UDC 512.5

A. Kianezhad (Islamic Azad Univ., Tehran, Iran),
A. J. Taherizadeh (Kharazmi Univ., Tehran, Iran)

FINITENESS PROPERTIES OF MINIMAX AND $\mathfrak{a}$-MINIMAX GENERALIZED LOCAL COHOMOLOGY MODULES

ВЛАСТИВОСТІ ФІНІТНОСТІ МІНІМАКСНИХ ТА $\mathfrak{a}$-МІНІМАКСНИХ УЗАГАЛЬНЕНИХ ЛОКАЛЬНИХ КОГОМОЛОГІЧНИХ МОДУЛІВ

Let R be a commutative Noetherian ring with nonzero identity, $\mathfrak{a}$ be an ideal of R, and M and N be two (finitely generated) R-modules. We prove that $H^i_{\mathfrak{a}}(M,N)$ is a minimax $\mathfrak{a}$-cofinite R-module for all $i<t$, $t\in\mathbb{N}_0$, if and only if $H^i_{\mathfrak{a}}(M,N)_p$ is a minimax $R_{\mathfrak{p}}$-module for all $i<t$. We also show that, under some conditions, $\operatorname{Hom}_R\left(\frac{R}{\mathfrak{a}}, H^t_{\mathfrak{a}}(M,N)\right)$ is minimax $(t\in\mathbb{N}_0)$. Finally, we investigate a necessary condition for $H^i_{\mathfrak{a}}(M,N)$ to be $\mathfrak{a}$-minimax.

Нехай R — комутативне нетерове кільце з ненульовою одиницею, $\mathfrak{a}$ — ідеал кільця R, а M та N — два (скінченнопороджених) R-модулі. Доведено, що $H^i_{\mathfrak{a}}(M,N)$ є мінімаксним $\mathfrak{a}$-кофінітним R-модулем для всіх $i<t$, $t\in\mathbb{N}_0$, тоді і тільки тоді, коли $H^i_{\mathfrak{a}}(M,N)_p$ є мінімаксним $R_{\mathfrak{p}}$-модулем для всіх $i<t$. Показано також, що за деяких умов $\operatorname{Hom}_R\left(\frac{R}{\mathfrak{a}}, H^t_{\mathfrak{a}}(M,N)\right)$ є мінімаксним $(t\in\mathbb{N}_0)$. Досліджено необхідні умови $\mathfrak{a}$-мінімаксності $H^i_{\mathfrak{a}}(M,N)$.

1. Introduction. Let R be a commutative Noetherian ring with nonzero identity, $\mathfrak{a}$ be an ideal of R and M, N be two R-modules. The generalized local cohomology was first introduced in the local case by Herzog [8] and in the general case by Bijan–Zadeh [4]. The ith generalized local cohomology module $H^i_{\mathfrak{a}}(M,N)$ is defined by

$$H^i_{\mathfrak{a}}(M,N)=\varinjlim_{n\in\mathbb{N}}\operatorname{Ext}^i_R\left(\frac{M}{\mathfrak{a}^nM},N\right)$$

for all $i\in\mathbb{N}_0$, where we use $\mathbb{N}_0$ (resp. $\mathbb{N}$) to denote the set of nonnegative (resp. positive) integers. With $M=R$, one clearly obtains the ordinary local cohomology modules $H^i_{\mathfrak{a}}(N)$ of N with respect to $\mathfrak{a}$, which was introduced by Grothendieck, see, for example, [6].

It is well known that the generalized local cohomology modules have some similar properties as ordinary local cohomology modules. We recall some properties of the generalized local cohomology modules which will be needed in this paper.

