UDC 517.5

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ON SOME MULTIDIMENSIONAL HILBERT-TYPE INEQUALITIES IN A DISCRETE CASE*

ПРО ДЕЯКІ БАГАТОВИМІРНІ НЕРІВНОСТІ ГІЛЬБЕРТОВОГО ТИПУ У ДИСКРЕТНОМУ ВИПАДКУ

Motivated by results of Huang, we derive a pair of discrete multidimensional Hilbert-type inequalities involving a homogeneous kernel of negative degree. We also establish conditions under which the constant factors involved in the established inequalities are the best possible. Finally, we consider some particular settings with homogeneous kernels and weight functions. In such a way we obtain generalizations of some results known from the literature.

З метою узагальнення результатів Хуанга отримано дві дискретні багатовимірні нерівності гільбертового типу з однорідним ядром від'ємного степеня. Також встановлено умови, за яких сталі множники, що входять до отриманих нерівностей, є найкращими з можливих. Розглянуто деякі конкретні випадки однорідних ядер та вагових функцій. Це дає змогу узагальнити деякі відомі результати.

1. Introduction. Hilbert's inequality is one of the most significant weighted inequalities in mathematical analysis and its applications. Through the years, Hilbert-type inequalities were discussed by numerous authors, who either reproved them using various techniques, or applied and generalized them in many different ways. For more details about Hilbert's inequality the reader is referred to [1] or [3].

Although classical, Hilbert's inequality is still of interest to numerous mathematicians. In this paper we refer to the recent paper [2], where Q. Huang obtained multidimensional discrete Hilbert-type inequality equipped with conjugate parameters. His result is contained in the following theorem.

Theorem 1.1. *Suppose that* $n \in \mathbb{N} \setminus \{1\}$, $p_i, r_i > 1$, $i = 1, \ldots, n$, $\sum_{i=1}^{n} \dfrac{1}{p_i} = \sum_{i=1}^{n} \dfrac{1}{r_i} = 1$, $\dfrac{1}{q_n} = 1 - \dfrac{1}{p_n}$, $\lambda > 0$, $0 < \alpha < 2$, $\beta \geq -\dfrac{1}{2}$, $\lambda\alpha \max\left\{\dfrac{1}{2-\alpha}, 1\right\} \leq \min_{1\leq i\leq n}\{r_i\}$, $a_{m_i}^{(i)} \geq 0$, $m_i \in \mathbb{N}$, *such that*

$$0 < \sum_{m_i=1}^{\infty} (m_i + \beta)^{p_i(1-\lambda\alpha/r_i)-1} \left(a_{m_i}^{(i)}\right)^{p_i} < \infty, \quad i = 1, \ldots, n.$$

Then the following two inequalities hold and are equivalent:

$$\sum_{m_n=1}^{\infty} \cdots \sum_{m_1=1}^{\infty} \frac{1}{\left[\sum_{i=1}^{n} (m_i + \beta)^{\alpha}\right]^{\lambda}} \prod_{i=1}^{n} a_{m_i}^{(i)} <$$

$$< \frac{\alpha^{1-n}}{\Gamma(\lambda)} \prod_{i=1}^{n} \Gamma\left(\frac{\lambda}{r_i}\right) \left(\sum_{m_i=1}^{\infty} (m_i + \beta)^{p_i\left(1-\frac{\lambda\alpha}{r_i}\right)-1} \left(a_{m_i}^{(i)}\right)^{p_i} \right)^{1/p_i},$$

* This research was supported by the Croatian Ministry of Science, Education, and Sports, under Research Grants 036-1170889-1054 (first author) and 058-1170889-1050 (second author).

$$\left[\sum_{m_n=1}^{\infty}(m_n+\beta)^{\frac{\lambda\alpha q_n}{r_n}-1}\left(\sum_{m_{n-1}=1}^{\infty}\ldots\sum_{m_1=1}^{\infty}\frac{\prod_{i=1}^{n-1}a_{m_i}^{(i)}}{\left[\sum_{i=1}^{n}(m_i+\beta)^{\alpha}\right]^{\lambda}}\right)^{q_n}\right]^{1/q_n} <$$

$$< \frac{\Gamma\left(\lambda/r_n\right)}{\alpha^{n-1}\Gamma(\lambda)}\prod_{i=1}^{n-1}\Gamma\left(\frac{\lambda}{r_i}\right)\left(\sum_{m_i=1}^{\infty}(m_i+\beta)^{p_i\left(1-\frac{\lambda\alpha}{r_i}\right)-1}\left(a_{m_i}^{(i)}\right)^{p_i}\right)^{1/p_i}.$$

The constant factor $\dfrac{\alpha^{1-n}}{\Gamma(\lambda)}\prod_{i=1}^{n}\Gamma\left(\dfrac{\lambda}{r_i}\right)$ *is the best possible.*

In the previous theorem Γ denotes the usual Gamma function. Besides, the best possible constant factor means that it can not be replaced with a smaller constant, so that the appropriate inequality still holds.

On the other hand Yang et al. [6], obtained the result which provides an unified treatment of multidimensional Hilbert-type inequality in the setting with conjugate exponents. All the measures are assumed to be σ-finite on measure space Ω.

