

PROJECTIVE METHOD FOR EQUATION OF RISK THEORY IN THE ARITHMETIC CASE

ПРОЕКТИВНИЙ МЕТОД ДЛЯ РІВНЯННЯ ТЕОРІЇ РИЗИКУ В АРИФМЕТИЧНОМУ ВИПАДКУ

We consider a discrete model of operation of an insurance company whose initial capital can take any integer value. In this statement, the problem of nonruin probability is naturally solved by the Wiener–Hopf method. Passing to generating functions and reducing the fundamental equation of risk theory to a Riemann boundary-value problem on the unit circle, we establish that this equation is a special one-sided discrete Wiener–Hopf equation whose symbol has a unique zero, and, furthermore, this zero is simple. On the basis of the constructed solvability theory for this equation, we justify the applicability of the projective method to the approximation of ruin probabilities in the spaces \mathbb{I}_1^+ and \mathbb{C}_0^+ . Conditions for the distributions of waiting times and claims under which the method converges are established. The delayed renewal process and stationary renewal process are considered, and approximations for the ruin probabilities in these processes are obtained.

Розглядається дискретна модель функціонування страхової компанії, початковий капітал якої може набувати довільного цілого значення. У такій постановці проблема обчислення ймовірності стійкості компанії природно розв'язується методом Вінера–Хопфа. При переході до твірних функцій і зведенні фундаментального рівняння теорії ризику до граничної задачі Рімана на одиничному колі з'ясовано, що розглядуване рівняння є особливим одностороннім дискретним рівнянням Вінера–Хопфа, символ якого має єдиний нуль і цей нуль є простим. На базі побудованої теорії розв'язності цього рівняння обґрунтовано застосування проективного методу до апроксимації ймовірностей банкрутства у просторах \mathbb{I}_1^+ і \mathbb{C}_0^+ . Отримано умови на розподіли часів очікування вимог і розмірів виплат для збіжності методу. Розглянуто процес відновлення із запізненням і стаціонарний процес відновлення, а також наближення для ймовірностей банкрутства у цих процесах.

1. Introduction. Consider the discrete ordinary renewal model for functioning of an insurance company in the arithmetic¹ case:

Assumptions. The renewal model is given by following conditions:

(a) *the claim sizes $\{Z_n\}_{n \in \mathbb{N}}$ are positive integer-valued independent identically distributed (iid) random variables (rvs), having common generating function $g_Z(z) = \sum_{n=1}^{\infty} q_n z^n$ and finite mean² $\mu = EZ_1$;*

(b) *the inter-arrival times $\{T_n\}_{n \in \mathbb{N}}$ are positive integer-valued iid rvs having common generating function $g_T(z) = \sum_{n=1}^{\infty} p_n z^n$ and finite mean $ET_1 = 1/\alpha$;*

(c) *the gross premium rate $c > \alpha\mu$, $c \in \mathbb{N}$;*

(d) *the sequences $\{Z_n\}_{n \in \mathbb{N}}$ and $\{T_n\}_{n \in \mathbb{N}}$ are independent of each other.*

Let $F_Z(v)$ and $F_T(v)$ be the distributions of the random variables Z_1 and T_1 , respectively. It is known that probability of solvency of the insurance company, $\varphi(u)$, with initial capital $u \geq 0$ in ordinary renewal process, in general case, satisfies the fundamental equation of risk theory [2, 6, 14]:

$$\varphi(u) - \int_0^{\infty} dF_T(v) \int_0^{u+cv} \varphi(u+cv-w) dF_Z(w) = 0, \quad u \in \mathbb{R}^+. \quad (1)$$

¹In the literature it is also used the term “periodic” [1].

²We denote by \mathbb{N} the set of positive integers.

At the derivation of this equation, nothing interferes us to consider u to be also negative. Really, the insurance company can start its activity having a debt, $u < 0$, i.e., being in the state of ruin. At the favorable concurrence of circumstances, the company can leave this state for the moment of arrival of first claim and more in it can return not. It happens, for example, when the random variable $u + cT_1 - Z_1$ accepts a nonnegative value and hereinafter the company will not be ruined. Therefore, it is reasonable to state the more general problem on the calculation of the nonruin probability for the company with initial capital $u \in \mathbb{R}$, being in the state of nonruin later on, that is, we are interested by the probability $\varphi(u)$ of the following event:

$$u + \sum_{k=1}^n (cT_k - Z_k) \geq 0 \quad \forall n \in \mathbb{N}, \quad u \in \mathbb{R}.$$

In such setting it is naturally to apply the Wiener–Hopf method [8, 16] for the solution of this problem.

For the nonruin probability $\varphi(u)$, $u \in \mathbb{R}$, introduce in consideration the probabilities $\varphi_{\pm}(u)$ by the formulas

$$\varphi_{\pm}(u) = H(\pm u)\varphi(u), \quad \varphi(u) = \varphi_+(u) + \varphi_-(u),$$

where $H(u)$ is the Heaviside function.

Following Feller, [2, 6, 13, 14], derive the equation for $\varphi_-(u)$,

$$\varphi_-(u) - \int_0^{\infty} dF_T(v) \int_0^{u+cv} \varphi_+(u+cv-w) dF_Z(w) = 0, \quad u \in \mathbb{R}^-. \quad (2)$$

Here we take into account that $\varphi_+(u) \equiv 0$ for $u < 0$, and therefore integration in internal integral, as a matter of fact, is over a set on which $u + cv - w > 0$.

Joining the equations (1) and (2), the equation for $\varphi_{\pm}(u)$, $u \in \mathbb{R}$, can be written in the form of one equation,

$$\varphi_+(u) + \varphi_-(u) - \int_0^{\infty} dF_T(v) \int_0^{u+cv} \varphi_+(u+cv-w) dF_Z(w) = 0, \quad u \in \mathbb{R}. \quad (3)$$

We are interested by the solution $\varphi(u)$ satisfying the conditions

$$\varphi_+(u) \nearrow 1 \quad \text{when } u \rightarrow +\infty, \quad \text{and} \quad \varphi_-(u) \searrow 0 \quad \text{when } u \rightarrow -\infty. \quad (4)$$

Denote by \mathbb{Z} the set of integers, $\mathbb{Z}^+ = \{0, 1, 2, \dots\}$, $\mathbb{Z}^- = \{-1, -2, \dots\}$.

Consider the linear space of all two-sided sequences of complex numbers $\xi = \{\xi_n\}_{n \in \mathbb{Z}}$. Denote by \mathbf{I}_1 the Banach space of all sequences of complex numbers $\xi = \{\xi_n\}_{n \in \mathbb{Z}}$ with the finite norm

$$\|\xi\|_{\mathbf{I}_1} = \sum_{n=-\infty}^{\infty} |\xi_n| < \infty$$

and by \mathbf{c} the space of all convergent sequences ξ which after introduction of the norm

$$\|\xi\|_{\mathbf{c}} = \sup_n |\xi_n| < \infty$$

also becomes the Banach space. Let $\mathbf{c}_0 \subset \mathbf{c}$ be the subspace of all sequences convergent to zero.