I) Let $0\longrightarrow N\longrightarrow M\longrightarrow L\longrightarrow 0$ be an exact sequence of R-modules. Then we have two long exact sequences (K is an arbitrary R-module):

$$0\longrightarrow H^0_{\mathfrak{a}}(K,N)\longrightarrow H^0_{\mathfrak{a}}(K,M)\longrightarrow H^0_{\mathfrak{a}}(K,L)\longrightarrow H^1_{\mathfrak{a}}(K,N)\longrightarrow\ldots$$

and

$$0\longrightarrow H^0_{\mathfrak{a}}(L,K)\longrightarrow H^0_{\mathfrak{a}}(M,K)\longrightarrow H^0_{\mathfrak{a}}(N,K)\longrightarrow H^1_{\mathfrak{a}}(L,K)\longrightarrow\ldots$$

of generalized local cohomology modules.

II) If N is an $\mathfrak{a}$-torsion R-module, then for all $i\in\mathbb{N}_0$, we have $H^i_{\mathfrak{a}}(M,N)\cong\operatorname{Ext}^i_R(M,N)$.

III) Let R' be a second commutative Noetherian ring with identity and let $f\colon R\longrightarrow R'$ be a flat ring homomorphism. Then there is an isomorphism $(i\in\mathbb{N}_0)$

$$H^i_{\mathfrak{a}}(M,N)\otimes_R R' \cong H^i_{\mathfrak{a}R'}(M\otimes_R R', N\otimes_R R').$$

The organization of the paper is an follows:

In Section 2, we study the minimax generalized local cohomology modules (Theorem 2.2 and Propositions 2.1 and 2.2). Also the minimaxness of $\operatorname{Hom}_R\left(\dfrac{R}{\mathfrak{a}}, \dfrac{H^i_{\mathfrak{a}}(M,N)}{K}\right)$, $t \in \mathbb{N}_0$, and $\operatorname{Hom}_R\left(\dfrac{R}{\mathfrak{a}}, \dfrac{H^t_{\mathfrak{a}}(M,N)}{K}\right)$ will be considered whenever M, N are finitely generated R-modules and K is a submodule of $H^t_{\mathfrak{a}}(M,N)$ (Theorem 2.2 and Corollary 2.1) which generalize [9] (Theorem 2.2).

In Section 3, we study the $\mathfrak{a}$-minimax generalized local cohomology modules. In Theorem 2.1, we show that whenever M, N are finitely generated and $\mathfrak{a}$-minimax R-modules such that $\operatorname{Ass}_R(N) \subseteq V(\mathfrak{a})$ and $pd(M) < \infty$, then $H^t_{\mathfrak{a}}(M,N)$ is $\mathfrak{a}$-minimax for all $i \geq 0$.

Throughout the paper R is a commutative Noetherian ring with nonzero identity.

2. Minimax generalized local cohomology modules. In this section we prove the minimaxness of some generalized local cohomology modules.

Definition 2.1. *Let N be an R-module. Then N is said to be a minimax module if there is a finitely generated submodule L of N such that $\dfrac{N}{L}$ is Artinian.*

The class of minimax modules includes all finite and all Artinian module. Moreover it is closed under taking submodules, quotients and extensions, i.e., it is a serre subcategory of the category of R-modules, cf. [12] and [13]. Of course this class is strictly larger than the class of all finite modules and Artinian modules as well, cf. [3] (Theorem 12). Also we note that a minimax R-module has only finitely many associated primes.

Lemma 2.1. *Let $N \xrightarrow{f} M \xrightarrow{g} L$ be an exact sequence of R-modules such that N and L are minimax. Then M is minimax.*

Proof. The result follows from the fact that the class of minimax R-modules is a serre subcategory of the category of R-modules, and the exact sequence

$$0 \longrightarrow f(N) \xrightarrow{\subseteq} M \xrightarrow{g} g(M) \longrightarrow 0.$$

Lemma 2.2. *Let M, N be two R-modules such that M is projective. Then $H^i_{\mathfrak{a}}(M,N) \cong H^i_{\mathfrak{a}}(\operatorname{Hom}_R(M,N))$ for all $i \in \mathbb{N}_0$.*

