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ON POLYMER EXPANSIONS FOR GENERALIZED GIBBS LATTICE SYSTEMS OF OSCILLATORS WITH TERNARY INTERACTION

ПРО ПОЛІМЕРНІ РОЗКЛАДИ ДЛЯ УЗАГАЛЬНЕНИХ ГІББСІВСЬКИХ ҐРАТКОВИХ СИСТЕМ ОСЦИЛЯТОРІВ З ТЕРНАРНОЮ ВЗАЄМОДІЄЮ

We propose a new short proof of the convergence of high-temperature polymer expansions for the thermodynamic limit of canonical correlation functions of classical and quantum Gibbs lattice systems of oscillators interacting via pair and ternary potentials and nonequilibrium stochastic systems of oscillators interacting via a pair potential with Gibbsian initial correlation functions.

Запропоновано нове коротке доведення збіжності високотемпературних полімерних розкладів термодинамічної границі канонічних кореляційних функцій ґраткових класичних та квантових гіббсівських систем осциляторів, що взаємодіють завдяки парному та тернарному потенціалам, а також нерівноважних стохастичних систем осциляторів, які взаємодіють завдяки парному потенціалу з гіббсівськими початковими кореляційними функціями.

We consider in the canonical ensemble generalized Gibbs systems on the lattice $\mathbb{Z}^d$, whose sites index variables from the measure space (Ω, P^0), with the potential energy U_c, which is a measurable function, expressed through the one-particle (external) potential $u(\omega)$ and the two-particle complex-valued potential $u_{x-y}(\omega_x, \omega_y)$

$$U_c(\omega_\Lambda) = \sum_{x\in\Lambda} u(\omega_x) + \sum_{x,y\in\Lambda} u_{x-y}(\omega_x, \omega_y),$$

where Λ is a finite set with the cardinality $|\Lambda|$. P^0 is a positive σ-finite measure (it is finite on compact sets if Ω is a complete metric space) and $P^0(\Omega) = \infty$.

The Gibbs canonical correlation functions are given by

$$\rho^\Lambda(\omega_X) = Z_\Lambda^{-1} \int e^{-\beta U_c(\omega_\Lambda)} P^0(d\omega_{\Lambda\setminus X}), \quad Z_\Lambda = \int e^{-\beta U_c(\omega_\Lambda)} P^0(d\omega_\Lambda) > 0, \quad \beta \in \mathbb{R}^+.$$

Here β is an inverse temperature, the integration is performed over $\Omega^{|\Lambda\setminus X|}$, $\Omega^{|\Lambda|}$, respectively, and $P^0(d\omega_X) = \prod_{x\in X} P^0(d\omega_x)$.

In [1, 2] we showed that three different choices of the measure space Ω and the potentials correspond to Gibbs classical, Gibbs quantum systems of lattice oscillators, interacting via the pair

$$u^0_{x-y}(q_x, q_y) = J_0(|x-y|) u_0(q_x, q_y)$$

and the factorized ternary

$$u_{x,y,z}(q_x, q_y, q_z) = J_1(|x-y|) J_1(|y-z|) u_1(q_x, q_y) u_1(q_y, q_z)$$

potentials such that $J_s \in L^1(\mathbb{Z}^d)$ and nonequilibrium gradient stochastic systems of lattice oscillators interacting via the pair potential $2u^0_{x-y}$ (their initial states are Gibbsian with a pair potential). The correlations functions of the latter systems are represented as correlation functions of a lattice diffusion Gibbs path oscillator system with pair and ternary interaction.The potential energy of the stochastic oscillators, which is present in the Smoluchowski equation (the forward Kolmogorov equation), is a sum of two terms generated by an external and pair potentials. The considered potentials are polynomials in oscillator variables $q_x \in \mathbb{R}$. The external potentials are bounded from below even polynomials $u(q)$ and $u_0(q), q \in \mathbb{R}$, of the $2n$ and $2n^0$-the degrees for classical, quantum and stochastic systems, respectively. Note that we used in [2] the following notations: $u^0(q) = 2u_0(q)$ and $u^0(q, q') = 2u_0(q, q')$.