Let E be either \mathbf{l}_1 or \mathbf{c}_0 and let m be some real number. Denote by E_m the space of all numerical sequences of the form $f = \{(1 + |n|^{-m}\xi_n)\}_{n \in \mathbb{Z}}$, where $\xi \in E$. Introduction of the norm $|f| = \|\xi\|_E$ converts E_m in the Banach space isometric and isomorphic to E [16]. In the case when m is a positive integer, the sequence $f = \{f_n\}_{n \in \mathbb{Z}}$ belongs to E_m if and only if $h_{(k)} = \{n^k f_n\}_{n \in \mathbb{Z}} \in E$, $k = 0, 1, \dots, m$.

Each of the spaces E_m has two distinguished subspaces: E_m^+ is the subspace of sequences $\xi^+ = \{\xi_n^+\}_{n \in \mathbb{Z}^+}$ (or $\{\xi_n\}_{n \in \mathbb{Z}^+}$) characterized by the condition $\xi_n^+ = 0$ for $n \in \mathbb{Z}^-$, and E_m^- is the subspace of sequences $\xi^- = \{\xi_n^-\}_{n \in \mathbb{Z}^-}$ (or $\{\xi_n\}_{n \in \mathbb{Z}^-}$) for which $\xi_n^- = 0$, $n \in \mathbb{Z}^+$.

Let $\mathbb{T} = \{t \in \mathbb{C} : |t| = 1\}$ denote the complex unit circle, $\mathbb{B}^+ = \{z : |z| < 1\}$ the complex unit disk and $\mathbb{B}^- = \{z : |z| > 1\}$ the complementary disk to $\mathbb{B}^+ \cup \mathbb{T}$.

For the space E_m of sequences $\xi = \{\xi_n\}_{n \in \mathbb{Z}}$ denote by \widehat{E}_m the space of generating functions (called also Laurent or Fourier transforms) of the form

$$\Xi(t) = \sum_{n=-\infty}^{\infty} \xi_n t^n, \quad t \in \mathbb{T}, \quad (5)$$

being generalized functions on \mathbb{T} (Schwartz distributions) [9].

Denote by W the Wiener algebra of all functions of the form (5) on \mathbb{T} , expanding in absolutely convergent Fourier series. Let W_m , $0 \leq m < \infty$, be the algebra of all functions of the form (5) for which $\{\xi_n\}_{n \in \mathbb{Z}} \in \mathbf{l}_{1,m}$, $W_0 = W$. As it is known, for $m \in \mathbb{N}$, the algebra W_m contains any m times differentiable function $\Xi(t)$ the derivatives of which belong to W [16]. In what follows, it will be useful for us the fact that if the point $t_0 \in \mathbb{T}$ is zero of the order m for $\Xi(t) \in W_m$, then $\Xi(t)$ is representable in the form $\Xi(t) = (t - t_0)^m b(t)$ with $b(t) \in W$ [16].

Denote by \widehat{E}_m^+ the subspace of \widehat{E}_m , consisting, generally speaking, of the Schwartz distributions of the form $\Xi^+(t) = \sum_{n=0}^{\infty} \xi_n t^n$, $t \in \mathbb{T}$, which are the boundary values of analytic functions in \mathbb{B}^+ , expandable into a Taylor series about z ,

$$\Xi^+(z) = \sum_{n=0}^{\infty} \xi_n z^n, \quad z \in \mathbb{B}^+,$$

and let \widehat{E}_m^- be the subspace of \widehat{E}_m , consisting, generally speaking, of the Schwartz distributions $\Xi^-(t) = \sum_{n=-\infty}^{-1} \xi_n t^n$, $t \in \mathbb{T}$, being the boundary values of analytic functions in \mathbb{B}^- , expandable into a Taylor series about $1/z$,

$$\Xi^-(z) = \sum_{n=-\infty}^{-1} \xi_n z^n, \quad z \in \mathbb{B}^-.$$

We say that $\Xi(t) = \Xi^+(t) + \Xi^-(t) \in \{\widehat{E}_m^+, \widehat{E}_m^-\}$ if $\Xi^+(t) \in \widehat{E}_m^+$ and $\Xi^-(t) \in \widehat{E}_m^-$, $t \in \mathbb{T}$.

Consider the sequence of nonruin probabilities $\varphi = \{\varphi_n\}_{n \in \mathbb{Z}} \in \mathbf{c}$. Following Feller, [2, 6, 13, 14], going over to the sums in the repeated Stieltjes integral in (3), the problem (3), (4) in the arithmetic case can be rewritten in the discrete form

$$A\varphi := \varphi_u^+ + \varphi_u^- - \sum_{v=1}^{\infty} q_v \sum_{k=1}^{u+cv} p_k \varphi_{u+cv-k}^+ = 0, \quad u \in \mathbb{Z}, \quad (6)$$

$$\varphi_u^+ \nearrow 1 \quad \text{when } u \rightarrow +\infty, \quad \text{and} \quad \varphi_u^- \searrow 0 \quad \text{when } u \rightarrow -\infty. \quad (7)$$

Here we take into account that $\varphi_u^+ = 0$ for $u \in \mathbb{Z}^-$, and therefore summation in internal sum, as a matter of fact, is proceeded on a set on which $u+cv-w \in \mathbb{Z}^+$. As we will see, the second condition in (7) is a direct consequence of the first one. We will say that the vector $\varphi = \varphi^+ + \varphi^-$ belongs to the class $\{\mathbf{c}^+, \mathbf{c}_0^-\}$, if $\varphi^+ \in \mathbf{c}^+$ and $\varphi^- \in \mathbf{c}_0^-$. Thus, we shall seek the solution of the problem (6), (7) in the space $\varphi \in \{\mathbf{c}^+, \mathbf{c}_0^-\}$.

That problem, called compound binomial model, was earlier considered in the monograph of A. Mel'nikov [15] only for stationary process and only in the case when $c = 1$. This model can be interpreted as a model with inter-arrival times T_n having the shifted geometrical distribution with generating function

$$g_T(z) = \frac{qz}{1 - (1-q)z}, \quad 0 < q = \alpha < 1. \quad (8)$$

A. Mel'nikov reduces the solution of such problem to the solution of infinite system of linear algebraic equations with a Toeplitz matrix of coefficients using some recurrence relations. The solution of the problem is received in the terms of a generating function only for $u \in \mathbb{Z}^+$.

In the present paper, going over to generating functions and reducing the equation (6) to a Riemann boundary-value problem on the unit circle \mathbb{T} , we will see that this equation turns out to be nonnormal³ one-sided discrete Wiener–Hopf equation. Nonnormality of the equation (6) imposes some difficulties on the construction of its solvability theory and additional restrictions on the distributions of T_n and Z_n for the convergence of the projective method for the approximate solution of the problem. Using the Wiener–Hopf method, the solvability theory for this equation is constructed, on the base of which the applicability of the projective method is justified, and the conditions on distributions of the waiting times and claims are obtained for the convergence of the method in the spaces \mathbf{I}_1^+ and \mathbf{c}_0^+ . Illustrative example is given.