Proof. For all $i \in \mathbb{N}_0$

$$H^i_{\mathfrak{a}}(M,N) = \varinjlim_{n\in\mathbb{N}} \operatorname{Ext}^i_R\left(\frac{M}{\mathfrak{a}^n M}, N\right) \cong \varinjlim_{n\in\mathbb{N}} \operatorname{Ext}^i_R\left(\frac{R}{\mathfrak{a}^n}\otimes_R M, N\right) \cong$$

$$\cong \varinjlim_{n\in\mathbb{N}} \operatorname{Ext}^i_R\left(\frac{R}{\mathfrak{a}^n}, \operatorname{Hom}(M,N)\right) \text{ (by [11], exercise 9.21)} \cong H^i_{\mathfrak{a}}(\operatorname{Hom}(M,N)).$$

The next theorem generalize [1] (Theorem 2.8).

Theorem 2.1. *Suppose that $t \in \mathbb{N}_0$ and M, N are two finitely generated R-modules such that $pd(M) < \infty$ and $H^i_{\mathfrak{a}}(N)$ is minimax for all $i < t$. Then the following are equivalent*:

(i) $H^i_{\mathfrak{a}}(M,N)$ *is minimax for all $i < t$,*

(ii) $H^i_{\mathfrak{a}}(M, N)$ *is minimax and* $\mathfrak{a}$*-cofinite for all* $i < t$,

(iii) $H^i_{\mathfrak{a}}(M, N)_{\mathfrak{p}}$ *is minimax for all* $i < t$ *and all* $p \in \operatorname{Spec}(R)$,

(iv) $H^i_{\mathfrak{a}}(M, N)_{\mathfrak{m}}$ *is minimax for all* $i < t$ *and all* $\mathfrak{m} \in \max(R)$.

Proof. (i) $\Rightarrow$ (ii). By induction on $n := pd(M)$. If $n = 0$, then the result follows from Lemma 2.3 and [2] (Theorem 2.8).

Next let $n > 0$. We have

$$H^0_{\mathfrak{a}}(M, N) = \varinjlim_{n\in\mathbb{N}} \operatorname{Hom}_R\left(\frac{M}{\mathfrak{a}^n M}, N\right) \cong \varinjlim_{n\in\mathbb{N}} \operatorname{Hom}_R\left(M \otimes_R \frac{R}{\mathfrak{a}^n}, N\right) \cong$$

$$\cong \varinjlim_{n\in\mathbb{N}} \operatorname{Hom}_R\left(\frac{R}{\mathfrak{a}^n}, \operatorname{Hom}(M, N)\right) \cong H^0_{\mathfrak{a}}(\operatorname{Hom}(M, N))$$

and so the result follows. Now suppose $i > 0$. From the short exact sequence $0 \longrightarrow L \longrightarrow \longrightarrow \bigoplus_{j=1}^n R \longrightarrow M \longrightarrow 0$ we get the exact sequence

$$H^i_{\mathfrak{a}}(N)^n \longrightarrow H^i_{\mathfrak{a}}(L, N) \longrightarrow H^{i+1}_{\mathfrak{a}}(M, N), \quad i \in \mathbb{N},$$

according to our assumption and Lemma 2.1 we deduce that $H^i_{\mathfrak{a}}(L, N)$ is minimax for all $i < t - 1$. By induction hypothesis $H^i_{\mathfrak{a}}(L, N)$ is $\mathfrak{a}$-cofinite for all $i < t - 1$. But from the exact sequence

$$H^{i-1}_{\mathfrak{a}}(L, N) \longrightarrow H^i_{\mathfrak{a}}(M, N) \longrightarrow H^i_{\mathfrak{a}}(N)^n.$$

and [2] (Theorem 2.8) and [10] (Corollary 4.4) we get the result.

(ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (iv) is obvious.

(iv) $\Rightarrow$ (i) The proof is similar to that for (i) $\Rightarrow$ (ii).

The next result follows by an standard argument.

