additive group G , and that the family of all coordinate projections $X = (Pr_g)_{g \in G}$ defined on a probability space $(\mathbb{R}^G, Ba(\mathbb{R}^G), p_\theta^G)$ is a G -process for every $\theta \in \mathbb{R}$. We cannot apply Theorem 3.1 in order to establish the validity of this fact since the family of shift measures $(p_\theta)_{\theta \in \mathbb{R}}$ does not satisfy the conditions of this theorem.

Example 3.3. Let p be a Cauchy Borel probability measure on \mathbb{R} . Then by Theorem 3.1, we deduce that for an arbitrary additive group G , the family of Borel G -powers $(p_\theta^G)_{\theta \in \mathbb{R}}$ of shift measures $(p_\theta)_{\theta \in \mathbb{R}}$ on \mathbb{R} is strongly separated and the family of all coordinate projections $X = (Pr_g)_{g \in G}$ defined on a probability space $(\mathbb{R}^G, \mathcal{B}(\mathbb{R}^G), p_\theta^G)$ is a G -process for every $\theta \in \mathbb{R}$. We cannot apply Theorem 3.2 in order to establish the validity of this fact since the integral $\int_{\mathbb{R}} x dF(x)$ does not converge.

Theorem 3.3. Let $(\Theta_i)_{i \in I}$ be a partition of the real axis \mathbb{R} , such that $\text{card}(\Theta_i) \leq \aleph_0$, where \aleph_0 is the cardinality of the set of all natural numbers. Let $(\alpha_\theta^{(i)})_{\theta \in \Theta_i}$ be a sequence of positive real numbers such that $\sum_{\theta \in \Theta_i} \alpha_\theta^{(i)} = 1$ for $i \in I$. Let μ_θ be a θ -shift of the Borel probability measure μ on \mathbb{R} with a strictly increasing continuous distribution function. Let G be an infinite additive group. For $i \in I$, we define a Borel probability measure λ_i on \mathbb{R}^G by $\lambda_i = \sum_{\theta \in \Theta_i} \alpha_\theta^{(i)} \mu_\theta^G$. Then $(\lambda_i)_{i \in I}$ is strongly separated and the family of all coordinate projections $X = (Pr_g)_{g \in G}$ defined on a probability space $(\mathbb{R}^G, \mathcal{B}(\mathbb{R}^G), \lambda_i)$ is a G -process for every $i \in I$.

Proof. By Lemma 2.2, the Baire measure μ_θ^G admits a unique Borel extension for $\theta \in \mathbb{R}$ for which we preserve the same notation. Note that $\lambda_i = \sum_{\theta \in \Theta_i} \alpha_\theta^{(i)} \mu_\theta^G$ will be a Borel probability measure on \mathbb{R}^G for $i \in I$.

By Lemma 2.9, the family of all coordinate projections $X = (Pr_g)_{g \in G}$ defined on a probability space $(\mathbb{R}^G, \mathcal{B}(\mathbb{R}^G), \lambda_i)$ is a G -process for every $i \in I$.

By Theorem 3.1, the family of Borel probability measures $(\mu_\theta^G)_{\theta \in \mathbb{R}}$ is strongly separated, i.e., there exists a family $(D_\theta)_{\theta \in \mathbb{R}}$ of elements of the σ -algebra $\cap_{\theta \in \mathbb{R}} \overline{\text{dom}(\mu_\theta^G)}$ such that:

- (i) $\mu_\theta^G(D_\theta) = 1$ for $\theta \in \mathbb{R}$;
- (ii) $D_{\theta_1} \cap D_{\theta_2} = \emptyset$ for all different parameters θ_1 and θ_2 from \mathbb{R} ;
- (iii) $\cup_{\theta \in \mathbb{R}} D_\theta = \mathbb{R}^G$.

We set $E_i = \cup_{\theta \in \Theta_i} D_\theta$ for $i \in I$. Now it is clear that $(E_i)_{i \in I}$ is a pairwise disjoint family of elements of the σ -algebra $\cap_{i \in I} \overline{\text{dom}(\lambda_i)}$ such that $\overline{\lambda_i}(E_i) = 1$ for $i \in I$.

Theorem 3.3 is proved.

Remark 3.3. Let us equip a set I with a discrete metric. Under the conditions of Theorem 3.3, by Lemma 2.5 we claim that a mapping $\tilde{\theta}: \mathbb{R}^G \rightarrow I$, defined by $\tilde{\theta}(x) = i$ for $x \in E_i$, is a consistent estimator of the parameter i .