Theorem 1.2. *Let $n \geq 2$ be an integer and let $p_1, \ldots, p_n$ be conjugate parameters such that $p_i > 1$, $i = 1, \ldots, n$. Let $K\colon \Omega^n \to \mathbf{R}$ and $\phi_{i,j}\colon \Omega \to \mathbf{R}$, $i, j = 1, \ldots, n$, be nonnegative measurable functions such that $\prod_{i,j=1}^{n}\phi_{ij}(x_j) = 1$. Then the following inequalities hold and are equivalent*:

$$\int\limits_{\Omega^n} K(x_1,\ldots,x_n)\prod_{i=1}^{n} f_i(x_i)d\mu_1(x_1)\ldots d\mu_n(x_n) \leq$$

$$\leq \prod_{i=1}^{n}\left(\int\limits_{\Omega} F_i(x_i)f_i^{p_i}(x_i)\phi_{ii}^{p_i}(x_i)d\mu_i(x_i)\right)^{1/p_i} \tag{1.1}$$

and

$$\int\limits_{\Omega} h(x_n)\left(\int\limits_{\Omega^{n-1}} K(x_1,\ldots,x_n)\prod_{i=1}^{n-1} f_i(x_i)d\mu_1(x_1)\ldots d\mu_{n-1}(x_{n-1})\right)^{q_n} d\mu_n(x_n) \leq$$

$$\leq \prod_{i=1}^{n-1}\left(\int\limits_{\Omega} F_i(x_i)f_i^{p_i}(x_i)\phi_{ii}^{p_i}(x_i)d\mu_i(x_i)\right)^{q_n/p_i}, \tag{1.2}$$

where

$$F_i(x_i) = \int\limits_{\Omega^{n-1}} K(x_1,\ldots,x_n)\times$$

$$\times\prod_{j=1, j\neq i}^{n}\phi_{ij}^{p_i}(x_j)\, d\mu_1(x_1)\ldots d\mu_{i-1}(x_{i-1})d\mu_{i+1}(x_{i+1})\ldots d\mu_n(x_n), \quad i = 1,\ldots,n,$$

$$h(x_n)=\phi_{nn}^{-q_n}(x_n)F_n^{1-q_n}(x_n) \qquad \textit{and} \qquad \frac{1}{q_n}=\sum_{i=1}^{n-1}\frac{1}{p_i}.$$

In the literature, the inequalities related to (1.1) are usually referred to as the Hilbert-type inequalities, while the inequalities related to (1.2) are usually called Hardy – Hilbert-type inequalities. For more details about their equivalence, the reader is referred to [6].

The main purpose of this paper is to generalize Theorem 1.1 in the view of Theorem 1.2. More precisely, in the sequel we derive the discrete forms of inequalities (1.1) and (1.2) containing the homogeneous kernel. Besides, considerable attention is dedicated to the investigation of the best possible constant factors in obtained inequalities, which can be achieved in some general settings. As an application, we also consider some particular settings of our general results which reduce to some recent results, known from the literature.

The techniques that will be used in the proofs are mainly based on classical real analysis. Further, throughout the whole paper, without further explanation, all the series and integrals are assumed to be convergent.

2. Main results. We start this section with the application of Theorem 1.2, which will give the discrete forms of inequalities (1.1) and (1.2). By using the notations as in the above mentioned theorem, we consider the case where $\Omega=\mathbb{N}$, the measures $\mu_i,\ i=1,\ldots,n$, are counting measures, and the kernel K is the nonnegative homogeneous function of degree $-\lambda,\ \lambda>0$.

In order to obtain the constant factors involved in the inequalities, we define the function $k\,(\beta_1,\ldots,\beta_{n-1})$ by

$$k\,(\beta_1,\ldots,\beta_{n-1}):=\int\limits_{(0,\infty)^{n-1}}K(1,t_1\ldots,t_{n-1})t_1^{\beta_1}\ldots t_{n-1}^{\beta_{n-1}}dt_1\ldots dt_{n-1}, \tag{2.1}$$

where we suppose that $k\,(\beta_1,\ldots,\beta_{n-1})<\infty$ for $\beta_1,\ldots,\beta_{n-1}>-1$ and $\beta_1+\ldots+\beta_{n-1}+n<\lambda+1$.

Further, let $A_{ij},\ i,j=1,\ldots,n$, be the real numbers satisfying

$$\sum_{i=1}^{n}A_{ij}=0, \quad j=1,2,\ldots,n. \tag{2.2}$$

We also define

$$\alpha_i=\sum_{j=1}^{n}A_{ij}, \quad i=1,2,\ldots,n. \tag{2.3}$$

Besides, we consider the discrete weighted functions involving real differentiable functions. More precisely, we have the following definition.

Definition 2.1. *Let $r\in\mathbb{R}$. We denote by $H(r)$ the set of all nonnegative differentiable functions $u\colon(0,\infty)\to\mathbb{R}$ satisfying the following conditions*:

(1) *u is strictly increasing on $(0,\infty)$ and there exists $x_0\in(0,\infty)$ such that $u(x_0)=1$;*

(2) *$\lim_{x\to\infty}u(x)=\infty$, $[u(x)]^r u'(x)$ is decreasing on $(0,\infty)$.*

Now, regarding the above notations and definitions, we are ready to state and prove our first general result.