Proposition 2.1. *Suppose that* M, N *are two* R*-modules such that* N *is finitely generated and for each* p *of* $\operatorname{supp}_R(N)$, $H^t_{\mathfrak{a}}\left(M, \frac{R}{\mathfrak{p}}\right)$ *be minimax. then* $H^t_{\mathfrak{a}}(M, N)$ *is minimax* $(t \in \mathbb{N}_0)$.

Proposition 2.2. *Let* M, N, L *be finitely generated* R*-modules such that* $pd(M)$ *and* $\dim(N)$ *is finite and* $\operatorname{supp}_R(L) \subseteq \operatorname{supp}_R(N)$. *Suppose* $H^i_{\mathfrak{a}}(M, N)$ *is minimax for all* $i \geq r$. *Then* $H^i_{\mathfrak{a}}(M, L)$ *is minimax for all* $i \geq r$, $r \in \mathbb{N}_0$.

Proof. By [4] (Proposition 5.5) $H^i_{\mathfrak{a}}(M, N) = 0$ for all $i > pd(M) + \dim N$. The proof is by decrasing induction on $i = r, r + 1, \ldots, pd(M) + \dim N + 1$. If $i = pd(M) + \dim(N) + 1$ then there is nothing to prove. So let $r \leq i \leq pd(M) + \dim(N)$. By Gruson's theorem there exist a chain $0 \longrightarrow L_0 \subseteq L_1 \subseteq \ldots \subseteq L_t = L$ such that the quotient $\dfrac{L_j}{L_{j-1}}$ a homomorphic image of a direct sum of finitely many copies of N for all $j = 1, 2, \ldots, t$. Consider the short exact sequences

$$0 \longrightarrow L_{j-1} \longrightarrow L_j \longrightarrow \frac{L_j}{L_{j-1}} \longrightarrow 0, \quad j = 1, 2, \ldots, t,$$

we may reduce to the case $t = 1$. So let there exsits a short exact sequence $0 \longrightarrow K \longrightarrow N^m \longrightarrow \longrightarrow L \longrightarrow 0$ in which $m > 0$ and K is a finitely generated R-module. This exact sequence induce the long exact sequence

$$\ldots \longrightarrow H^i_{\mathfrak{a}}(M, N^m) \longrightarrow H^i_{\mathfrak{a}}(M, L) \longrightarrow H^{i+1}_{\mathfrak{a}}(M, K) \longrightarrow \ldots . \tag{2.1}$$

Since $\operatorname{supp}_R(K) \subseteq \operatorname{Supp}_R(N)$, by induction hypothesis $H^{i+1}_{\mathfrak{a}}(M, K)$ is minimax for all $i = r, \ldots, pd(M) + \dim N$. Moreover $H^i_{\mathfrak{a}}(M, N^m) \cong H^i_{\mathfrak{a}}(M, N)^m$ is minimax. Thus from the exact sequence (2.1) and Lemma 2.1 the result follows.

The next two results generalized [9] (Theorem 2.2 and Corollary 2.3).

Theorem 2.2. *Let M, N be two finitely generated R-modules such that $H^i_{\mathfrak{a}}(M, N)$ and $H^i_{\mathfrak{a}}(N)$ is minimax for all $i < t$, $t \in \mathbb{N}_0$, and $pd(M) < \infty$. Then $\operatorname{Hom}_R\left(\frac{R}{\mathfrak{a}}, H^t_{\mathfrak{a}}(M, N)\right)$ are minimax and so $\operatorname{Ass} H^t_{\mathfrak{a}}(M, N)$ is finite.*

Proof. If $t = 0$, then,

$$\operatorname{Ext}^0_R\left(\frac{R}{\mathfrak{a}}, \operatorname{Hom}_R(M, N)\right) \cong \operatorname{Hom}_R\left(\frac{R}{\mathfrak{a}}, \operatorname{Hom}_R(M, N)\right)$$

which is finitely generated. Furthermore

$$\operatorname{Hom}_R\left(\frac{R}{\mathfrak{a}}, H^0_R(M, N)\right) \cong \operatorname{Hom}_R\left(\frac{R}{\mathfrak{a}}, H^0_{\mathfrak{a}}(\operatorname{Hom}(M, N)\right) \cong \operatorname{Hom}_K\left(\frac{R}{\mathfrak{a}}, \operatorname{Hom}_R(M, N)\right).$$