The paper is organized as follows. In Section 2, we investigate solvability of the problem (6), (7) reducing the equation (6) to a Riemann boundary-value problem which is solved by the factorization method, and obtain exact solution of the problem (6), (7) in terms of the generating functions. In Section 3, the formulas for nonruin probabilities in accompanying delayed renewal (stationary) process are also given in terms of generating functions. Solvability theory of the problem for delayed ordinary renewal process in arithmetic case slightly differs of that in the nonarithmetic case. It concerns the value $\varphi^s(0) = 1 - \frac{\alpha\mu}{c}$ in the nonarithmetic case, which, in arithmetic case for $c = 1$, is accepted for the value $u = -1$, and, in the general case, when $c \in \mathbb{N}$ the values φ_u^s , $u = -1, -2, \dots, -c$, are expressed in the terms of c th roots of unity. The formulas for the solution of the problem (6), (7) in the stationary case are also given. Earlier some results on given problem was announced by the author in [3]. Section 4 presents the results from [16] on the convergence of projective method for degenerated discrete Wiener–Hopf equation, and in Section 5, relying on these

³In functional analysis, the terms ‘non-Noetherian’, ‘singular’, ‘nonelliptic’, or ‘degenerated’ are also used.

results, we obtain sufficient conditions on the distributions of random variables T_n and Z_n for the convergence of the projective method in the spaces I_1^+ and c_0^+ . Approximation for ruin probabilities in delayed process is also given. Illustrative example is considered.

2. Solvability of the fundamental equation. Consider the sequence of nonruin probabilities $\varphi = \{\varphi_n\}_{n \in \mathbb{Z}} \in \{c^+, c_0^-\}$, and define the generating functions $\Phi(t) \in \{\widehat{c}^+, \widehat{c}_0^-\}$, $t \in \mathbb{T}$, for the sequence φ by the formula

$$\Phi(t) = \sum_{u=-\infty}^{+\infty} \varphi_u t^u, \quad t \in \mathbb{T},$$

considered as a generalized function on \mathbb{T} . In reality, we are interested by the convergent series

$$\Phi^+(z) = \sum_{u=0}^{+\infty} \varphi_u z^u, \quad z \in \mathbb{B}^+, \quad \Phi^-(z) = \sum_{u=-\infty}^{-1} \varphi_u z^u, \quad z \in \mathbb{B}^-.$$

Theorem 1. *If the conditions of Assumptions is fulfilled, then the symbol $A(t)$ of the operator A in (6) is given by the formula*

$$A(t) = 1 - g_T(t^{-c}) g_Z(t) \in W_1, \quad t \in \mathbb{T}. \quad (9)$$

Proof. Going over to generating functions, reduce the equation (6) to a Riemann boundary-value problem. We have

$$\Phi^+(t) + \Phi^-(t) - \sum_{u=-\infty}^{\infty} \left(\sum_{v=1}^{\infty} q_v \sum_{k=1}^{u+cv} p_k \varphi_{u+cv-k}^+ \right) t^u = 0, \quad t \in \mathbb{T}.$$

Interchanging order of summation in the sum term, we obtain

$$\begin{aligned} \sum_{u=-\infty}^{\infty} \left(\sum_{v=1}^{\infty} q_v \sum_{k=1}^{u+cv} p_k \varphi_{u+cv-k}^+ \right) t^u &= \sum_{v=1}^{\infty} q_v \sum_{k=1}^{\infty} p_k \sum_{u=k-cv}^{\infty} \varphi_{u-(k-cv)}^+ t^u = \\ &= \sum_{v=1}^{\infty} q_v t^{-cv} \sum_{k=1}^{\infty} p_k t^k \sum_{u=k-cv}^{\infty} \varphi_{u-(k-cv)}^+ t^{u-(k-cv)} = \\ &= \sum_{v=1}^{\infty} q_v t^{-cv} \sum_{k=1}^{\infty} p_k t^k \sum_{n=0}^{\infty} \varphi_n^+ t^n = g_T(t^{-ct}) g_Z(t) \Phi^+(t), \quad t \in \mathbb{T}. \end{aligned}$$

Thus, the equation (6) is reduced to the Riemann boundary-value problem [7, 8, 16],

$$(1 - g_T(t^{-c}) g_Z(t)) \Phi^+(t) = -\Phi^-(t), \quad t \in \mathbb{T}. \quad (10)$$

According to [7, 8, 16], the symbol $A(t)$ is the coefficient at $\Phi^+(t)$.

Note that we seek a solution to (10) at additional conditions: $\Phi^-(\infty) = 0$ and $\Phi^+(t)$ has simple pole with the residue $+1$ at $t_0 = 1$, since $\varphi_n \rightarrow 1$ when $n \rightarrow \infty$.

The belonging $A(t) \in W_1$ follows from the existence of finite means for the random variables T_n and Z_n . Thus, $A(t)$ is differentiable function on \mathbb{T} .

Theorem 1 is proved.

It is interesting to observe that $g_T(t^{-ct})g_Z(t)$ is the generating function for the random variable $U = Z_1 - cT_1$.

In this way, the equation (6) can be considered as one-sided Wiener–Hopf type discrete equation

$$\sum_{j=0}^{\infty} a_{k-j} \varphi_j^+ = 0, \quad k \in \mathbb{Z}^+, \quad (11)$$

the coefficient matrix of which, $\{a_{k-j}\}_{k,j \in \mathbb{Z}^+}$, is determined by the decomposition of the symbol $A(t)$ into Fourier series

$$A(t) = \sum_{j=-\infty}^{\infty} a_j t^j, \quad t \in \mathbb{T}.$$

Observe that at solution of the Wiener–Hopf equation, the introduction in consideration of the function $\Phi^-(z)$, $z \in \mathbb{B}^- \cup \mathbb{T}$, is a successful artificial method [8, 10, 17]. As in our case, the function $\Phi^-(z)$, $z \in \mathbb{B}^-$, bears, in addition, the completely definite probabilistic sense load as the component of generating function $\Phi(t)$, $t \in \mathbb{T}$.

Let $[\arg A(t)]_{\mathbb{T}}$ be the increment of the argument of $A(t)$ when t passes \mathbb{T} in positive direction (counter-clockwise) and

$$\text{ind}_{\mathbb{T}} A(t) := \frac{1}{2\pi} [\arg A(t)]_{\mathbb{T}}.$$

It should be noted the following properties of the symbol $A(t)$.