Theorem 2.1. *Let* $p_1, \ldots, p_n$ *be conjugate parameters such that* $p_i > 1$, $i = 1, \ldots, n$, *and let* $\dfrac{1}{q_n} = \sum_{i=1}^{n-1} \dfrac{1}{p_i}$. *Let* $K\colon (0,\infty)^n \to \mathbb{R}$ *be nonnegative homogeneous function of degree* $-\lambda$, $\lambda > 0$, *strictly decreasing in each variable, and let* A_{ij}, $i, j = 1, \ldots, n$, *and* α_i, $i = 1, \ldots, n$, *be real parameters satisfying* (2.2) *and* (2.3). *If* $a_{m_i}^{(i)} \geq 0$, $m_i \in \mathbb{N}$, *and* $u_i \in H(p_i A_{ij})$, $i, j = 1, \ldots, n$, $i \neq j$, *then we have the following equivalent inequalities*:

$$\sum_{m_n=1}^{\infty} \ldots \sum_{m_1=1}^{\infty} K(u_1(m_1), \ldots, u_n(m_n)) \prod_{i=1}^{n} a_{m_i}^{(i)} \leq$$

$$\leq L \prod_{i=1}^{n} \left(\sum_{m_i=1}^{\infty} [u_i(m_i)]^{n-\lambda-1+p_i\alpha_i} [u_i'(m_i)]^{1-p_i} \left(a_{m_i}^{(i)}\right)^{p_i} \right)^{1/p_i}, \tag{2.4}$$

$$\left[\sum_{m_n=1}^{\infty} [u_n(m_n)]^{(1-q_n)(n-1-\lambda)-q_n\alpha_n} \times \right.$$

$$\left. \times \left(\sum_{m_{n-1}=1}^{\infty} \ldots \sum_{m_1=1}^{\infty} K(u_1(m_1), \ldots, u_n(m_n)) \prod_{i=1}^{n-1} a_{m_i}^{(i)} \right)^{q_n} \right]^{1/q_n} \leq$$

$$\leq L \prod_{i=1}^{n-1} \left(\sum_{m_i=1}^{\infty} [u_i(m_i)]^{n-\lambda-1+p_i\alpha_i} [u_i'(m_i)]^{1-p_i} \left(a_{m_i}^{(i)}\right)^{p_i} \right)^{1/p_i}, \tag{2.5}$$

where

$$L = k(p_1A_{12}, \ldots, p_1A_{1n})^{1/p_1} k(\lambda - n - p_2(\alpha_2 - A_{22}), p_2A_{23}, \ldots, p_2A_{2n})^{1/p_2} \ldots$$

$$\ldots k(p_nA_{n2}, \ldots, p_nA_{n,n-1}, \lambda - n - p_n(\alpha_n - A_{nn}))^{1/p_n}, \tag{2.6}$$

and $p_iA_{ij} > -1$, $i \neq j$, $p_i(A_{ii} - \alpha_i) > n - \lambda - 1$.

Proof. Rewrite the inequality (1.1) for the counting measure on $\mathbb{N}$,

$$(\phi_{ij} \circ u_j)(m_j) = \big[u_j(m_j)\big]^{A_{ij}} \big[u_j'(m_j)\big]^{1/p_i}, \quad i \neq j,$$

$$(\phi_{ii} \circ u_i)(m_i) = \big[u_i(m_i)\big]^{A_{ii}} \big[u_i'(m_i)\big]^{1/p_i-1},$$

and the sequences $\big(a_{m_i}^{(i)}\big)$, $i = 1, \ldots, n$. Obviously, these substitutions are well defined, since u_i, $i = 1, \ldots, n$, are injective functions. Thus, in the above setting we have

$$\sum_{m_n=1}^{\infty} \ldots \sum_{m_1=1}^{\infty} K\big(u_1(m_1), \ldots, u_n(m_n)\big) \prod_{i=1}^{n} a_{m_i}^{(i)} \leq$$

$$\leq \prod_{i=1}^{n} \left(\sum_{m_i=1}^{\infty} \big[u_i(m_i)\big]^{p_iA_{ii}} \big[u_i'(m_i)\big]^{1-p_i} (F \circ u_i)(m_i) \left(a_{m_i}^{(i)}\right)^{p_i} \right)^{1/p_i}, \tag{2.7}$$

where

$$(F \circ u_i)(m_i) = \sum_{m_n=1}^{\infty} \ldots \sum_{m_{i+1}=1}^{\infty} \sum_{m_{i-1}=1}^{\infty} \sum_{m_1=1}^{\infty} K\big(u_1(m_1), \ldots, u_n(m_n)\big) \times$$

$$\times \left(\prod_{j=1, j \neq i}^{n} [u_j(m_j)]^{p_i A_{ij}} u_j'(m_j) \right).$$