The spaces Ω for these systems are represented as $\Omega^0 \times \Omega_*$ ($\omega \in \Omega\colon \omega = (\omega^0, \omega_*), \omega^0 \in \Omega^0, \omega_* \in \in \Omega^*$) such that $\Omega^0 = \Omega_* = \mathbb{R}$ for classical Gibbs systems and $\Omega^0 = \mathbb{R} \times C(\mathbb{R}^+)$, $\Omega_* = C(\mathbb{R}^+)$ for the remaining systems, where $C(\mathbb{R}^+)$ is the space of continuous functions on $\mathbb{R}^+$.The measure P^0 is factorized on $\Omega^0 \times \Omega_*$, i.e., $P^0 = \tilde{P} \otimes P_0$ and P_0 is a probability measure. Besides the following equalities are true: $\tilde{P}(dq) = dq$, $P_0(ds) = (\sqrt{2\pi})^{-1} e^{-s^2/2} ds$ for Gibbs classical systems, $\tilde{P}(dw) = dq P^{\beta}_{q,q}(dw)$, $\tilde{P}(dw) = dq P_q(dw)$ for Gibbs quantum and stochastic systems, respectively, where $P_q(dw)$ is the Wiener measure and $P^{\beta}_{q,q}(dw)$ is the conditional Wiener measure concentrated on continuous paths starting and arriving into a point q at a "time" β (see the Remark 1 in the end of the paper).

The goal of this paper is to find the thermodynamic limit $\Lambda \to \mathbb{Z}^d$ of the correlation functions and simplify the technique proposed for that in [1, 2] based on proving of a convergence of the polymer high-temperature expansion for the correlation functions (see the Remark 2). The expansion is given by

$$\rho_{\Lambda,X}(\omega_X) = \bar{z}^{|X|} e^{\beta \sum\limits_{x \in X} u(\omega_x)} \rho^{\Lambda}(\omega_X) = \sum_{Y \in \Lambda} \bar{\rho}_{\Lambda}(X \cup Y) \int P(d\omega_Y) F_{\omega_X}(\omega_Y),$$

where $F_{\omega_X}(\omega_Y)$ are the truncated Boltzmann functions satisfying the KS recursion relation with the interaction potential energy, $\bar{\rho}_{\Lambda}(X) = \bar{z}^{|X|} Z_{\Lambda \setminus X} Z_{\Lambda}^{-1}$ and

$$P(d\omega_Y) = \bar{z}^{-|Y|} \prod_{y \in Y} e^{-\beta u(\omega_y)} P^0(d\omega_y), \quad \bar{z} = \int e^{-\beta u(\omega)} P^0(d\omega).$$

To perform the thermodynamic limit one has to guarantee the absolute convergence of this expansion uniformly in Λ. The important role in this is played by the superstability condition for he real part of the pair potential

$$|\operatorname{Re} u_{x-y}(\omega_x, \omega_y)| \le \frac{1}{2} J'(|x-y|)(v(\omega_x) + v(\omega_y)), \qquad \int e^{\gamma\beta v^{1+\zeta}(\omega) - \beta u(\omega)} P^0(d\omega) < \infty, \qquad (1)$$

where $|x|$ is the Euclidean norm of x, J', v, $\zeta \ge 0$, $\gamma > 0$, $||J'||_1 = \sum_x J'_x < \infty$ ($J' \in \mathbb{Z}^d$) and the summation is performed over $\mathbb{Z}^d$. To formulate our results we need the following notations:

$$B = \operatorname*{ess\,sup}_{\omega} \sum_x b_x(\omega), \quad D = \int e^{\beta \tilde{v}(\omega)} P(d\omega),$$

$$b_x(\omega) = e^{-\beta(\tilde{v}(\omega) - ||J'||v(\omega))||\sqrt{J}||_1^{-1} \sqrt{J(|x|)}} \int e^{\beta \tilde{v}(\omega_x)} |e^{-\beta u_x(\omega, \omega_x)} - 1| P(d\omega_x).$$