Next let $t > 0$. The proof is by induction on $n := pd_R(M) \geq 0$. If $n = 0$, then by Lemma 2.2 $H^i_{\mathfrak{a}}(M, N) \cong H^i_{\mathfrak{a}}(\operatorname{Hom}(M, N))$ for all i, and

$$\operatorname{Hom}_R\left(\frac{R}{\mathfrak{a}}, H^t_{\mathfrak{a}}(M, N)\right) \cong \operatorname{Hom}_R\left(\frac{R}{\mathfrak{a}}, H^t_{\mathfrak{a}}(\operatorname{Hom}(M, N))\right).$$

So since $\operatorname{Ext}^t_R\left(\frac{R}{\mathfrak{a}}, \operatorname{Hom}_R(M, N)\right)$ is finitely generated and $H^i_{\mathfrak{a}}(\operatorname{Hom}(M, N))$ is minimax for all $i < t$, the result will be deduced from [9] (Theorem 2.2).

Now suppose that $n > 0$. There is a short exact sequence

$$0 \longrightarrow K \longrightarrow R^n \longrightarrow M \longrightarrow 0$$

from which we get the long exact sequcence

$$\operatorname{Hom}_R\left(\frac{R}{\mathfrak{a}}, H^{t-1}_{\mathfrak{a}}(K, N)\right) \longrightarrow \operatorname{Hom}_R\left(\frac{R}{\mathfrak{a}}, H^t_{\mathfrak{a}}(M, N)\right) \longrightarrow \operatorname{Hom}_R\left(\frac{R}{\mathfrak{a}}, H^t_{\mathfrak{a}}(N)^n\right) \longrightarrow \ldots . \tag{2.2}$$

Next, $H^i_{\mathfrak{a}}(K, N)$ is minimax for all $i < t - 1$, by the exact sequence

$$\ldots \longrightarrow H^i_{\mathfrak{a}}(N)^n \longrightarrow H^i_{\mathfrak{a}}(K, N) \longrightarrow H^{i+1}_{\mathfrak{a}}(M, N) \longrightarrow \ldots$$

the minimaxness of $H^i_{\mathfrak{a}}(N)$ and $H^i_{\mathfrak{a}}(M, N)$ for all $i < t - 1$ and Lemma 2.1. Thus by induction hypothesis $\operatorname{Hom}_R\left(\frac{R}{\mathfrak{a}}, H^{t-1}_{\mathfrak{a}}(K, N)\right)$ is minimax. In addition $\operatorname{Hom}_R\left(\frac{R}{\mathfrak{a}}, H^t_{\mathfrak{a}}(N)\right)$ is minimax by [9] (Theorem 2.2). Now we deduce the result from exact sequence (2.2) and Lemma 2.1.

***Corollary* 2.1.** *With the same assumptions as in* (2.2), *let K be an R-submodule of $H^t_{\mathfrak{a}}(M, N)$ such that $\operatorname{Ext}^1_R\left(\frac{R}{\mathfrak{a}}, K\right)$ is minimax. Then $\operatorname{Hom}_R\left(\frac{R}{\mathfrak{a}}, \frac{H^t_{\mathfrak{a}}(M, N)}{K}\right)$ is a minimax module. In particular $\operatorname{Ass}\left(H^t_{\mathfrak{a}}(M, N)/K\right)$ is finite.*