Our next task is to estimate the functions $(F \circ u_i)(m_i)$, $i = 1, \ldots, n$. Since the kernel K is strictly decreasing in each variable and $u_i \in H(p_i A_{ij})$, $i \neq j$, we conclude that the functions $F_i \circ u_i$, $i = 1, \ldots, n$, are strictly decreasing. Hence, we have

$$(F_1 \circ u_1)(m_1) \leq \int\limits_{(0,\infty)^{n-1}} K\big(u_1(m_1), u_2(x_2), \ldots, u_n(x_n)\big) \times$$

$$\times \prod_{j=2}^{n} \Big([u_j(x_j)]^{p_1 A_{1j}} u_j'(x_j) \Big) dx_2 \ldots dx_n, \tag{2.8}$$

since the left-hand side of this inequality is obviously the lower Darboux sum for the integral on the right-hand side of inequality. Further, by using the substitution $t_i = u_i(x_i)$, $i = 2, \ldots, n$, from (2) we get

$$(F_1 \circ u_1)(m_1) \leq \int\limits_{(0,\infty)^{n-1}} K(u_1(m_1), t_2, \ldots, t_n) \prod_{j=2}^{n} t_j^{p_1 A_{1j}} dt_2 \ldots dt_n,$$

wherefrom by using the homogeneity of the kernel K and the obvious change of variables, we have

$$(F_1 \circ u_1)(m_1) \leq \int\limits_{(0,\infty)^{n-1}} \big[u_1(m_1)\big]^{-\lambda} K\big(1, t_2/u_1(m_1), \ldots, t_n/u_1(m_1)\big) \times$$

$$\times \prod_{j=2}^{n} t_j^{p_1 A_{1j}} dt_2 \ldots dt_n =$$

$$= \big[u_1(m_1)\big]^{n-1-\lambda+p_1\left(\alpha_1 - A_{11}\right)} k\big(p_1 A_{12}, \ldots, p_1 A_{1n}\big).$$

By using the same arguments as for the function $F_1 \circ u_1$, we also get

$$(F_2 \circ u_2)(m_2) \leq \int\limits_{(0,\infty)^{n-1}} K(t_1, u_2(m_2), t_3, \ldots, t_n) \prod_{j=1, j \neq 2}^{n} t_j^{p_2 A_{2j}} dt_1 dt_3 \ldots dt_n. \tag{2.9}$$

Now, let J denotes the right-hand side of the inequality (2.9). It is easy to see that the transformation of variables

$$t_1 = u_2(m_2) \frac{1}{v_2}, \qquad t_i = u_2(m_2) \frac{v_i}{v_2}, \quad i = 3, \ldots, n,$$

yields

$$\frac{\partial(t_1, t_3, \ldots, t_n)}{\partial(v_2, v_3, \ldots, v_n)} = \big[u_2(m_2)\big]^{n-1} v_2^{-n},$$

where $\dfrac{\partial(t_1, t_3, \ldots, t_n)}{\partial(v_2, v_3, \ldots, v_n)}$ denotes the Jacobian of the transformation.

Now, by using the homogeneity of the kernel K and the above change of variables, we have

$$J = \int\limits_{(0,\infty)^{n-1}} t_1^{-\lambda} K\big(1, u_2(m_2)/t_1, t_3/t_1 \ldots, t_n/t_1\big) \prod_{j=1, j\neq 2}^{n} t_j^{p_2 A_{2j}} dt_1 dt_3 \ldots dt_n =$$

$$= \int\limits_{(0,\infty)^{n-1}} [u_2(m_2)]^{-\lambda} v_2^{\lambda} K(1, v_2, \ldots, v_n) [u_2(m_2)]^{p_2(\alpha_2 - A_{22})} \times$$

$$\times v_2^{-p_2(\alpha_2 - A_{22})} v_2^{p_2 A_{23}} \ldots v_n^{p_2 A_{2n}} [u_2(m_2)]^{n-1} v_2^{-n} dv_2 dv_3 \ldots dv_n =$$

$$= [u_2(m_2)]^{n-1-\lambda+p_2(\alpha_2 - A_{22})} \int\limits_{(0,\infty)^{n-1}} v_2^{\lambda - n - p_2(\alpha_2 - A_{22})} \prod_{j=3}^{n} v_j^{p_2 A_{2j}} dv_2 \ldots dv_n =$$

$$= [u_2(m_2)]^{n-1-\lambda+p_2(\alpha_2 - A_{22})} k(\lambda - n - p_2(\alpha_2 - A_{22}), p_2 A_{23}, \ldots, p_2 A_{2n}).$$

Hence, inequality (2.9) and the above equality yield

$$(F_2 \circ u_2)(m_2) \leq [u_2(m_2)]^{n-1-\lambda+p_2(\alpha_2 - A_{22})} k\big(\lambda - n - p_2(\alpha_2 - A_{22}), p_2 A_{23}, \ldots, p_2 A_{2n}\big).$$

In a similar manner we obtain

$$(F_i \circ u_i)(m_i) \leq [u_i(m_i)]^{n-1-\lambda+p_i(\alpha_i - A_{ii})} \times$$

$$\times k\big(p_i A_{i2}, \ldots, p_i A_{i,i-1}, s - n - p_i(\alpha_i - A_{ii}), p_i A_{i,i+1}, \ldots, p_i A_{in}\big),$$

for $i = 3, \ldots, n$. This completes the proof of inequality (2.4).