Theorem 1. *Let* (1) *hold,* $J(|x|) \geq 0$, $||\sqrt{J}||_1 < \infty$ *and*

$$\lim_{\beta \to 0} B = 0, \quad \lim_{\beta \to .0} BD = 0.$$

Then for sufficiently small β *there exists the thermodynamic limit* $\bar{\rho}$ *of* $\bar{\rho}_\Lambda$, *the polymer cluster expansion for the correlation functions* $\rho^\Lambda(\omega_X)$ *converges absolutely uniformly in* Λ, *their thermodynamic limit is also represented by the polymer cluster expansion in which the summation is performed over* $\mathbb{Z}^d$ *and the thermodynamic limit of* $\bar{\rho}$ *is substituted instead of* $\bar{\rho}_\Lambda$. *Moreover there exist positive constants* c, C *such that* $c^{-|X|} \exp\left\{-\beta \sum_{x \in X} \tilde{v}(\omega_x)\right\} |\rho^\Lambda(\omega_X)| \leq C$.

The proof of this theorem is standard [3, 4]: at first one proves the bound

$$\sum_{Y:|Y|=m} \int P(d\omega_Y) |F_{\omega_X}(\omega_Y)| \leq e^{|X|}(eB)^m e^{\beta \sum\limits_{x \in X} \tilde{v}(\omega_x)}$$

and then shows that the polymer correlation functions $\bar{\rho}_\Lambda$, whose sequence satisfies the polymer KS equation, are appropriately bounded. Note that due to the law of conservation of probability in the stochastic systems $\bar{\rho}_\Lambda$ coincides with the same expression for the Gibbs initial distribution generated by a pair potential and an external field, that is, the existence of its thermodynamic limit follows from the results of [3].

We have to choose such $\tilde{v}, J$ that the condition of the Theorem 1 is satisfied. The following lemma gives the first step in this direction.

Lemma 1. *Let* (1) *hold,* $\zeta > 0$,

$$\operatorname{Im} u_{x-y}(\omega_x, \omega_y) = \tilde{J}(|x-y|)\phi(\omega_x, \omega_y), \quad ||\tilde{J}||_1 < \infty, \tag{2}$$

and

$$\tilde{v}(\omega) = (v^{1+\zeta}(\omega) + \beta^{r-1} b(\omega))\gamma, \quad b^2(\omega) = \int |\phi(\omega, \omega')|^2 P(d\omega'), \quad 0 < r < 1. \tag{3}$$

Let also $D(\gamma)$ *coincide with* D *and* J *in the expression for* B *in the Theorem* 1 *be given by* $J = J' + \tilde{J}$. *Then there exist positive constants* a_1, a_2 *such that*

$$B \leq a_1 \beta^{1-r} \sqrt{D(2\gamma)} + a_2 \beta^{\frac{\zeta}{1+\zeta}} D(2\gamma + 2^{-1}|J'|_0).$$

The proof of the lemma will be given in the end of the paper.

Corollary. *The conditions for* B, D *in the Theorem* 1 *are satisfied if* $D(\gamma)$ *is bounded in* $\beta > 0$ *in a neighborhood of zero.*

In the three considered cases the real part of the pair potential and the external field u depend only on ω^0. This fact and the fact that the measure P^0 is factorized on $\Omega^0 \times \Omega_*$ yield the result

$$D = \int e^{\gamma \beta v^{1+\zeta}(\omega^0)} D_*(\omega^0) P'(d\omega^0), \quad P'(d\omega^0) = \left(\int e^{-\beta u(\omega^0)} \tilde{P}(d\omega^0)\right)^{-1} e^{-\beta u(\omega^0)} \tilde{P}(d\omega^0),$$

where

$$D_*(\omega^0) = \int e^{\beta^r \gamma b(\omega^a, \omega_*)} P_0(d\omega_*).$$

Now we are going to prove that D is finite for the considered systems. In order to do this we have to write down the explicit expressions of the potentials and characterize their properties.