Theorem 2. *If conditions of Assumptions is fulfilled, then:*

- 1) the point $t_0 = 1$ is the unique zero of the symbol $A(t)$ on \mathbb{T} , and this zero is simple;
- 2) $|1 - A(t)| \leq 1$, $t \in \mathbb{T}$;
- 3) for the function $B(t) = A(t)/(1 - t) \in W$, we have $B(t) \neq 0$, $t \in \mathbb{T}$, and

$$\text{ind}_{\mathbb{T}} B(t) = -1; \quad (12)$$

- 4) for the symbol $A(t)$, the following factorization exists:

$$A(t) = A^+(t) \frac{1-t}{t} A^-(t), \quad t \in \mathbb{T}, \quad (13)$$

with

$$A^{\pm}(z) \neq 0, \quad z \in \mathbb{B}^{\pm} \cup \mathbb{T}, \quad A^{\pm}(t) \in W^{\pm}, \quad t \in \mathbb{T}, \quad A^+(1) = 1, \quad A^-(1) = -\frac{c}{\alpha} + \mu;$$

- 5) the unique solution of the problem (6), (7) is generated by the functions

$$\Phi^+(z) = \frac{1}{(1-z)A^+(z)}, \quad z \in \mathbb{B}^+ \cup \mathbb{T}, \quad \Phi^-(z) = -\frac{A^-(z)}{z}, \quad z \in \mathbb{B}^- \cup \mathbb{T}. \quad (14)$$

Proof. 1. Uniqueness of zero of $A(t)$, $t \in \mathbb{T}$, follows from the fact that the steps of the random variables T_n and Z_n are assumed to be equal 1 [6]. The first order of the root $t_0 = 1$ follows from the L'Hospital rule and the condition $c > \alpha\mu$:

$$\lim_{t \rightarrow 1} \frac{A(t)}{1-t} = \lim_{t \rightarrow 1} ((g_T(t^{-c}))' g_Z(t) + g_T(t^{-c}) g_Z'(t)) = -\frac{c}{\alpha} + \mu \neq 0.$$

So, we have to do with the Riemann problem (10) in exceptional case and the corresponding equation (6) is of nonnormal type [7, 8, 16].

2. The inequality $|1 - A(t)| \leq 1$, $t \in \mathbb{T}$, follows from the corresponding property of generating functions. Geometrical sense of this inequality is that the plot Γ of the symbol $\zeta = A(t)$, $t \in \mathbb{T}$, is situated in the unite disk with the center in the point $\zeta_1 = 1$ and this plot is tangent to the axis of ordinates at the point $\zeta_0 = 0$ in the plane of the complex variable ζ .

3. Since the point $t_0 = 1 \in \mathbb{T}$ is the simple root of $A(t) \in W_1$, it follows that $A(t)$ is representable in the form $A(t) = (t-1)B(t)$ with $B(t) \in W$ [16].

Next, the following equality holds:

$$\text{ind}_{\mathbb{T}} A(t) = -\frac{1}{2}. \quad (15)$$

Really,

$$(A(e^{i\tau}))' |_{\tau=0} = i(c/\alpha - \mu), \quad \tau \in (-\pi, \pi],$$

what means that the tangent vector to Γ at $\zeta_0 = 0$ is directed along the positive direction on the axis of ordinates when $\tau \in (-\pi, \pi]$ bypasses from $-\pi$ to π , i.e., in negative direction (clockwise) with respect to the domain bounded by the curve Γ . This implies (15), since the $A(t)$ at $t_0 = 1$ has the simple root, $A(t)$ has not other roots on \mathbb{T} , and the plot Γ of the symbol $A(t)$ is smooth.

On the other hand we have

$$\text{ind}_{\mathbb{T}} A(t) = \text{ind}_{\mathbb{T}} (1-t) + \text{ind}_{\mathbb{T}} B(t) = \frac{1}{2} + \text{ind}_{\mathbb{T}} B(t),$$

whence, in view of (15), we have (12).

4. Introduce into consideration the function $C(t) = A(t) \frac{t}{1-t}$ which has the following properties: $C(t) \neq 0$, $t \in \mathbb{T}$, $\text{ind}_{\mathbb{T}} C(t) = 0$, $C(t)$ is smooth on $\mathbb{T} \setminus \{1\}$ and continuous⁴ on \mathbb{T} .

Since the point $t_0 = 1$ is a simple root of the function $A(t) \in W_1$ on \mathbb{T} , the function $C(t)$ belongs to the space W , [16], Section 5.1.3, Corollary 1.4, and can be factored in the form

$$C(t) = C^+(t) \cdot C^-(t), \quad t \in \mathbb{T},$$

where $C^{\pm}(t) \in W^{\pm}$ and, consequently, are continuous, $C^{\pm}(z) \neq 0$, $z \in \mathbb{B}^{\pm} \cup \mathbb{T}$. As it is known, $C^{\pm}(t)$ are defined up to constant factors.

On the other hand, since $C(t)$ is smooth on $\mathbb{T} \setminus \{1\}$, the factors $C^{\pm}(t)$ can be expressed in terms of the Cauchy type integral on $t \in \mathbb{T} \setminus \{1\}$ [7]

⁴Smoothness is, generally speaking, violated at $t_0 = 1$, and $C(t)$ can be nothing but continuous at $t_0 = 1$.

$$C^{\pm}(t) = \exp \left[\frac{1}{2} \ln[C(t)] \pm \frac{1}{2\pi i} \text{v.p.} \int_{\mathbb{T}} \frac{\ln[C(\tau)]}{\tau - t} d\tau \right], \quad t \in \mathbb{T} \setminus \{1\}.$$

Whether these formulas work for the point $t_0 = 1$ remains the open problem, but since $C^{\pm}(t)$ are continuous on \mathbb{T} , the values $C^{\pm}(1)$ can be determined by continuity.

Setting

$$A^+(t) = \frac{C^+(t)}{C^+(1)}, \quad A^-(t) = C^-(t)C^+(1), \quad t \in \mathbb{T},$$

we obtain the existence of the unique factorization of the form (13) with $A^+(1) = 1$. The value $A^-(1) = -\frac{c}{\alpha} + \mu$ follows from the condition $A^+(1) = 1$ by the L'Hospital rule applied to the representation $A(t) = A^+(t) \frac{1-t}{t} A^-(t)$ at the point $t_0 = 1$.

5. Note that in [7], the Riemann problem (10) is written in classical form as

$$\Phi^+(t) = -\frac{1}{A(t)} \Phi^-(t), \quad t \in \mathbb{T},$$

and we seek the solution of the problem in exceptional case with coefficient $(-1/A(t))$ having a simple pole at $t_0 = 1$, and on the solution of the problem additional conditions are imposed: $\Phi^+(t)$ has a simple pole at $t_0 = 1$, $\Phi^-(t) \in W^-$ on \mathbb{T} and $\Phi^-(\infty) = 0$. Using the results of the Gakhov monograph [7] on exceptional case of the Riemann problem, observing that in our case (in notation of [7]) $\kappa = 1$, $p = 1$, we obtain that the Riemann problem (10) has one linear independent solution of the form

$$\Phi_1^+(z) = \frac{1}{(1-z)C^+(z)}, \quad z \in \mathbb{B}^+, \quad \Phi_1^-(z) = -\frac{C^-(z)}{z}, \quad z \in \mathbb{B}^-,$$

which generate one linear independent solution of the equation (6), belonging to the space $\{\mathbf{c}^+, \mathbf{c}_0^-\}$. Then the solution

$$\Phi^+(z) = \frac{1}{(1-z)A^+(z)}, \quad z \in \mathbb{B}^+, \quad \Phi^-(z) = -\frac{A^-(z)}{z}, \quad z \in \mathbb{B}^-,$$

generates unique solution of the problem (6), (7). Let us prove this.