The following theorem is a simple consequence of Theorem 3.2.

Theorem 3.4. Let $(\Theta_i)_{i \in I}$ be a partition of the real axis \mathbb{R} such that $\text{card}(\Theta_i) \leq \aleph_0$, where \aleph_0 is the cardinality of the set of all natural numbers. Let $(\alpha_\theta^{(i)})_{\theta \in \Theta_i}$ be a sequence of positive real numbers such that $\sum_{\theta \in \Theta_i} \alpha_\theta^{(i)} = 1$ for $i \in I$. Let μ_θ be a θ -shift of the Borel probability measure μ on \mathbb{R} such that the integral $\int_{\mathbb{R}} x dF(x)$ is finite, where F is the distribution function defined by μ . Let G be an infinite additive group. For $i \in I$, we define a Baire probability measure on \mathbb{R}^G by $\lambda_i = \sum_{\theta \in \Theta_i} \alpha_\theta^{(i)} \mu_\theta^G$. Then $(\lambda_i)_{i \in I}$ is strongly separated and the family of all coordinate projections $X = (Pr_g)_{g \in G}$ defined on a probability space $(\mathbb{R}^G, Ba(\mathbb{R}^G), \lambda_i)$ is a G -process for every $i \in I$.

Remark 3.4. Let us equip the set I with a discrete metric. Under the conditions of Theorem 3.4, by Lemma 2.5 we deduce that there exists a consistent estimator of the parameter i .

4. On consistent estimators of a useful signal in the linear one-dimensional stochastic model when the expectation of the transformed signal is not defined. Suppose that Θ is a vector subspace of the infinite-dimensional topological vector space of all real-valued sequences $\mathbb{R}^{\mathbb{N}}$ equipped with the product topology.

In the information transmission theory we consider the linear one-dimensional stochastic system

$$(\xi_k)_{k \in \mathbb{N}} = (\theta_k)_{k \in \mathbb{N}} + (\Delta_k)_{k \in \mathbb{N}}, \quad (4.1)$$

where $(\theta_k)_{k \in \mathbb{N}} \in \Theta$ is a sequence of useful signals, $(\Delta_k)_{k \in \mathbb{N}}$ is sequence of independent identically distributed random variables (the so-called generalized “white noise”) defined on the some probability space (Ω, \mathcal{F}, P) and $(\xi_k)_{k \in \mathbb{N}}$ is a sequence of transformed signals. Let μ be a Borel probability measure on \mathbb{R} defined by a random variable Δ_1 . Then the \mathbb{N} -power of the measure μ denoted by $\mu^{\mathbb{N}}$ coincides with the Borel probability measure on $\mathbb{R}^{\mathbb{N}}$ defined by the generalized “white noise”, i.e.,

$$(\forall X)(X \in \mathcal{B}(\mathbb{R}^{\mathbb{N}}) \rightarrow \mu^{\mathbb{N}}(X) = P(\{\omega: \omega \in \Omega \ \& \ (\Delta_k(\omega))_{k \in \mathbb{N}} \in X\})),$$

where $\mathcal{B}(\mathbb{R}^{\mathbb{N}})$ is the Borel σ -algebra of subsets of $\mathbb{R}^{\mathbb{N}}$.

In the information transmission theory, the general decision is that the Borel probability measure λ , defined by the sequence of transformed signals $(\xi_k)_{k \in \mathbb{N}}$ coincides with $(\mu^{\mathbb{N}})_{\theta_0}$ for some $\theta_0 \in \Theta$ provided that

$$(\exists \theta_0)(\theta_0 \in \Theta \rightarrow (\forall X)(X \in \mathcal{B}(\mathbb{R}^{\mathbb{N}}) \rightarrow \lambda(X) = (\mu^{\mathbb{N}})_{\theta_0}(X))),$$

where $(\mu^{\mathbb{N}})_{\theta_0}(X) = \mu^{\mathbb{N}}(X - \theta_0)$ for $X \in \mathcal{B}(\mathbb{R}^{\mathbb{N}})$.