For the classical Gibbs systems the real-valued part of the complex potential is given by ($\omega = (q, s) \in \mathbb{R}^2$)

$$\operatorname{Re} u_{x-y}(\omega_x, \omega_y) = u_{x-y}(q_x, q_y) = J_0(|x-y|)u_0(q_x, q_y) - J_1^2(|x-y|)u_1^2(q_x, q_y).$$

And in its turn for Gibbs quantum and stochastic systems ($\omega = (q, w) \in \mathbb{R} \times C(\mathbb{R}^+)$) it is given, respectively, by

$$\operatorname{Re} u_{x-y}(\omega_x, \omega_y) = \beta^{-1} \int\limits_0^\beta u_{x-y}(w_x(\tau) - w_y(\tau))d\tau,$$

$$\operatorname{Re} u_{x-y}(\omega_x, \omega_y) = u^0_{x-y}(q_x, q_y) + u_{1,x-y}(w_x(t\beta^{-1}), w_y(t\beta^{-1})) + \int\limits_0^{t\beta^{-1}} u_{2,x-y}(w_x(\tau) - w_y(\tau))d\tau,$$

where t is the time. For our purposes here the explicit expressions for the the pair potential $u_{2,x}$ is not needed (it is expressed in terms of the interaction pair potential $u_{0,x}$) and we advise readers to find them in the Section 3 in [2].

For the classical Gibbs systems $\tilde{J} = J_1$ and the following equality is true:

$$\phi(q_x, s_x, q_y, s_y) = \frac{1}{\sqrt{2}}(s_x + s_y)u_1(q_x, q_y).$$

The potential $\phi(\omega_x, \omega_y)$ is expressed in terms of the stochastic integrals in $w_x^*, w_y^* \in C(\mathbb{R}^+)$

$$\phi(w_x, w_x^*, w_y, w_y^*) = 4^{-1}(\beta^{-1}2)^\kappa \left[\int\limits_0^{t'} dw_x^*(\tau)u'(w_x(\tau), w_y(\tau)) + \int\limits_0^{t'} dw_y^*(\tau)u'(w_y(\tau), w_x(\tau)) \right], \tag{3'}$$

where $u'(q, q') = u_1(q, q')$, $\kappa = 1$, $t' = \beta$, $\tilde{J} = J_1$ and $u'(q, q') = 2\partial u_0(q, q')$, $\kappa = 0, t' = t\beta^{-1}$, $\tilde{J} = J_0$, for quantum and stochastic systems, respectively (u_0, u_1 are symmetric functions).

We impose as in [1, 2] the conditions

$$|u_0(q, q')| \le \frac{1}{2}(v_0(q) + v_0(q')), \quad |u'(q, q')| \le \frac{1}{2}(v'(q) + v'(q')). \tag{4}$$

They are more general than their Kunz version considered in [1] (there is a product of the square roots of the two functions instead of the half of their sums in [1]). Here $v_0(q)$, $v'(q)$ are positive polynomials in $|q|$ of the $2m_0$-th, m_1-th degrees, respectively, such that for classical and quantum systems $m_0 < n$, $2m_1 < n$. Note that the similar inequalities hold for $\partial u_0(q, q')$, $\partial^2 u_0(q, q')$ in which the functions from the right-hand side which coincide with polynomials in $|q|$ with the degrees $2m_0 - 1$, $2m_0 - 2$, respectively. This follows from the inequality $a^l b^k \le 2^{k+l}(a^{k+l} + b^{k+l})$, $a, b > 0$ since the potentials $u_0(q, q')$, $u_1(q, q')$ are linear combinations of the elementary polynomials $q^l q'^k$, $l + k \in \mathbb{Z}^+$. As a result $m_1 = 2m_0 - 1$ for the stochastic systems for which $m_0 < n^0$. These conditions allow one to prove the following theorem.

Theorem 2. *Let* (4) *hold. Let also the inequality* $1+\zeta<\dfrac{n}{m}$, $m=\max(m_0,2m_1)$ *hold for the classical and quantum systems and the inequalities* $1+\zeta<\min\left(\dfrac{2n^0-1}{n^0+m_0-1},\dfrac{n^0}{m_0}\right)$, $t\le t_0\beta^2$ *hold for the stochastic systems, where* t *is the time and* t_0 *is a positive constant. Then* $D(\gamma)$ *is bounded in* β *in a neighborhood of zero for the classical systems if* $r=\dfrac{m_1}{n}$. *The same is true for the quantum and stochastic oscillator systems if* $r=\dfrac{1}{2}+\dfrac{m_1}{n}$.

Hence we proved the main result of our paper.