The proof of the inequality (2.5) follows from the inequality (1.2), by using the same estimates as in the first part of the proof.

The next problem we are dealing with in this section is to determine the conditions under which the constant factor L, defined by (2.6), is the best possible in inequalities (2.4) and (2.5). Considering Theorem 1.1, we see that the appropriate constant factor does not include any exponent. Bearing in mind that fact, we shall find the conditions under which the constant L reduces to the form without any exponents.

In order to obtain the constant factor without exponents, it is natural to impose the following conditions on the parameters A_{ij}:

$$p_j A_{ji} = \lambda - n - p_i(\alpha_i - A_{ii}), \qquad i, j = 1, 2, \ldots, n, \quad i \neq j. \tag{2.10}$$

If the parameters A_{ij} satisfy the set of conditions (2.10), then the constant L from Theorem 2.1 reduces to the form

$$L^* = k(\widetilde{A}_2, \ldots, \widetilde{A}_n), \tag{2.11}$$

where we use the abbreviations

$$\widetilde{A}_i = p_j A_{ji}, \qquad i, j = 1, 2, \ldots, n, \quad i \neq j. \tag{2.12}$$

Regarding the set of conditions (2.10), it is easy to see that the parameters $\widetilde{A}_i$ satisfy the relation

$$\sum_{i=1}^{n} \widetilde{A}_i = \lambda - n. \tag{2.13}$$

Furthermore, by using (2.2) and (2.12), we have the following relationship between the parameters A_{ii} and $\widetilde{A}_i$, $i = 1, 2, \ldots, n$:

$$A_{ii} = -A_{1i} - A_{2i} - \ldots - A_{i-1,i} - A_{i+1,i} - \ldots - A_{ni} =$$

$$= -\frac{\widetilde{A}_i}{p_1} - \frac{\widetilde{A}_i}{p_2} - \ldots - \frac{\widetilde{A}_i}{p_{i-1}} - \frac{\widetilde{A}_i}{p_{i+1}} - \ldots - \frac{\widetilde{A}_i}{p_n} =$$

$$= \widetilde{A}_i \left(\frac{1}{p_i} - 1 \right). \tag{2.14}$$

Now, taking into account the relations (2.11), (2.12), and (2.14), the inequalities (2.4) and (2.5) with the parameters A_{ij}, $i, j = 1, 2, \ldots, n$, satisfying the set of conditions (2.10), become

$$\sum_{m_n=1}^{\infty} \ldots \sum_{m_1=1}^{\infty} K(u_1(m_1), \ldots, u_n(m_n)) \prod_{i=1}^{n} a_{m_i}^{(i)} \leq$$

$$\leq L^* \prod_{i=1}^{n} \left(\sum_{m_i=1}^{\infty} [u_i(m_i)]^{-1-p_i \widetilde{A}_i} [u_i'(m_i)]^{1-p_i} \left(a_{m_i}^{(i)} \right)^{p_i} \right)^{1/p_i}, \tag{2.15}$$

and

$$\left[\sum_{m_n=1}^{\infty} [u_n(m_n)]^{(1-q_n)(-1-p_n \widetilde{A}_n)} \times \right.$$

$$\left. \times \left(\sum_{m_{n-1}=1}^{\infty} \ldots \sum_{m_1=1}^{\infty} K(u_1(m_1), \ldots, u_n(m_n)) \prod_{i=1}^{n-1} a_{m_i}^{(i)} \right)^{q_n} \right]^{1/q_n} \leq$$

$$\leq L^* \prod_{i=1}^{n-1} \left(\sum_{m_i=1}^{\infty} [u_i(m_i)]^{-1-p_i \widetilde{A}_i} [u_i'(m_i)]^{1-p_i} \left(a_{m_i}^{(i)} \right)^{p_i} \right)^{1/p_i}, \tag{2.16}$$

where the constant factor L^* is defined by (2.11).

Now we are ready to prove that the constant factor L^* is the best possible in both inequalities (2.15) and (2.16). That is the content of the following theorem.

Theorem 2.2. *If the parameters* A_{ij}, $i, j = 1, \ldots, n$, *satisfy the conditions* (2.2) *and* (2.10), *then the constant factor* L^* *is the best possible in both inequalities* (2.15) *and* (2.16).

Proof. It is enough to show that the constant factor L^* is the best possible in inequality (2.15), since (2.15) and (2.16) are equivalent inequalities.

For that sake, we consider the real sequences $\widetilde{a}_{m_i}^{(i)} = [u_i(m_i)]^{\widetilde{A}_i - \varepsilon/p} u_i'(m_i)$, where $\varepsilon > 0$ is sufficiently small number. Since $u_i \in H(\widetilde{A}_i)$, $i = 1, \ldots, n$, we may assume that u_i is strictly increasing on $(0, \infty)$ and that there exists $x_0 \in (0, \infty)$ such that $u_i(x_0) = 1$.