Consider the series

$$\frac{1}{A^+(z)} = \sum_{n=0}^{\infty} a_n z^n, \quad z \in \mathbb{B}^+ \cup \mathbb{T}, \quad a_n = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{dt}{A^+(t)t^{n+1}}, \quad n \in \mathbb{Z}^+,$$

$$A^-(z) = \sum_{n=-\infty}^0 a_n z^n, \quad z \in \mathbb{B}^- \cup \mathbb{T}, \quad a_n = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{A^-(t) dt}{t^{n+1}}, \quad n \in \mathbb{Z}^- \cup \{0\}.$$

Note that from the equality $A^+(1) = 1$ we have

$$\frac{1}{A^+(1)} = \sum_{n=0}^{\infty} a_n = 1.$$

Expansion for $\Phi^+(z)$ is of the following form:

$$\Phi^+(z) = \frac{1}{(1-z)A^+(z)} = \sum_{n=0}^{\infty} \left(\sum_{k=0}^n a_k \right) z^n, \quad z \in \mathbb{B}^+.$$

Assuming⁵

$$\varphi_n = \sum_{k=0}^n a_k, \quad n \in \mathbb{Z}^+,$$

we obtain

$$\lim_{n \rightarrow +\infty} \varphi_n = \lim_{n \rightarrow +\infty} \sum_{k=0}^n a_k = 1.$$

Expansion for $\Phi^-(z)$ is of the form

$$\Phi^-(z) = -\frac{A^-(z)}{z} = -\sum_{n=-\infty}^0 a_n z^{n-1} = -\sum_{n=-\infty}^{-1} a_{n-1} z^n, \quad z \in \mathbb{B}^-.$$

Assume

$$\varphi_n = -a_{n-1}, \quad n \in \mathbb{Z}^-.$$

The series for $A^-(t)$, $t \in \mathbb{T}$, converges at $t_0 = 1$, since

$$A^-(1) = \sum_{n=-\infty}^0 a_n = -\frac{c}{\alpha} + \mu.$$

This implies that

$$\lim_{n \rightarrow -\infty} \varphi_n = 0.$$

Theorem 2 is proved.

Especially simply the Wiener–Hopf method works when the random variables T_n and Z_n have the rational generating functions. This happens to be the case for such distributions as uniform discrete, binomial, geometrical, negative binomial with entire exponent (all shifted in right in a reasonable way). In these cases, the symbol $A(t)$ is a rational function which can be factored in explicit form.

It may be noted that the equation (6) and the Riemann problem is not equivalent each other. For example, the solution of the Riemann problem (10)

⁵The formulas for the Taylor coefficients of analytic function in \mathbb{B}^+ , having simple pole on \mathbb{T} , can be considered as a Cauchy type integral. If the function $1/A^+(t)$ is, in addition, supposed Hölder-continuous at the point $t_0 = 1$, then the coefficients φ_n can be computed by the following formulas:

$$\varphi_n = -\frac{1}{2A^+(1)} - \frac{1}{2\pi i} \text{p.v.} \int_{\mathbb{T}} \frac{dt}{A^+(t)(1-t)t^{n+1}}, \quad n \in \mathbb{Z}^+, \quad A^+(1) = 1.$$

$$\left\{ \frac{1}{A^+(t)}, -\frac{(1-t)A^-(t)}{t} \right\}$$

does not generate a solution of the equation (6) since for the function

$$\Phi_1^-(z) = -\frac{(1-z)A^-(z)}{z}$$

the condition $\Phi_1^-(\infty) = 0$ is not valid. Thus, the homogeneous equation (6) has not any more solutions in arbitrary other space except the solution (14). It opens the way to the search of approximate solution of the problem (6), (7) reducing the equation (6) in the interval $[0, \infty)$ to an equivalent nonhomogeneous equation in the space \mathbf{c}_0^+ .

Example. Assume $c = 2$, and let T_n be shifted negative binomial random variable having generating function

$$g_T(t) = \frac{49}{64} \frac{t}{(1-t/8)^2}, \quad \alpha = \frac{7}{9}, \quad g_T(t^{-c}) = \frac{49t^2}{(8t^2-1)^2},$$

and Z_n be shifted negative binomial random variable with generating function

$$g_Z(t) = \frac{9}{16} \frac{t}{(1-t/4)^2}, \quad \mu = \frac{5}{3}.$$

Then

$$\begin{aligned} A(t) &= \frac{(t-1)(64t^5 - 448t^4 + 560t^3 + 247t^2 - 8t - 16)}{(8t^2-1)^2(t-4)^2} = \\ &= A^+(t) \frac{1-t}{t} A^-(t), \quad t \in \mathbb{T}, \end{aligned}$$

where

$$A^+(t) = \frac{(t-2.168948920)(t-5.160935390)}{.5404356589(t-4)^2},$$

$$A^-(t) = -34.58788218 \frac{t(t^2 + .5550511502t + .09918770222)(t - .2251668399)}{(8t^2-1)^2},$$

and

$$\Phi^+(z) = .5404356589 \frac{(z-4)^2}{(z-2.168948920)(z-5.160935390)(1-z)}, \quad z \in \mathbb{B}^+,$$

$$\Phi^-(z) = 34.58788218 \frac{(z^2 + .5550511502z + .09918770222)(z - .2251668399)}{(8z^2-1)^2}, \quad z \in \mathbb{B}^-.$$

Expanding the functions $\Phi^\pm(z)$ into series about $z^{\pm 1}$, respectively, we obtain the solution of the problem (6), (7). Corresponding probabilities are adduced in the Table at the end of the paper.

3. Delayed renewal processes. In addition to the sequence $\{T_n\}$, $n \in \mathbb{N}$, introduce a positive integer-valued random variable S_0 with some generating function $g_{S_0}(z) = \sum_{n=1}^{\infty} r_n z^n$ and consider the variables

$$S_n = S_0 + T_1 + T_2 + \dots + T_n$$

called the renewal epochs [6]. The renewal process $\{S_n\}$ is called *pure* if $S_0 = 0$ and *delayed* otherwise. The expected number of renewal epochs on $[0, n]$ equals

$$V(n) = \sum_{k=0}^{\infty} \mathbf{P}\{S_k \leq n\}, \quad n \in \mathbb{Z}^+,$$

and has the generating function

$$g_V(z) = \frac{g_{S_0}(z)}{1-z} \left(1 + \sum_{k=1}^{\infty} [g_T(z)]^k \right) = \frac{g_{S_0}(z)}{(1-z)(1-g_T(z))}, \quad z \in \mathbb{B}^+. \quad (16)$$

It is known, [6], that

$$\lim_{n \rightarrow \infty} [V(n+1) - V(n)] = \alpha.$$

It follows from this that $V(n) \sim \alpha n$ as $n \rightarrow \infty$. It is natural to ask whether $g_{S_0}(z)$ can be chosen as to get the identity $V(n) = \alpha n$, $n \in \mathbb{Z}^+$, meaning a constant renewal rate.

Noticing that the generating function for the sequence $\{\alpha n\}$, $n \in \mathbb{Z}^+$, is given by the formula

$$g_{\{\alpha n\}}(z) = \alpha \sum_{n=1}^{\infty} n z^n = \frac{\alpha z}{(1-z)^2}, \quad z \in \mathbb{B}^+,$$

and equating $g_V(z) = g_{\{\alpha n\}}(z)$, we obtain

$$g_{S_0}(z) = \frac{\alpha z(1-g_T(z))}{1-z}, \quad z \in \mathbb{B}^+ \cup \mathbb{T}, \quad (17)$$

which is the generating function of a proper probability distribution⁶ and so the answer is affirmative:

With the initial random variable S_0 having generating function (17) the renewal rate is constant, $V(n) = \alpha n$, $n \in \mathbb{Z}^+$.