Theorem 3. *Let* (4), *the conditions for* ζ, r, t *in the Theorem* 2 *and for* J *in the Theorem* 1 *and Lemma* 1 *hold. Then the conclusion of the Theorem* 1 *is true for* $\bar{\rho}_\Lambda$ *and the correlation functions* $\rho^\Lambda(\omega_X)$ *of the classical, quantum and stochastic oscillator systems.*

Proof of Theorem **2**. For the classical and quantum systems (1) is valid with $J'=J_0+J^2$ and $v(q)=v_0(q)+v_1^2(q)$, $v(w)=\beta^{-1}\displaystyle\int_0^\beta v(w(\tau))d\tau$, respectively. The second condition in (1) will be satisfied if $1+\zeta<\dfrac{n}{m}$. For the classical systems we have

$$b^2(q,s)=(2\sqrt{2\pi})^{-1}\left(\int e^{-\beta u(q)}dq'\right)^{-1}\int(s+s')^2u_1^2(q,q')e^{-\frac{s'^2}{2}}e^{-\beta u(q')}dq'ds',$$

where the integration is performed over $\mathbb{R}^2$ and

$$b^2(q,s)\le(c_4^2s^2+c_3^2)v'^2(q)+\beta^{-\frac{2m_1}{n}}(c_1^2s^2+c_0^2),$$

where c_3, c_4 does not depend on β and c_1, c_0 are bounded functions in nonnegative finite β. Here we used the inequality $(s+s')^2\le 2(s^2+s'^2)$ and rescaled the variables in the corresponding integrals in the numerators and denominators by $\beta^{-\frac{1}{2n}}$. As a result

$$b(q,s)\le(c_4|s|+c_3)v'(q)+\beta^{-\frac{m_1}{n}}(c_1|s|+c_0).$$

Hence we can put $r=\dfrac{m_1}{n}$ and $D_*(q)$ is easily estimated as a Gaussian integral. That is

$$D_*(q)\le 2\exp\left\{\frac{1}{2}\left[c_3\gamma\beta^{\frac{m_1}{n}}v'(q)+\gamma^2(\beta^{\frac{m_1}{n}}c_4v'(q)+c_1)^2\right]+c_0\gamma\right\}.$$

As a result the condition $1+\zeta<\dfrac{n}{m}$ guarantees that

$$D=\int e^{\gamma\beta v^{1+\zeta}(q)}D_*(q)P'(dq)$$

is finite for nonnegative finite β after a rescaling of the variables in the corresponding integrals in the numerator and denominator by $\beta^{-\frac{1}{2n}}$. This concludes the proof for the classical systems.

Now let's consider the quantum and stochastic systems characterized, respectively, by the external potentials $u(q),2u_0(q)$. In an estimate of $D_*(\omega^0)$, based on the estimates from the Section 2 from [3], we will use the inequalities

$$\eta_-q^{2n^0}-\bar{u}\le u_0(q)\le\eta_+q^{2n^0}+\bar{u},\quad \eta_-q^{2n}-\bar{u}\le u(q)\le\eta_+q^{2n}+\bar{u},\quad \eta+\le\frac{3}{2}\eta_-.$$

The proof is based on application of the following lemma.

Lemma 2. *Let $b(w,w^*)$ correspond to ϕ in $(3')$. Then*

$$\int e^{\gamma\beta^r b(w,w^*)} P_0(dw^*) \le \kappa_* \left[\sqrt{I_P} + e^{\sigma\beta^{2(1-\kappa)}\int_0^{t'} w^{2n}(\tau)d\tau}\right],$$

$$I_P = \int P'(dw) e^{2\sigma\beta^{2(1-\kappa)}\int_0^{t'} w^{2n}(\tau)d\tau},$$

where $2\sigma < \eta_-, \sigma < 2n^2\eta_-^2$ for the quantum $(\kappa = 1)$ and stochastic systems $(\kappa = 0)$, respectively, $n = 2n^0 - 1$ for the stochastic systems, κ_* does not depend on w and is an entire function of $(\beta^{2r-2\frac{m_1}{n}(1-\kappa)} t'^{-2\kappa+1-\frac{m_1}{n}})^{\frac{1}{2}}$.