3. $\mathfrak{a}$-Minimax modules and generalized local cohomology modules. Recall that for an R-module $M,$ the Goldie dimension of M is defined as the cardinal of the set of indecomposable submodules of $E(M)$, which appear in a decomposition of $E(M)$ into direct sum of indecomposable submodules. The notation $G.\dim(M)$ is used for Goldie dimension of M. For a prime ideal p of R $\mu^0(p, M)$ denotes the 0th Bass number of M with respect to p. It is known that $\mu^0(p, M) > 0$ if and only if $p \in \operatorname{Ass}_R(M)$. Therefore by definition of Goldie dimension, $G\dim M = \sum_{p\in \operatorname{Ass}(M)} \mu^0(p, M)$. Also for an ideal $\mathfrak{a}$ of R and R-module M the $\mathfrak{a}$-relative Goldie dimension of M is defined as

$$G\dim_{\mathfrak{a}} M := \sum_{p\in V(\mathfrak{a})} \mu^0(p, M).$$

The $\mathfrak{a}$-relative Goldie dimension of an R-module has been introduced and studied in [7].

Definition 3.1. *Let $\mathfrak{a}$ be an ideal of R. An R-module M is said to be minimax with respect to $\mathfrak{a}$ or $\mathfrak{a}$- minimax if the $\mathfrak{a}$-relative Goldie dimension of any quotient module of M is finite, i.e., for any submodule N of M, $G\dim_{\mathfrak{a}}\left(\frac{M}{N}\right) < \infty$.*

The concept of $\mathfrak{a}$-minimax modules was introduced and studied in [2]. By [2] (Proposition 2.6) if M is an $\mathfrak{a}$-minimax R-module such that $\operatorname{Ass} M \subseteq V(\mathfrak{a})$, then $H^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-minimax for all $i \geq 0$.

Now we intend to generalize the result to obtain another similar result for generalized local cohomology modules.

Theorem 3.1. *Let M and N be two R-modules such that M is finitely generated and N is $\mathfrak{a}$-minimax with $\operatorname{Ass}_R(N) \subseteq V(\mathfrak{a})$ and $p(M) < \infty$. Then $H^i_{\mathfrak{a}}(M, N)$is $\mathfrak{a}$-minimax for all $i \geq 0$.*

Proof. By induction on $n := pdM$. If $n = 0$, then $H^i_{\mathfrak{a}}(M, N) \cong H^i_{\mathfrak{a}}(\operatorname{Hom}(M, N))$ by Lemma 2.2 and $\operatorname{Hom}_R(M, N)$ is $\mathfrak{a}$-minimax by [2] (Corollary 2.5). Also $\operatorname{Ass}(\operatorname{Hom}_R(M, N)) = \operatorname{Ass}_R(N)\cap$ $\cap \operatorname{Supp}_R(M) \subseteq V(\mathfrak{a})$, by [5] (Ch. 4, § 2.1, Proposition 10) the result follows from [2] (Proposition 2.6). Next, let $n > 0$ and suppose that the result is true for $n - 1$. From the short exact sequence $0 \longrightarrow L \longrightarrow R^k \longrightarrow M \longrightarrow 0$ (in which L is a finitely generated R-module) we get the following long exact sequence:

$$\ldots \longrightarrow H^{i-1}_{\mathfrak{a}}(L, N) \longrightarrow H^i_{\mathfrak{a}}(M, N) \longrightarrow H^i_{\mathfrak{a}}(N)^k \longrightarrow \ldots . \tag{3.1}$$

By induction hypothesis $H^{i-1}_{\mathfrak{a}}(L, N)$ is $\mathfrak{a}$-minimax for all $i \geq 1$. Moreover $H^i_{\mathfrak{a}}(N)^k$ is $\mathfrak{a}$-minimax by [2] (Proposition 2.6 and Corollary 2.4). Next we deduce from (3.1) and [2] (Proposition 2.3) that $H^i_{\mathfrak{a}}(M, N)$ is $\mathfrak{a}$- minimax for all $i \geq 1$. For $i = 0$, we have $H^0_{\mathfrak{a}}(M, N) \cong H^0_{\mathfrak{a}}(\operatorname{Hom}_R(M, N))$ which is $\mathfrak{a}$-minimax by [2] (Corollary 2.5 and Proposition 2.3).