Therefore, by considering integral sums, we have

$$\frac{1}{\varepsilon} = \int_1^\infty [u_i(x)]^{-1-\varepsilon} d[u_i(x)] < \sum_{m_i=1}^\infty [u_i(m_i)]^{-1-\varepsilon} u_i'(m_i) =$$

$$= \sum_{m_i=1}^\infty [u_i(m_i)]^{-1-p_i\widetilde{A}_i} [u_i'(m_i)]^{1-p_i} \left(\widetilde{a}_{m_i}^{(i)}\right)^{p_i} <$$

$$< \vartheta_i(1) + \int_1^\infty [u_i(x)]^{-1-\varepsilon} d[u_i(x)] = \vartheta_i(1) + \frac{1}{\varepsilon},$$

where the function ϑ_i is defined by $\vartheta_i(x) = [u_i(x)]^{-1-\varepsilon} u_i'(x)$. In other words, the following relation is valid:

$$\sum_{m_i=1}^\infty [u_i(m_i)]^{-1-p_i\widetilde{A}_i} [u_i'(m_i)]^{1-p_i} \left(\widetilde{a}_{m_i}^{(i)}\right)^{p_i} = \frac{1}{\varepsilon} + O(1), \quad i = 1, \ldots, n. \tag{2.17}$$

Now, let us suppose that there exists a positive constant M, smaller than L^*, such that the inequality (2.15) is still valid, if we replace L^* with M. Hence, if we insert relations (2.17) in inequality (2.15), with the constant M instead of L^*, we get

$$\widetilde{I} := \sum_{m_n=1}^\infty \ldots \sum_{m_1=1}^\infty K\big(u_1(m_1), \ldots, u_n(m_n)\big) \prod_{i=1}^n \widetilde{a}_{m_i}^{(i)} < \frac{1}{\varepsilon}(M + o(1)). \tag{2.18}$$

On the other hand, let us estimate the left-hand side of inequality (2.15). Namely, by inserting the above defined sequences $\left(\widetilde{a}_{m_i}^{(i)}\right)_{m_i \in \mathbb{N}}$ in the left-hand side of inequality (2.15), we easily get the inequality

$$\widetilde{I} > \int_1^\infty [u_1(x_1)]^{\widetilde{A}_1 - \varepsilon/p_1} \left(\int_1^\infty \ldots \int_1^\infty K(u_1(x_1), \ldots, u_n(x_n)) \times \right.$$

$$\left. \times \prod_{i=2}^n [u_i(x_i)]^{\widetilde{A}_i - \varepsilon/p_i} d[u_2(x_2)] \ldots d[u_n(x_n)] \right) d[u_1(x_1)]. \tag{2.19}$$

Further, let J denotes the right-hand side of the inequality (2.19). By using the substitution $t_i = \dfrac{u_i(x_i)}{u_1(x_1)}$, $i = 2, \ldots, n$, we find that

$$J = \int_1^\infty [u_1(x_1)]^{-1-\varepsilon} \left[\int_{1/u_1(x_1)}^\infty \ldots \int_{1/u_1(x_1)}^\infty K(1, t_2, \ldots, t_n) \prod_{i=2}^n t_i^{\widetilde{A}_i - \varepsilon/p_i} dt_2 \ldots dt_n \right] d[u_1(x_1)].$$

Now, considering the obtained expression for J, we easily get inequality

$$J \geq \int_1^\infty [u_1(x_1)]^{-1-\varepsilon} \left[\int_{(0,\infty)^{n-1}} K(1, t_2, \ldots, t_n) \prod_{i=2}^n t_i^{\widetilde{A}_i - \varepsilon/p_i} dt_2 \ldots dt_n \right] d[u_1(x_1)] -$$

$$- \int_1^\infty [u_1(x_1)]^{-1-\varepsilon} \sum_{j=2}^n I_j(u_1) d[u_1(x_1)], \tag{2.20}$$

where for $j = 2, \ldots, n$, $I_j(u_1)$ is defined by

$$I_j(u_1) = \int_{D_j} K(1, t_2, \ldots, t_n) \prod_{i=2}^n t_i^{\widetilde{A}_i - \varepsilon/p_i} dt_2 \ldots dt_n,$$

and $D_j = \left\{ (t_2, t_3, \ldots, t_n);\ 0 < t_j \leq \dfrac{1}{u_1(x_1)},\ 0 < t_k < \infty, k \neq j \right\}$.

Without losing generality, it is enough to estimate the integral $I_2(x_1)$. Obviously, since $1 - t_2^\varepsilon \to 1$ $(t_2 \to 0^+)$, there exists the constant $C \geq 0$ such that $1 - t_2^\varepsilon \leq C$ $(t_2 \in (0, 1])$. Now, by using the well-known Fubini's theorem, it follows that

$$0 \leq \varepsilon \int_1^\infty [u_1(x_1)]^{-1-\varepsilon} I_2(u_1) d[u_1(x_1)] =$$

$$= \varepsilon \int_1^\infty [u_1(x_1)]^{-1-\varepsilon} \left[\int_{(0,\infty)^{n-2}} \int_0^{1/u_1(x_1)} K(1, t_2, \ldots, t_n) \prod_{i=2}^n t_i^{\widetilde{A}_i - \varepsilon/p_i} dt_2 \ldots dt_n \right] d[u_1(x_1)] =$$