The following statement takes place:

The ordinary renewal process in arithmetic case is stationary if and only if the inter-arrival times T_n have the shifted geometrical distribution with the generating function (8).

This shifted geometrical distributions is analog of exponential distribution for the nonarithmetic case.

For the problem (6), (7), we consider *the accompanying delayed stationary renewal process* $\{S_n\}_{n \in \mathbb{Z}^+}$ with generating function (17) for S_0 . Then the generating function for the nonruin probabilities in such process is built as follows:

$$\Phi_s(t) = g_{S_0}(t^{-c}) g_Z(t) \Phi^+(t) = \frac{\alpha t^{-c}(1-g_T(t^{-c})) g_Z(t) \Phi^+(t)}{1-t^{-c}} =$$

⁶Using the L'Hospital rule, we can prove that $g_{S_0}(z)$ is extendable on \mathbb{T} with $g_{S_0}(1) = 1$.

$$= \frac{\alpha(1 - g_T(t^{-c})) g_Z(t) \Phi^+(t)}{t^c - 1}, \quad t \in \mathbb{T}, \quad (18)$$

similarly as it is done in the nonarithmetic case.

Using (10), we can exclude $g_T(t^{-c})$ from the last expression,

$$\Phi_s(t) = \frac{\alpha(1 - g_Z(t))\Phi^+(t) + \alpha\Phi^-(t)}{t^c - 1}, \quad t \in \mathbb{T}. \quad (19)$$

Let

$$\Phi_s(t) = \sum_{u=-\infty}^{\infty} \varphi_u^s t^u, \quad t \in \mathbb{T}.$$

Our immediate task is obtaining formulas for $\Phi_s^\pm(z)$, $z \in \mathbb{B}^\pm$. For this end we must solve the jump problem for the function $\Phi_s(t)$ [7]

$$\Phi_s(t) = \Phi_s^+(t) + \Phi_s^-(t) \in \{\widehat{\mathbf{c}}^+, \widehat{\mathbf{c}}_0^-\}, \quad t \in \mathbb{T},$$

where $\Phi_s^+(t)$ has simple poles at the c th roots of unity, $\Phi_s^-(t) \in W^-$, $t \in \mathbb{T}$, and $\Phi_s^-(\infty) = 0$.

Rewrite the formula (19) in the equivalent form

$$\Phi_s(t) = \frac{\alpha(1 - g_Z(t))\Phi^+(t) + p_{c-1}(t)}{t^c - 1} + \frac{\alpha\Phi^-(t) - p_{c-1}(t)}{t^c - 1}, \quad t \in \mathbb{T}, \quad (20)$$

where $p_{c-1}(t)$ is a specially chosen polynomial

$$p_{c-1}(t) = p_{c-1}t^{c-1} + p_{c-2}t^{c-2} + \dots + p_0$$

with the coefficients p_k , $k = 0, 1, \dots, c-1$, satisfying the system of linear equations with the Vandermonde matrix,

$$p_{c-1}(\varepsilon_k) = \alpha\Phi^-(\varepsilon_k), \quad k = 0, 1, \dots, c-1, \quad (21)$$

where ε_k are the c th roots of unity.

Then from (20) we obtain

$$\Phi_s^+(t) = \frac{\alpha(1 - g_Z(t))\Phi^+(t) + p_{c-1}(t)}{1 - t^c} \in \widehat{\mathbf{c}}^+, \quad t \in \mathbb{T}, \quad (22)$$

$$\Phi_s^-(t) = \frac{\alpha\Phi^-(t) - p_{c-1}(t)}{1 - t^c} \in \widehat{\mathbf{c}}_0^-, \quad t \in \mathbb{T}. \quad (23)$$

Representing $\Phi_s^-(t)$ in the form

$$\Phi_s^-(t) = -\frac{\alpha\Phi^-(t)}{t^c} \frac{1}{1 - \frac{1}{t^c}} + \frac{p_{c-1}(t)}{t^c} \frac{1}{1 - \frac{1}{t^c}}, \quad t \in \mathbb{T}, \quad (24)$$

we see that $p_{c-1}(t)$ is representable in the form

$$p_{c-1}(t) = \varphi_{-1}^s t^{c-1} + \varphi_{-2}^s t^{c-2} + \dots + \varphi_{-c}^s$$

with the coefficients $\varphi_{-k}^s = p_{c-k}$, $k = 1, 2, \dots, c$, which are the first c coefficients of the expansion of $\Phi_s^-(z)$ into series about $1/z$, $z \in \mathbb{B}^-$.

In the simplest case $c = 1$, from the system (21), we obtain the relation

$$\varphi_{-1}^s = \alpha \Phi^-(1) = \alpha \left(\frac{c}{\alpha} - \mu \right) = 1 - \alpha \mu$$

that is the analog of the known equality

$$\varphi(0) = 1 - \frac{\alpha \mu}{c}$$

for the nonruin probability $\varphi(u)$ in the nonarithmetic case at $c = 1$ [2, 13, 14].

Since the function $\Phi_s^-(t)$ is continuous on \mathbb{T} , the Taylor-series expansion for it is not the problem.

The function $\Phi_s^+(z)$ has the simple poles at the c th roots of unity. Therefore the expansion for $\Phi_s^+(z)$ is constructed in two steps. Since the numerator of (22),

$$A_s^+(t) = \alpha(1 - g_Z(t)) \Phi^+(t) + p_{c-1}(t)$$

is continuous on \mathbb{T} , we construct the expansion for it

$$A_s^+(t) = \sum_{n=0}^{\infty} a_n^s z^n, \quad z \in \mathbb{B}^+ \cup \mathbb{T},$$

and then the expansion for $\Phi_s^+(z)$ is constructed by the expansion of the ratio⁷

$$\frac{A_s^+(z)}{1 - z^c} = \frac{\sum_{n=0}^{\infty} a_n^s z^n}{1 - z^c} = \sum_{n=0}^{\infty} \varphi_n^s z^n.$$

Example (continued). The system (21) for φ_{-1}^s and φ_{-2}^s in our case has the form

$$\begin{aligned} \varphi_{-1}^s + \varphi_{-2}^s &= \alpha \Phi^-(1), \\ -\varphi_{-1}^s + \varphi_{-2}^s &= \alpha \Phi^-(-1), \end{aligned}$$

the solution of which is $\varphi_{-1}^s = .5348541581$, $\varphi_{-2}^s = .1688495471$. Then

$$\Phi_s^+(z) = \frac{\alpha(1 - g_Z(z)) \Phi^+(z) + .5348541581z + .1688495471}{1 - z^2},$$

$$\Phi_s^-(z) = \frac{.08000291975z + .01274595223}{z^2 + .7071067812z + .125} + \frac{.4548512383z - .1089541931}{z^2 - .7071067812z + .125}.$$

Corresponding probabilities are adduced in Table at the end of the paper.