Here we consider a particular case of the above model when a vector space of useful signals Θ has the form

$$\Theta = \{(\theta, \theta, \dots) : \theta \in \mathbb{R}\}.$$

For $\theta \in \mathbb{R}$, a measure $\mu_\theta^{\mathbb{N}}$, defined by

$$\mu_\theta^{\mathbb{N}} = \mu_\theta \times \mu_\theta \times \dots,$$

where μ_θ is a θ -shift of μ (i.e., $\mu_\theta(X) = \mu(X - \theta)$ for $X \in \mathcal{B}(\mathbb{R})$), is called the \mathbb{N} -power of the θ -shift of μ on \mathbb{R} . It is obvious that $\mu_\theta^{\mathbb{N}} = (\mu^{\mathbb{N}})_{(\theta, \theta, \dots)}$.

Following Lemma 2.7, the sample mean is a consistent estimator of a parameter $\theta \in \mathbb{R}$ (in the sense of almost everywhere convergence) for the family $(\mu_\theta^{\mathbb{N}})_{\theta \in \mathbb{R}}$ if the first order absolute moment of μ is finite. We have a different picture when the first order absolute moment of μ is not defined. In that case, Lemma 2.7 cannot be used. Unfortunately, we could not find in the literature any method that would allow us to estimate a useful signal for model (4.1) when the expectation of the transformed signal is not defined. In the remaining part of the paper we resolve this problem.

Definition 4.1. A Borel measurable function $T_n: \mathbb{R}^n \rightarrow \mathbb{R}$, $n \in \mathbb{N}$, is called a consistent estimator of a parameter θ (in the sense of everywhere convergence) for the family $(\mu_\theta^{\mathbb{N}})_{\theta \in \mathbb{R}}$ if the condition

$$\mu_\theta^{\mathbb{N}}(\{(x_k)_{k \in \mathbb{N}} : (x_k)_{k \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}} \& \lim_{n \rightarrow \infty} T_n(x_1, \dots, x_n) = \theta\}) = 1$$

holds for each $\theta \in \mathbb{R}$.

Definition 4.2. A Borel measurable function $T_n: \mathbb{R}^n \rightarrow \mathbb{R}$, $n \in \mathbb{N}$, is called a consistent estimator of a parameter θ (in the sense of convergence in probability) for the family $(\mu_\theta^{\mathbb{N}})_{\theta \in \mathbb{R}}$ if the condition

$$\lim_{n \rightarrow \infty} \mu_\theta^{\mathbb{N}}\left(\{(x_k)_{k \in \mathbb{N}} : (x_k)_{k \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}} \& |T_n(x_1, \dots, x_n) - \theta| > \epsilon\}\right) = 0$$

holds for every $\epsilon > 0$ and $\theta \in \mathbb{R}$.

Definition 4.3. A Borel measurable function $T_n: \mathbb{R}^n \rightarrow \mathbb{R}$, $n \in \mathbb{N}$, is called a consistent estimator of a parameter θ (in the sense of convergence in distribution) for the family $(\mu_\theta^{\mathbb{N}})_{\theta \in \mathbb{R}}$ if the condition

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^{\mathbb{N}}} f(T_n(x_1, \dots, x_n)) d\mu_\theta^{\mathbb{N}}((x_k)_{k \in \mathbb{N}}) = f(\theta)$$

holds for every continuous bounded real-valued function f on \mathbb{R} .

Remark 4.1. Based on [6, p. 272] (see Theorem 2), for the family $(\mu_\theta^{\mathbb{N}})_{\theta \in \mathbb{R}}$ we make the following conclusions:

(a) the existence of a consistent estimator of a parameter θ in the sense of everywhere convergence implies the existence of a consistent estimator of a parameter θ in the sense of convergence in probability;

(b) the existence of a consistent estimator of a parameter θ in the sense of convergence in probability implies the existence of a consistent estimator of a parameter θ in the sense of convergence in distribution.

Theorem 4.1. Let F be a strictly increasing continuous distribution function on \mathbb{R} and μ be a Borel probability measure on \mathbb{R} defined by F . For $\theta \in \mathbb{R}$, we set $F_\theta(x) = F(x - \theta)$, $x \in \mathbb{R}$, and denote by μ_θ the Borel probability measure on \mathbb{R} defined by F_θ (obviously, this is an equivalent definition of the θ -shift of μ). Then a function $T_n: \mathbb{R}^n \rightarrow \mathbb{R}$, defined by

$$T_n(x_1, \dots, x_n) = -F^{-1}(n^{-1}\#(\{x_1, \dots, x_n\} \cap (-\infty; 0])) \quad (4.2)$$

for $(x_1, \dots, x_n) \in \mathbb{R}^n$, $n \in \mathbb{N}$, is a consistent estimator of a parameter θ for the family $(\mu_\theta^{\mathbb{N}})_{\theta \in \mathbb{R}}$ in the sense of almost everywhere convergence.