The proof repeats all the steps of the proof of Lemma 2.1 in [5] which coincides with the Lemma 2 for quantum systems. Note that m_1 corresponds to n_1 in [5]. For quantum and stochastic systems the function κ_* is bounded in nonnegative finite β if $2r - 1 - \dfrac{m_1}{n} \ge 0$ and therefore one can put $r = \dfrac{1}{2} + \dfrac{m_1}{n}$. Note that for the stochastic systems $2m_1 < 2(n-1)$ following from $m_1 = 2m_0 - 1$, $m_0 < n^0$.

Now in order to show that D is bounded for nonnegative finite β one has to establish the same fact for I_P. It was done for quantum systems in [5] (see the proof of the Lemma 2.2) with the help of the Golden – Thompson and Jensen inequalities, applied for the numerator and denominator, and a rescaling of the simple integrals in a variable from $\mathbb{R}$. For them $u(w) = \beta^{-1}\displaystyle\int_0^\beta u(w(\tau))d\tau$, where $u(q)$, $q \in \mathbb{R}$, is the external potential from the expression of the potential energy of quantum systems. The analogous simplified technique will be applied for the stochastic systems. Instead of the Golden – Thompson inequality the law of conservation of probability will be utilized by us and a rescaling of continuous (Wiener) paths as in [2] will not be considered. For the stochastic systems we have $\omega = (q, w)$ and

$$u(q,w) = u_0(q) + u_1(w(t')) + \int_0^{t'} u_2(w(\tau))d\tau,$$

where

$$u_1(q) = (2\eta_0 - 1)u_0(q) + u^1(q), \quad u_2(q) = -\partial^2 u_0(q) + \beta(\partial u_0(q))^2, \quad \eta_0 > \frac{1}{2},$$

and u^1 is an even polynomial of the degree less than $2n^0$. The function v is given by (see the proof of the Proposition 4.1 in [2])

$$v(q,w) = v_0(q) + v_1(w(t')) + \int_0^{t'} v_2(w(\tau))d\tau, \quad v_2 = v_2' + \beta v_2'',$$

where v_0, v_1, v_2', $v_2''(q)$ are positive polynomials with the degrees $2m_0 < 2n^0$, $2m_0$, $2(m_0 - 1)$, $2(n^0 + m_0 - 1)$, respectively (n^0, m_0 are denoted by n, m, respectively, in the Section 2 in [2]). From elementary inequality $(a_1 + \ldots + a_k)^{\frac{n}{m}} \le k^n(a_1^n + \ldots + a_k^n)^{\frac{1}{m}} \le k^n(a_1^{\frac{n}{m}} + \ldots + a_k^{\frac{n}{m}})$, where $n, m \in \mathbb{Z}^+$, $n > m$, $a_j \ge 0$ and the Helder inequality it follows that

$$v^{1+\zeta}(q,w) \le 3^{\langle 1+\zeta\rangle}\left(v_0^{1+\zeta}(q) + v_1^{1+\zeta}(w(t')) + t'^{\zeta}\int\limits_0^{t'} v_2^{1+\zeta}(w(\tau))d\tau\right),$$

where $\langle 1+\zeta\rangle$ is the numerator integer in the fractional representation of $1+\zeta$ and $1+\zeta <$ $< \min\left(\dfrac{2n^0-1}{n^0+m_0-1}, \dfrac{n^0}{m_0}\right)$. The last condition implies the second inequality in (1).

The following law of conservation of probability holds

$$\int dq \int e^{-\beta u(q,w|a)} P_q(dw) = \int e^{-\beta(u^1(q)+2\eta_0 a u_0(q))}dq, \tag{5}$$

where $u(q,w|a)$ is equal to $u(q,w)$ if one substitutes au_0 instead of u_0 into its expression. It follows from the gradient character of the Smoluchowski equation. We remind that the function under the sign of the integral in q in the left-hand side of this equality coincides with the solution of the Smoluchowski equation with the initial function which coincides with the function under the sign of the integral in its right-hand side. Let $a = 2^{-1}$ and in addition $2\sigma + n^2\eta_+^2 < 4n^2\eta_-^2$ then

$$e^{\gamma\beta v^{1+\zeta}(q,w)} e^{-\beta(u(q,w)-u(q,w|\frac{1}{2}))} e^{2\beta^2\sigma\int_0^{t'} w^{2n}(\tau)d\tau} \le e^{C(\beta,t')}, \tag{6}$$