⁷If the function $A_s^+(t)$ is supposed Hölder-continuous at the c th roots of unity on \mathbb{T} , then the coefficients φ_n^s can be computed by the formulas

$$\varphi_n^s = -\frac{\sum_{k=0}^{c-1} A_s^+(\varepsilon_k)}{2\varepsilon_k^{n+1}} - \frac{1}{2\pi i} \text{p.v.} \int_{\mathbb{T}} \frac{A_s^+(t) dt}{(1-t)t^{n+1}}, \quad n \in \mathbb{Z}^+.$$

4. Nonruin probabilities for stationary process. Assume now that the inter-arrival times T_n have the shifted geometrical distribution with the generating function (8), i.e., *the renewal process is stationary* and hence $\Phi_s(t) = \Phi(t)$. Then from (22) and (23) we obtain the equations for $\Phi_s^\pm(z)$:

$$\Phi_s^+(z) = \frac{\alpha(1 - g_Z(z))\Phi_s^+(z) + p_{c-1}(z)}{1 - z^c}, \quad z \in \mathbb{B}^+, \quad (25)$$

$$\Phi_s^-(z) = \frac{\alpha\Phi_s^-(z) - p_{c-1}(z)}{1 - z^c}, \quad z \in \mathbb{B}^-, \quad (26)$$

whence

$$\Phi_s^+(z) = \frac{\varphi_{-1}^s z^{c-1} + \varphi_{-2}^s z^{c-2} + \dots + \varphi_{-c}^s}{1 - z^c - \alpha(1 - g_Z(z))}, \quad z \in \mathbb{B}^+, \quad (27)$$

$$\Phi_s^-(z) = \frac{\varphi_{-1}^s z^{c-1} + \varphi_{-2}^s z^{c-2} + \dots + \varphi_{-c}^s}{1 - \alpha - z^c}, \quad z \in \mathbb{B}^-. \quad (28)$$

If we put $z = 0$ in (27), we obtain the formula⁸ for φ_0^s ,

$$\varphi_0^s = \Phi_s^+(0) = \frac{\varphi_{-c}^s}{1 - \alpha}.$$

In the simplest case when $c = 1$, we obtain⁹

$$\Phi_s^+(z) = \frac{1 - \alpha\mu}{1 - z - \alpha(1 - g_Z(z))}, \quad z \in \mathbb{B}^+ \cup \mathbb{T}, \quad (29)$$

and

$$\varphi_0^s = \frac{1 - \alpha\mu}{1 - \alpha}.$$

This formula was derived by A. Mel'nikov in [15] by other method.

5. The projective method. Since to construct explicitly the factorization of the symbol frequently is rather difficultly, the approximate methods have particular importance. For approximate solution of fundamental equation of risk theory in arithmetic case we propose to use the projective method (called in the literature also finite section method or natural reduction method) [16].

Consider discrete Wiener – Hopf equation in general case,

$$\sum_{j=0}^{\infty} a_{k-j} \xi_j = \eta_k^+, \quad k \in \mathbb{Z}^+. \quad (30)$$

Here it is assumed that

- 1) $\sum_{j=-\infty}^{\infty} |a_j| < \infty$,
- 2) $\eta^+ = \{\eta_j^+\}_{j \in \mathbb{Z}^+}$ is given and such that the equation (30) is solvable,
- 3) $\xi^+ = \{\xi_j\}_{j \in \mathbb{Z}^+}$ is the unknown vector in the space E^+ , where E^+ is any of the spaces I_1^+ or \mathbf{c}_0^+ .

⁸Remind that $\alpha = q < 1$ and $g_Z(0) = 0$.

⁹In [15] this formula is written in some cumbersome form.

For discrete Wiener–Hopf equation (30) the projective method consists in following [10–12, 16, 17]:

The equation (30) is replaced by the system of $n + 1$ equations with $n + 1$ unknown

$$\sum_{j=0}^n a_{k-j} \tilde{\xi}_j = \eta_k^+, \quad k = 0, 1, \dots, n, \quad (31)$$

and solution of this curtained (reduced) system,

$$\tilde{\xi}^{(n)} = \{\tilde{\xi}_0, \tilde{\xi}_1, \dots, \tilde{\xi}_n\},$$

is considered as the approximation to the solution of initial equation (30).

If the condition of normality on the symbol,

$$A(t) = \sum_{j=-\infty}^{\infty} a_j t^j \neq 0, \quad t \in \mathbb{T},$$

is not fulfilled (as in our case), the projective process (31) in the space \mathbf{I}_1^+ , generally, already does not converge any more for all that $\eta^+ \in \mathbf{I}_1^+$ for which the equation (30) is solvable [16]. We are interested by the conditions at which the projective method works for the problem (6), (7).

Let $t_0 = 1$ be unique zero of the symbol $A(t)$ on \mathbb{T} and this zero is simple. Put

$$C(t) = A(t) (t^{-1} - 1)^{-1}, \quad t \in \mathbb{T}.$$

The following theorem is the simple consequence of the Theorem 0.4 proved in [16].

Theorem 3 (convergence in \mathbf{I}_1^+). *Let $C(t) \in W_1$.*

(a) *If $C(t) \neq 0$ and $\text{ind}_{\mathbb{T}} C(t) = 0$, then the projective method (31) converges in the space \mathbf{I}_1^+ for all $\eta^+ \in \mathbf{I}_{1,1}^+$.*

(b) *In addition, if $C(t) \in W_{1+\delta}$ and $\eta^+ \in \mathbf{I}_{1,1+\delta}^+$, $\delta > 0$, then it is valid the estimate*

$$\|\xi - \tilde{\xi}^{(n)}\|_{\mathbf{I}_1^+} = O(n^{-\delta}), \quad n \rightarrow \infty.$$

The following theorem is the simple consequence of Theorem 4.1 and Theorem 4.2, Ch. 11 [16].

Theorem 4 (convergence in \mathbf{c}_0^+). *Let $C(t) \in W$, $C(t) \neq 0$.*

(a) *If $\text{ind}_{\mathbb{T}} C(t) = 0$, then the projective method for the system (30) converges in the space \mathbf{c}_0^+ .*

(b) *In addition, if $C(t) \in W_{1+\delta}$, and $\eta^+ \in \mathbf{c}_{1+\delta}^+$, $\delta > 0$, then it is valid the estimate*

$$\|\xi - \tilde{\xi}^{(n)}\|_{\mathbf{c}_0^+} = O(n^{-\delta}), \quad n \rightarrow +\infty.$$

6. Convergence of the projective method for the fundamental equation. Here it is convenient to introduce the ruin probability

$$\psi_u^+ = 1 - \varphi_u^+, \quad u \in \mathbb{Z}^+.$$

Then the problem (6), (7) (only for $u \in \mathbb{Z}^+$) can be rewritten in the form of inhomogeneous equation for ψ_u^+ :

$$\psi_u^+ - \sum_{v=1}^{\infty} q_v \sum_{k=1}^{u+cv} p_k \psi_{u+cv-k}^+ = 1 - \sum_{v=1}^{\infty} q_v \sum_{k=1}^{u+cv} p_k, \quad u \in \mathbb{Z}^+, \quad (32)$$

$$\lim_{u \rightarrow +\infty} \psi_u^+ = 0. \quad (33)$$

It follows from results of S.Prössdorf [16] that the problem (32), (33) has an unique solution in the spaces \mathbf{c}_0^+ and \mathbf{I}_1^+ . Using Theorems 3 and 4, we can receive the conditions on the random variables T_n and Z_n , and consequently, on the symbol $A(t)$ and on the right part $\eta^+ = \{\eta_u^+\}_{u \in \mathbb{Z}^+}$ of the equation (32) with

$$\eta_u^+ = 1 - \sum_{v=1}^{\infty} q_v \sum_{k=1}^{u+cv} p_k, \quad u \in \mathbb{Z}^+, \quad (34)$$

for the convergence of projective process (31) for the problem (32), (33) in the spaces \mathbf{I}_1^+ and \mathbf{c}_0^+ .