Proof. It is clear that T_n is a Borel measurable function for $n \in \mathbb{N}$. For $\theta \in \mathbb{R}$, we set

$$A_\theta = \{(x_k)_{k \in \mathbb{N}}: (x_k)_{k \in \mathbb{N}} \text{ is } \mu_\theta\text{-uniformly distributed on } \mathbb{R}\}.$$

By Corollary 2.4, we obtain $\mu_\theta^{\mathbb{N}}(A_\theta) = 1$ for $\theta \in \mathbb{R}$.

For $\theta \in \mathbb{R}$, we have

$$\begin{aligned} & \mu_\theta^{\mathbb{N}} \left(\left\{ (x_k)_{k \in \mathbb{N}}: (x_k)_{k \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}} \ \& \ \lim_{n \rightarrow \infty} T_n(x_1, \dots, x_n) = \theta \right\} \right) = \\ & = \mu_\theta^{\mathbb{N}} \left(\left\{ (x_k)_{k \in \mathbb{N}}: (x_k)_{k \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}} \ \& \ \lim_{n \rightarrow \infty} F^{-1}(n^{-1}\#(\{x_1, \dots, x_n\} \cap (-\infty; 0])) = -\theta \right\} \right) = \\ & = \mu_\theta^{\mathbb{N}} \left(\left\{ (x_k)_{k \in \mathbb{N}}: (x_k)_{k \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}} \ \& \ \lim_{n \rightarrow \infty} n^{-1}\#(\{x_1, \dots, x_n\} \cap (-\infty; 0]) = F(-\theta) \right\} \right) = \\ & = \mu_\theta^{\mathbb{N}} \left(\left\{ (x_k)_{k \in \mathbb{N}}: (x_k)_{k \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}} \ \& \ \lim_{n \rightarrow \infty} n^{-1}\#(\{x_1, \dots, x_n\} \cap (-\infty; 0]) = F_\theta(0) \right\} \right) \geq \\ & \geq \mu_\theta^{\mathbb{N}}(A_\theta) = 1. \end{aligned}$$

Theorem 4.1 is proved.

The following corollaries are simple consequences of Theorem 4.1 and Remark 4.1.

Corollary 4.1. An estimator T_n defined by (4.2) is a consistent estimator of a parameter θ for the family $(\mu_\theta^{\mathbb{N}})_{\theta \in \mathbb{R}}$ in the sense of convergence in probability.

Corollary 4.2. An estimator T_n defined by (4.2) is a consistent estimator of a parameter θ for the family $(\mu_\theta^{\mathbb{N}})_{\theta \in \mathbb{R}}$ in the sense of convergence in distribution.

Remark 4.2. Combining Lemma 2.7 and Theorem 4.1, we get the validity of the condition

$$\begin{aligned} & \mu_\theta^{\mathbb{N}} \left(\left\{ (x_k)_{k \in \mathbb{N}}: (x_k)_{k \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}} \ \& \ - \lim_{n \rightarrow \infty} F^{-1}(n^{-1}\#(\{x_1, \dots, x_n\} \cap (-\infty; 0])) = \right. \right. \\ & \left. \left. = \lim_{n \rightarrow \infty} n^{-1} \sum_{k=1}^n x_k = \theta \right\} \right) = 1 \end{aligned}$$

for $\theta \in \mathbb{R}$ when μ is equivalent to the linear standard Gaussian measure on \mathbb{R} , the first order absolute moment of μ is finite and the first order moment of μ is equal to zero.

Definition 4.4. Following [1], the family $(\mu_\theta^{\mathbb{N}})_{\theta \in \mathbb{R}}$ is called strongly separated in the usual sense if there exists a family $(Z_\theta)_{\theta \in \mathbb{R}}$ of Borel subsets of $\mathbb{R}^{\mathbb{N}}$ such that

- (i) $\mu_\theta^{\mathbb{N}}(Z_\theta) = 1$ for $\theta \in \mathbb{R}$;
- (ii) $Z_{\theta_1} \cap Z_{\theta_2} = \emptyset$ for all different parameters θ_1 and θ_2 from \mathbb{R} ;
- (iii) $\cup_{\theta \in \mathbb{R}} Z_\theta = \mathbb{R}^{\mathbb{N}}$.