$$= \varepsilon \int_{(0,\infty)^{n-2}} \int_0^1 K(1, t_2, \ldots, t_n) \prod_{i=2}^n t_i^{\widetilde{A}_i - \varepsilon/p_i} \left(\int_1^{1/t_2} t_1^{-1-\varepsilon} dt_1 \right) dt_2 \ldots dt_n =$$

$$= \varepsilon \int_{(0,\infty)^{n-2}} \int_0^1 K(1, t_2, \ldots, t_n) \prod_{i=2}^n t_i^{\widetilde{A}_i - \varepsilon/p_i} \left(\frac{1}{\varepsilon} (1 - t_2^\varepsilon) \right) dt_2 \ldots dt_n \leq$$

$$\leq C \int_{(0,\infty)^{n-2}} \int_0^1 K(1, t_2, \ldots, t_n) \prod_{i=2}^n t_i^{\widetilde{A}_i - \varepsilon/p_i} dt_2 \ldots dt_n \leq$$

$$\leq C \int\limits_{(0,\infty)^{n-1}} K(1,t_2,\ldots,t_n)\prod_{i=2}^{n} t_i^{\widetilde{A}_i-\varepsilon/p_i}dt_2\ldots dt_n =$$

$$= Ck\left(\widetilde{A}_2-\frac{\varepsilon}{p_2},\ldots,\widetilde{A}_n-\frac{\varepsilon}{p_n}\right) < \infty.$$

Further, regarding the above derived relation and inequality (2.20), we have that

$$\widetilde{I} \geq \frac{1}{\varepsilon}k\left(\widetilde{A}_2-\frac{\varepsilon}{p_2},\ldots,\widetilde{A}_n-\frac{\varepsilon}{p_n}\right) - o(1). \tag{2.21}$$

Finally, by comparing the relations (2.18) and (2.21), we conclude that $L^* \leq M$ when $\varepsilon \to 0^+$, which is an obvious contradiction. Hence, it follows that the constant factor L^* is the best possible in (2.15).

Clearly, the constant factor L^* is also the best possible in the inequality (2.16) since the equivalence keeps the best possible constant.

Theorem 2.2 is proved.

3. Some applications. This section is dedicated to the applications of our general results, i.e., Theorems 2.1 and 2.2, to some particular choices of homogeneous kernels $K\colon (0,\infty)^n \to \mathbb{R}$, differentiable functions $u_i\colon (0,\infty) \to \mathbb{R}$, $i = 1,\ldots,n$, and real parameters $\widetilde{A}_i$, $i = 1,\ldots,n$, defined in the previous section.

Here, we shall be concerned with the homogeneous function

$$K_1(x_1,\ldots,x_n) = \frac{1}{(x_1+\ldots+x_n)^\lambda}, \quad \lambda > 0.$$

Note that the kernel K_1 is symmetric, strictly decreasing in each variable, and

$$k(\beta_1-1,\ldots,\beta_{n-1}-1) = \int\limits_{(0,\infty)^{n-1}} \frac{\prod_{i=1}^{n-1} t_i^{\beta_i-1}}{\left(1+\sum_{i=1}^{n-1} t_i\right)^\lambda}dt_1\ldots dt_{n-1} =$$

$$= \frac{\Gamma\left(\lambda-\sum_{i=1}^{n-1}\beta_i\right)\prod_{i=1}^{n-1}\Gamma(\beta_i)}{\Gamma(\lambda)}, \tag{3.1}$$

where we used the integral formula derived in the paper [4]. Now, in the above described setting, as an immediate consequence of Theorems 2.1 and 2.2, we get the following result.

***Corollary* 3.1.** *Suppose the parameters* q_n, p_i, A_{ij}, $i,j = 1,\ldots,n$, *and the functions* u_i: $(0,\infty) \to \mathbb{R}$, $i = 1,\ldots,n$, *are defined as in statement of Theorem* 2.1. *If the parameters* A_{ij}, $i,j = 1,\ldots,n$, *fulfill the set of conditions* (2.10), *then the inequalities*

$$\sum_{m_n=1}^{\infty}\cdots\sum_{m_1=1}^{\infty}\frac{\prod_{i=1}^{n} a_{m_i}^{(i)}}{\left(\sum_{i=1}^{n} u_i(m_i)\right)^\lambda} \leq$$

$$\leq L_1\prod_{i=1}^{n}\left(\sum_{m_i=1}^{\infty}[u_i(m_i)]^{-1-p_i\widetilde{A}_i}[u_i'(m_i)]^{1-p_i}\left(a_{m_i}^{(i)}\right)^{p_i}\right)^{1/p_i} \tag{3.2}$$

and

$$\left[\sum_{m_n=1}^{\infty}[u_n(m_n)]^{(1-q_n)(-1-p_n\widetilde{A}_n)}\times\right.$$

$$\left.\times\left(\sum_{m_{n-1}=1}^{\infty}\cdots\sum_{m_1=1}^{\infty}\frac{\prod_{i=1}^{n}a_{m_i}^{(i)}}{\left(\sum_{i=1}^{n}u_i(m_i)\right)^{\lambda}}\right)^{q_n}\right]^{1/q_n}\leq$$