For this end we must clarify the probabilistic nature of the right part η^+ and its properties.

Theorem 5. *The right part η^+ of the equation (32) is the right tail of the random variable $U = Z_1 - cT_1$.*

Proof is obvious.

Corollary 1. *If $A(t) \in W_k$, then $\eta^+ \in \mathbf{I}_{1,k-1}^+$.*

Proceeding from Theorem 5, Theorem 3 in the case of the problem (32), (33) can be formulated in the following form:

Theorem 6 (convergence in \mathbf{I}_1^+). *Let the conditions of Assumptions be fulfilled.*

(a) *In addition, if the random variables T_n and Z_n have finite variances then the projective process (31) for the problem (32), (33) converges in the space \mathbf{I}_1^+ .*

(b) *If the random variables T_n and Z_n have finite $(2 + \delta)$ th moments, $\delta > 0$, then it is valid the estimate*

$$\|\psi - \tilde{\psi}^{(n)}\|_{\mathbf{I}_1^+} = O(n^{-\delta}), \quad n \rightarrow +\infty.$$

Proof. The existence of finite variances for T_n and Z_n implies $A(t) \in W_2$, $C(t) \in W_1$, and simultaneously $\eta^+ \in l_{1,1}^+$, $t \in \mathbb{T}$ [16] (Ch. 5, Theorem 1.7). Besides we have $C(t) \neq 0$, $\text{ind}_{\mathbb{T}} C(t) = 0$. By Theorem 4 this implies the statement (a).

It follows from the existence of the finite $(2 + \delta)$ th moments for the random variables T_n and Z_n that $A(t) \in W_{2+\delta}$, $C(t) \in W_{1+\delta}$ and $\eta^+ \in \mathbf{I}_{1,1+\delta}^+$. By Theorem 4 this implies the statement (b).

Theorem 6 is proved.

Theorem 4 for the problem (32), (33) can be formulated as follows:

Theorem 7 (convergence in \mathbf{c}_0^+). *Let the conditions of Assumptions be fulfilled.*

(a) *The projective method (31) for the problem (32), (33) converges in the space \mathbf{c}_0^+ .*

(b) *In addition, if the random variables T_n and Z_n have finite $(2 + \delta)$ th moments, $\delta > 0$, then it is valid the estimate*

$$\|\psi - \tilde{\psi}^{(n)}\|_{\mathbf{c}_0^+} = O(n^{-\delta}), \quad n \rightarrow +\infty.$$

Proof. (a) From existence of finite means for the random variables T_n and Z_n it follows that $A(t) \in W_1$, and, consequently, $C(t) \in W$ [16]. By Theorem 4 this implies the statement (a).

The statement (b) is proved in a way analogous to that used for the proof of the statement (b) of Theorem 6.

Theorem 7 is proved.

Note that the projective method converges in the space c_0^+ without superposition of somehow additional conditions on the random variables T_n and Z_n except the conditions of Assumptions. Such quality of projective method removes the problem of “large” claims [5].

Exact and approximate nonruin probabilities from Example

u	φ_u	$\tilde{\varphi}_u$	φ_u^s	$\tilde{\varphi}_u^s$
-10	.1233318169e-3	.1232544182e-3	.1121618097e-3	.1121656751e-3
-9	.5508161983e-3	.5507966082e-3	.5029119846e-3	.5029293048e-3
-8	.8270422905e-3	.8265289772e-3	.7554169265e-3	.7554429513e-3
-7	.3568781788e-2	.3568646637e-2	.3278631163e-2	.3278744079e-2
-6	.5339440364e-2	.5336180872e-2	.4908314995e-2	.4908484016e-2
-5	.2184827192e-1	.2184736324e-1	.2027173160e-1	.2027242975e-1
-4	.3250033924e-1	.3248103939e-1	.3018635667e-1	.3018739538e-1
-3	.1211703164	.1211644271	.1145153112	.1145192552
-2	.1782812446	.1781810546	.1688495471	.1688553493
-1	.5404356589	.5403995867	.5348541581	.5348725793
0	.7724782018	.7725048388	.7696659284	.7696924
1	.8920702933	.8921010541	.8907273961	.89075812
2	.9496515748	.9496843208	.9490234591	.949056191
3	.9766729524	.9767066317	.9763816202	.976415293
4	.9892229558	.9892570645	.9890882995	.9891224045
5	.9950269424	.9950612554	.9949647935	.9949991059
6	.9977063305	.9977407385	.9976776636	.9977120670
7	.9989423371	.9989767060	.9989291175	.9989635043
8	.9995123308	.9995470499	.9995062351	.9995409313
9	.9997751528	.9998084191	.9997723421	.9998057237
10	.9998963325	.9999325190	.9998950366	.9999315536

Example (continued). Since $\Phi(t) \in \{\widehat{\mathbf{c}}^+, \widehat{\mathbf{c}}_0^-\}$, $t \in \mathbb{T}$, the approximation $\widetilde{\Phi}_{(n)}^+(t) \in \widehat{\mathbf{c}}^+$ for $\Phi^+(t)$ is selected in the form

$$\widetilde{\Phi}_{(n)}^+(t) = \sum_{k=0}^n \left(1 - \widetilde{\psi}_k^{(n)}\right) t^k + \frac{t^{n+1}}{1-t}, \quad t \in \mathbb{T}.$$

Then the generating function $\widetilde{\Phi}_{(n)}^-(t)$ for the vector $\widetilde{\varphi}_{(n)}^-$ is constructed by the formula¹⁰

$$\widetilde{\Phi}_{(n)}^-(t) = \left(-A(t)\widetilde{\Phi}_{(n)}^+(t)\right)^-, \quad t \in \mathbb{T},$$

and the approximate generating function for the approximate probabilities $\widetilde{\varphi}^s$ in delayed renewal process is constructed using the formulas of Section 3, in which the exact solution $\Phi^+(t)$ should be replaced by the approximate solution $\widetilde{\Phi}_{(n)}^+(t)$.

The comparison of exact and approximate values of nonruin probabilities in the Table shows sufficiently accuracy of the method.

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¹⁰By $(\cdot)^-$ we mean a projection on the subspace of the boundary values $\Phi^-(t)$, $t \in \mathbb{T}$, of the functions analytical in \mathbb{B}^- , $\Phi^-(\infty) = 0$.