Definition 4.5. Following [1], a Borel measurable function $T: \mathbb{R}^{\mathbb{N}} \rightarrow \mathbb{R}$ is called an infinite sample consistent estimator of a parameter θ for the family $(\mu_\theta^{\mathbb{N}})_{\theta \in \mathbb{R}}$ if the following condition:

$$(\forall \theta) \left(\theta \in \mathbb{R} \rightarrow \mu_\theta^{\mathbb{N}} \left(\left\{ (x_k)_{k \in \mathbb{N}} : (x_k)_{k \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}} \ \& \ T((x_k)_{k \in \mathbb{N}}) = \theta \right\} \right) = 1 \right)$$

is fulfilled.

Remark 4.3. The existence of an infinite sample consistent estimator of a parameter θ for the family $(\mu_\theta^{\mathbb{N}})_{\theta \in \mathbb{R}}$ implies that the family $(\mu_\theta^{\mathbb{N}})_{\theta \in \mathbb{R}}$ is strongly separated in a usual sense. Indeed, if we set $Z_\theta = \{(x_k)_{k \in \mathbb{N}} : (x_k)_{k \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}} \ \& \ T((x_k)_{k \in \mathbb{N}}) = \theta\}$ for $\theta \in \mathbb{R}$, then all the conditions of Definition 2.5 will be satisfied.

Theorem 4.2. Let F be a strictly increasing continuous distribution function on \mathbb{R} and μ be the Borel probability measure on \mathbb{R} defined by F . For $\theta \in \mathbb{R}$, we set $F_\theta(x) = F(x - \theta)$, $x \in \mathbb{R}$, and denote by μ_θ the Borel probability measure on \mathbb{R} defined by F_θ . Then the estimators $\overline{\lim} \widetilde{T}_n := \inf_n \sup_{m \geq n} \widetilde{T}_m$ and $\underline{\lim} \widetilde{T}_n := \sup_n \inf_{m \geq n} \widetilde{T}_m$ are infinite sample consistent estimators of a parameter θ for the family $(\mu_\theta^{\mathbb{N}})_{\theta \in \mathbb{R}}$, where $\widetilde{T}_n: \mathbb{R}^{\mathbb{N}} \rightarrow \mathbb{R}$ is defined by

$$(\forall (x_k)_{k \in \mathbb{N}}) \left((x_k)_{k \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}} \rightarrow \widetilde{T}_n((x_k)_{k \in \mathbb{N}}) = -F^{-1}(n^{-1} \#(\{x_1, \dots, x_n\} \cap (-\infty; 0])) \right). \quad (4.3)$$

Proof. Following [6, p. 189], the functions $\overline{\lim} \widetilde{T}_n$ and $\underline{\lim} \widetilde{T}_n$ are Borel measurable. By Corollary 2.4, we have $\mu_\theta^{\mathbb{N}}(A_\theta) = 1$ for $\theta \in \mathbb{R}$, which implies

$$\begin{aligned} & \mu_\theta^{\mathbb{N}} \left(\left\{ (x_k)_{k \in \mathbb{N}} : (x_k)_{k \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}} \ \& \ \overline{\lim} \widetilde{T}_n(x_k)_{k \in \mathbb{N}} = \theta \right\} \right) \geq \\ & \geq \mu_\theta^{\mathbb{N}} \left(\left\{ (x_k)_{k \in \mathbb{N}} : (x_k)_{k \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}} \ \& \ \overline{\lim} \widetilde{T}_n(x_k)_{k \in \mathbb{N}} = \underline{\lim} \widetilde{T}_n(x_k)_{k \in \mathbb{N}} = \theta \right\} \right) \geq \\ & \geq \mu_\theta^{\mathbb{N}}(A_\theta) = 1, \end{aligned}$$

where

$$A_\theta = \{(x_k)_{k \in \mathbb{N}} : (x_k)_{k \in \mathbb{N}} \text{ is } \mu_\theta\text{-uniformly distributed on } \mathbb{R}\}$$

for $\theta \in \mathbb{R}$.

From the last relation it follows that $\overline{\lim} \widetilde{T}_n$ is the infinite sample consistent estimator of a parameter θ for the family $(\mu_\theta^{\mathbb{N}})_{\theta \in \mathbb{R}}$.

Using the above scheme, we can established the validity of an analogous fact for the estimator $\underline{\lim} \widetilde{T}_n$.

Theorem 4.2 is proved.