Proposition 3.1. *Let M be a finitely generated R-module and N an arbitrary R-module such that $\operatorname{Hom}_R(M, N)$ is $\mathfrak{a}$-minimax and $\operatorname{supp}_R(M) \subseteq V(\mathfrak{a})$. Furthermore let $t \in \mathbb{N}_0$ and for each $i \neq t$, $H^i_{\mathfrak{a}}(M, N)$ be $\mathfrak{a}$-minimax. Then $H^t_{\mathfrak{a}}(M, N)$ is $\mathfrak{a}$-minimax.*

Proof. By induction on $t \geq 0$. If $t = 0$, then $H^0_{\mathfrak{a}}(M, N) \cong H^0_{\mathfrak{a}}(\operatorname{Hom}_R(M, N))$, and $\operatorname{Hom}_R(M, N)$ is $\mathfrak{a}$-minimax by our assumption and

$$\operatorname{Ass}_R(\operatorname{Hom}_R(M, N)) = \operatorname{Ass}_R(N) \cap \operatorname{Supp}_R(M) \subseteq V(\mathfrak{a}),$$

the result follows from [2] (Proposition 2.6).

Next let $t > 0$ and assume that the result holds for $t - 1$. Set $E := E(N)$, $L := \frac{E}{N}$ and consider the short exact sequence $0 \longrightarrow N \longrightarrow E \longrightarrow L \longrightarrow 0$. Since $H^i_{\mathfrak{a}}(M, E) = 0$ for all $i > 0$, we deduce that

$$H^i_{\mathfrak{a}}(M,L) \cong H^{i-1}_{\mathfrak{a}}(M,N).$$

Thus it follows from the hypothesis that for all $i \neq t-1$ $H^i_{\mathfrak{a}}(M,L)$ is $\mathfrak{a}$-minimax. Hence by induction hypothesis $H^{t-1}_{\mathfrak{a}}(M,L)$ is $\mathfrak{a}$-minimax and consequently by (2.1) $H^t_{\mathfrak{a}}(M,N)$ is $\mathfrak{a}$-minimax as well.

1. *Aghapournahr M., Melkersson L.* Finiteness properties of minimax and coatomic local cohomology modules // Arch. Math. – 2010. – **94**. – P. 519 – 528.
2. *Azam J., Naghipour R., Vakili B.* Finiteness properties of local cohomology modules for $\mathfrak{a}$-minimax modules // Proc. Amer. Math. Soc. – 2009. – **137**. – P. 439 – 448.
3. *Belshoff R., Enochs E. E., Rozas J. R. G.* Generalized Matlis duality // Proc. Amer. Math. Soc. – 2000. – **128**. – P. 1307 – 1312.
4. *Bijan-Zadeh M. H.* A common generalization of local cohomology theories // Glasgow Math. J. – 1980. – **21**. – P. 173 – 181.
5. *Bourbaki N.* Elements of mathematics commutative algebra. – Springer-Verlag, 1989.
6. *Brodmann M. P., Sharp R. Y.* Local cohomology, an algebraic introduction with geometric applications. – Cambridge Univ. Press, 1998.
7. *Divaani-Azar K., Esmakhani M. A.* Artinianess of local cohomology modules of ZD-modules // Communs Algebra. – 2005. – **33**. – P. 2857 – 2863.
8. *Herzog J.* Komplex Auflösungen und Dualität in der lokalen Algebra // Habilitations Chrift. – Univ. Regensburg, 1970.
9. *Lorestani K. B., Schandi P., Yassemi S.* Artinian local cohomology modules // Can. Math. Bull. – 2007. – **50**. – P. 598 – 602.
10. *Melkersson L.* Modules cofinite with respect to an ideal // J. Algebra. – 2005. – **285**. – P. 649 – 668.
11. *Rotman J.* An introduction to homological algebra. – Acad. Press, 1979.
12. *Rudolf P.* On minimax and related modules // Can. J. Math. – 1992. – **49**. – P. 154 – 166.
13. *Zochinger H.* Minimax modules // J. Algebra. – 1986. – **102**. – P. 1 – 32.

Received 11.03.12