where

$$C(\beta,t') = \beta(t'\beta^{-1}C_0 + t'^{\zeta+1}(C_1+\beta^{-2}C_2) + C_3 + C_4 t'), \quad \beta < 1,$$

and C_j are constants. Here one has to use the fact that the coefficient before $\displaystyle\int_0^{t'} w^{2n}(\tau)\,d\tau$ in the exponent in the left-hand side of (6) is negative, and apply the inequality $q^k \le \varepsilon q^l + \varepsilon^{-\frac{k}{l-k}}$ (see the proof below) for $q > 0, k \le l$ and small $\varepsilon < 1$. That is

$$\int\limits_0^{t'} w^k(\tau)\,d\tau \le \beta\epsilon \int\limits_0^{t'} w^l(\tau)d\tau + (\beta\epsilon)^{-\frac{k}{l-k}} t'.$$

C_3 is the contribution of terms depending on $q, w(t')$, C_0 is the contribution of the terms in $\displaystyle\int_0^{t'} u_2(w(\tau))d\tau$ containing the second derivative of u_0, since $\beta^{-\frac{m_0-1}{2n^0-m_0}} \le \beta^{-1}$. The integrals of v_2'', v_2' contributed the constants C_1, $C_2 > 0$, respectively, since $\beta^{-\frac{(m_0-1)(1+\zeta)}{n-(m_0-1)(1+\zeta)}} \le \beta^{-\frac{n^0}{2n^0-1-n^0}} \le \beta^{-2}$. The terms containing the first derivative of u_0 in the integral $\displaystyle\int_0^{t'} u_2(w(\tau))d\tau$ contribute C_4. (6) is valid since we choose ε such that sum of all the terms in the exponent in the left-hand side of (5) with powers in q, w less than the senior powers in $u(q,w) - u\left(q, w\middle|\dfrac{1}{2}\right)$ yield the negative coefficients before the terms with the senior powers . Using these bounds, (6) and the law of conservation of probability we see that ($\sqrt{I_P} \le I_P$)

$$D \le 2\kappa_* e^{C(\beta,t')} \int e^{-\beta(u^1(q)+\eta_0 u_0(q))}dq \left(\int e^{-\beta(u^1(q)+2\eta_0 u_0(q))}dq\right)^{-1}.$$

Hence D is bounded in nonnegative finite β (recall that $t' = t\beta^{-1}$) if $t \le t_0\beta^2$ since $\dfrac{2+\zeta}{1+\zeta} \le 2$, where t_0 is independent of β.

The inequality $q^k \le \varepsilon q^n + \varepsilon^{-\frac{k}{n-k}}$ is proved if one proves that for $q \ge \varepsilon^{-\frac{1}{n-k}}$ the inequality $q^k \le \varepsilon q^n$ holds. This inequality is true if $(a + \varepsilon^{-\frac{1}{n-k}})^k \le \varepsilon(a + \varepsilon^{-\frac{1}{n-k}})^n$ for $a \ge 0$. The last inequality is checked comparing the coefficients near a^l in its both sides. They equal $C_k^l \varepsilon^{-\frac{k-l}{n-k}}$, $C_n^l \varepsilon\varepsilon^{-\frac{n-l}{n-k}} = C_n^l \varepsilon^{-\frac{k-l}{n-k}}$, respectively. Hence all the terms in its left-hand side are smaller than the corresponding terms in its right-hand side. This concludes the proof of the theorem.

***Proof of Lemma* 1.** From

$$|e^a - 1| = |e^{i\operatorname{Im} a}(e^{\operatorname{Re} a} - 1) + (e^{i\operatorname{Im} a} - 1)| \le |\operatorname{Re} a| e^{|\operatorname{Re} a|} + 2|\operatorname{Im} a|,$$