$$\leq L_1\prod_{i=1}^{n-1}\left(\sum_{m_i=1}^{\infty}[u_i(m_i)]^{-1-p_i\widetilde{A}_i}[u_i'(m_i)]^{1-p_i}\left(a_{m_i}^{(i)}\right)^{p_i}\right)^{1/p_i}, \tag{3.3}$$

where $L_1=\dfrac{\prod_{i=1}^{n}\Gamma(\widetilde{A}_i+1)}{\Gamma(\lambda)}$, *hold for all nonnegative real sequences* $\left(a_{m_i}^{(i)}\right)_{m_i\in\mathbb{N}}$ *and are equivalent. Moreover, the constant factor* L_1 *is the best possible in both inequalities* (3.2) *and* (3.3).

***Remark* 3.1.** Note that inequalities (3.2) and (3.3) involve the parameters $\widetilde{A}_i$, $i=1,2,\ldots,n$, since the parameters A_{ij}, $i,j=1,2,\ldots,n$, satisfy the set of conditions (2.10).

The following remark describes the connection between our Corollary 3.1 and Theorem 1.1 in detail.

***Remark* 3.2.** It is obvious that our Corollary 3.1 is the generalization of Theorem 1.1 from the Introduction (see also [2]). Namely if we substitute the power functions $u_i(x_i)=(x_i+\beta)^{\alpha}$ and the parameters $\widetilde{A}_i=\dfrac{\lambda}{r_i}-1$, $i=1,\ldots,n$, in Corollary 3.1 we get the inequalities from Theorem 1.1 with the best possible constant factor $\dfrac{\alpha^{1-n}}{\Gamma(\lambda)}\prod_{i=1}^{n}\Gamma\left(\dfrac{\lambda}{r_i}\right)$.

We conclude this paper with yet another consequence of Corollary 3.1, known from the literature.

***Remark* 3.3.** Let

$$A_{ii}=\frac{(n-\lambda)(p_i-1)}{p_i^2}\qquad\text{and}\qquad A_{ij}=\frac{\lambda-n}{p_ip_j},\qquad i,j=1,2,\ldots,n,\quad i\neq j, \tag{3.4}$$

where p_i, $i=1,2,\ldots,n$, are conjugate exponents. These parameters are symmetric and

$$\sum_{i=1}^{n}A_{ij}=\sum_{j=1}^{n}A_{ij}=\frac{(n-\lambda)(p_i-1)}{p_i^2}+\sum_{j=1,j\neq i}^{n}\frac{\lambda-n}{p_ip_j}=\frac{n-\lambda}{p_i}\left(1-\sum_{j=1}^{n}\frac{1}{p_j}\right)=0.$$

Moreover, the above defined parameters satisfy the set of conditions (2.10), so the resulting relations will include the best possible constant factors.

Now, for the above choice of parameters A_{ij} defined by (3.4), and the functions $u_i(x_i)=x_i$, the inequalities (3.2) and (3.3) respectively read

$$\sum_{m_n=1}^{\infty}\cdots\sum_{m_1=1}^{\infty}\frac{\prod_{i=1}^{n}a_{m_i}^{(i)}}{\left(\sum_{i=1}^{n}m_i\right)^{\lambda}}\leq L_2\prod_{i=1}^{n}\left(\sum_{m_i=1}^{\infty}{m_i}^{n-1-\lambda}\left(a_{m_i}^{(i)}\right)^{p_i}\right)^{1/p_i} \tag{3.5}$$

and

$$\left[\sum_{m_n=1}^{\infty} m_n^{(1-q_n)(p_n-\lambda-1)} \left(\sum_{m_{n-1}=1}^{\infty} \cdots \sum_{m_1=1}^{\infty} \frac{\prod_{i=1}^{n} a_{m_i}^{(i)}}{\left(\sum_{i=1}^{n} m_i\right)^{\lambda}}\right)^{q_n}\right]^{1/q_n} \leq$$

$$\leq L_2 \prod_{i=1}^{n-1} \left(\sum_{m_i=1}^{\infty} m_i^{n-1-\lambda} \left(a_{m_i}^{(i)}\right)^{p_i}\right)^{1/p_i}, \tag{3.6}$$

where $L_2 = \dfrac{1}{\Gamma(\lambda)} \prod_{i=1}^{n} \Gamma\left(\dfrac{p_i+\lambda-n}{p_i}\right)$. Note that the condition $\lambda \leq \min_{1\leq i\leq n}\{p_i\}$ must be satisfied, so that the function u_i belongs to the set $H(p_i A_{ij})$, $i, j = 1, 2, \ldots, n$ (see the statement of Theorem 2.1). Moreover, since we consider the Gamma function with positive argument, inequalities (3.5) and (3.6) hold under condition $n - \min_{1\leq i\leq n}\{p_i\} \leq \lambda \leq \min_{1\leq i\leq n}\{p_i\}$.

Finally, let us mention that our inequality (3.5) is a discrete variant of the appropriate integral result from paper [4].

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Received 08.07.11