Remark 4.4. By Remark 4.3 and Theorem 4.2, we deduce that the family $(\mu_\theta^{\mathbb{N}})_{\theta \in \mathbb{R}}$ is strongly separated in the usual sense. Since each Borel subset of $\mathbb{R}^{\mathbb{N}}$ is an element of the σ -algebra $S := \cap_{\theta \in \mathbb{R}} \text{dom}(\overline{\mu}_\theta)$, we claim that Theorem 4.2 extends the result of Lemma 2.6.

Example 4.1. Let $\mu_{(\theta,1)}$ be a linear Gaussian measure on \mathbb{R} with parameters $(\theta, 1)$. Let $[\cdot]$ denote the integer part of a real number. Since a sequence of real numbers $(\pi \times k - [\pi \times k])_{k \in \mathbb{N}}$ is uniformly distributed on $(0, 1)$ (see [2, p. 17], Example 2.1), by Lemma 2.4 we claim that a sequence $(x_k)_{k \in \mathbb{N}}$ defined by

$$x_k = F^{-1}(\pi \times k - [\pi \times k]) + \theta$$

is a $\mu_{(\theta,1)}$ -equidistributed sequence on \mathbb{R} , where F denotes a linear standard Gaussian distribution function on \mathbb{R} with parameters $(\theta, 1)$.

It is obvious that $(x_k)_{k \in \mathbb{N}}$ is a certain realization of model (4.1), where $(\Delta_k)_{k \in \mathbb{N}}$ is a sequence of independent Gaussian random variables on \mathbb{R} .

In the sequel we use the notation introduced above:

- (i) n is a number of trials;
- (ii) T_n is the estimator defined by formula (4.2);
- (iii) \bar{X}_n is the sample average;
- (iv) θ is a “useful signal”.

We have considered the construction of the one-dimensional linear stochastic model (4.1) for $\theta = 1$. Below we present the numerical results obtained by Microsoft Excel.

Table 4.1

n	T_n	\bar{X}_n	θ	n	T_n	\bar{X}_n	θ
50	0.994457883	1.146952654	1	550	1.04034032	1.034899747	1
100	1.036433389	1.010190601	1	600	1.036433389	1.043940988	1
150	1.022241387	1.064790041	1	650	1.03313984	1.036321771	1
200	1.036433389	1.037987511	1	700	1.030325691	1.037905202	1
250	1.027893346	1.045296447	1	750	1.033578332	1.03728633	1
300	1.036433389	1.044049728	1	800	1.03108705	1.032630945	1
350	1.030325691	1.034339407	1	850	1.033913784	1.037321098	1
400	1.036433389	1.045181911	1	900	1.031679632	1.026202323	1
450	1.031679632	1.023083495	1	950	1.034178696	1.036669278	1
500	1.036433389	1.044635371	1	1000	1.036433389	1.031131694	1

Note that the results of computations presented in Table 4.1 do not contradict Remark 4.2, which asserts that under the conditions of Theorem 4.1, the estimators T_n and \bar{X}_n coincide and both are consistent estimators of the “useful signal” θ .

We have also considered similar model constructions when F is a Cauchy distribution function on \mathbb{R} . The results of relevant numerical computations are given in the next table.

Table 4.2

n	T_n	\bar{X}_n	θ	n	T_n	\bar{X}_n	θ
50	1.20879235	2.555449288	1	550	1.017284476	41.08688757	1
100	0.939062506	1.331789564	1	600	1.042790358	41.30221291	1
150	1.06489184	71.87525566	1	650	1.014605804	38.1800532	1
200	1.00000000	54.09578271	1	700	1.027297114	38.03399768	1
250	1.06489184	64.59240343	1	750	1.012645994	35.57956117	1
300	1.021166379	54.03265563	1	800	1.015832638	35.25149408	1
350	1.027297114	56.39846672	1	850	1.018652839	33.28723503	1
400	1.031919949	49.58316089	1	900	1.0070058	31.4036155	1
450	1.0070058	44.00842613	1	950	1.023420701	31.27321466	1
500	1.038428014	45.14322051	1	1000	1.012645994	29.73405416	1

We see that the results of numerical computations in Table 4.2 do not contradict Theorem 4.2, which asserts that T_n is a consistent estimator of the parameter $\theta = 1$. These results also show that our attempt to estimate a useful signal by the sample average in our model is not successful. These computational results seem natural because the mean and the variance do not exist for Cauchy random variables.

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Received 20.05.11,
after revision – 16.01.13