(1) and the superstability condition it follows that (note that $\omega_0^0 = \omega^0$)

$$|e^{-\beta u_x(\omega,\omega_x)} - 1| \le \beta|\operatorname{Re} u_x(\omega,\omega_x)| e^{\beta|\operatorname{Re} u_x(\omega,\omega_x)|} + 2\beta|\operatorname{Im} u_x(\omega,\omega_x)| \le$$

$$\le \beta\left[\frac{1}{2}J'(|x|)(v(\omega_x^0) + v(\omega^0)) e^{\frac{\beta}{2}J'(|x|)(v(\omega_x^0)+v(\omega^0))} + 2\tilde{J}(|x|)\phi(\omega_x,\omega)\right].$$

Let b_x^0 and b_x^1 are the contributions to b_x of the first and second terms in the square brackets. Let also B^0 and B^1 be the corresponding parts from B. Then the Schwartz inequality gives

$$b_x^1(\omega) \le 2\beta\sqrt{D(2\gamma)}\tilde{J}(|x|) e^{-\gamma\beta^r ||\sqrt{J}||_1^{-1}\sqrt{J(|x|)}b(\omega)} b(\omega) e^{-\beta(\gamma v^{1+\zeta}(\omega) - ||J'||v(\omega))||\sqrt{J}||_1^{-1}\sqrt{J(|x|)}}.$$

That is

$$B^1 \le 2\sqrt{D(2\gamma)}\gamma^{-1}\beta^{1-r}\kappa_1\kappa||\sqrt{J}||_1^2, \quad \kappa_n = \sup_{a\ge 0} a^n e^{-a},$$

$$\kappa = \sup_{a\ge 0, x} e^{-\beta(\gamma a^{1+\zeta} - ||J'||a)||\sqrt{J}||_1^{-1}\sqrt{J(x)}} = e^{\beta||\sqrt{J}||_1^{-1}|\sqrt{J}|_0\left(\frac{||J'||\gamma^{-\frac{1}{1+\zeta}}}{1+\zeta}\right)^{1+\frac{1}{z}}}, \quad |J|_0 = \sup_x J(|x|).$$

Multiplying b_x^0 by $e^{\gamma\beta v^{1+\zeta}(\omega_x^0)} e^{-\gamma\beta v^{1+\zeta}(\omega_x^0)}$ we obtain, using the equalities $1 - \dfrac{1}{2(1+\zeta)} = \dfrac{1+2\zeta}{2(1+\zeta)}$, $\dfrac{1}{2} - \dfrac{1}{2(1+\zeta)} = \dfrac{\zeta}{2(1+\zeta)}$, the following inequality:

$$B^0 \le 2^{-1}\beta^{\frac{\zeta}{1+\zeta}}\gamma^{-\frac{1}{1+\zeta}}[\kappa_1\kappa(0)||J'||_1 + \kappa_0\kappa(1)||J^{\frac{1+2\zeta}{2(1+\zeta)}}||_1] D(2\gamma + 2^{-1}|J'|_0),$$

where

$$\kappa(n) = \sup_{a\ge 0} a^n \exp\left\{-a^{1+\zeta} + (\gamma^{-1}\beta^{\zeta}||\sqrt{J}||_1)^{\frac{1}{1+\zeta}}(2^{-1}|J^{\frac{1+2\zeta}{2(1+\zeta)}}|_0 + ||J'||_1|J^{\frac{\zeta}{2(1+\zeta)}}|_0)a\right\}.$$

We have $B = B^0 + B^1$ and the last bounds for B^0, B^1 prove the lemma.

***Remark* 1.** We assume that the generator of the semigroup, producing the Wiener measure, coincides with ∂^2, where ∂^2 is the operator of the second derivative. This obliges us to assume that the mass of the quantum particles is equal to $\dfrac{1}{2}$. The transition to the quantum systems with the mass equal to 1, considered in [1], is produced by a simple rescaling of variables.

***Remark* 2.** Our main result (Theorem 3) permits us to obtain the conclusions of the Theorems 3 – 4.1 in [1], Theorem 2.1 in [2] easily. The conditions and the proof of the Theorem 3 are simpler than those of the theorems in [1, 2] based on a rescaling of wiener paths. The proposed in this paper technique is based on another choice of $\tilde{v}$, the bound for B in the Lemma 1, on the bound of the Lemma 2, the Golden – Thompson inequality (for the quantum systems) and the law of conservation of probability (for the stochastic systems). No rescaling of wiener paths is needed in this paper. The grand canonical analog of the result of [2] can be found in [6].

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