UDC 517.5

G. Tephnadze (Tbilisi State Univ., Georgia)

STRONG CONVERGENCE OF TWO-DIMENSIONAL WALSH – FOURIER SERIES

СИЛЬНА ЗБІЖНІСТЬ ДВОВИМІРНИХ РЯДІВ УОЛША – ФУР'Є

We prove that certain means of quadratic partial sums of the two-dimensional Walsh – Fourier series are uniformly bounded operators acting from the Hardy space H_p to the space L_p for $0 < p < 1$.

Доведено, що певні середні квадратичних часткових сум двовимірних рядів Уолша – Фур'є є рівномірно обмеженими операторами, що діють із простору Харді H_p у простір L_p при $0 < p < 1$.

1. Introduction. It is known [7, p. 125] that the Walsh – Paley system is not a Schauder basis in $L_1(G)$. Moreover (see [8]), there exists a function in the dyadic Hardy space $H_1(G)$, the partial sums of which are not bounded in $L_1(G)$. However, in Simon [9] the following strong convergence result was obtained for all $f \in H_1$:

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{\|S_k f - f\|_1}{k} = 0,$$

where $S_k f$ denotes the kth partial sum of the Walsh – Fourier series of f (for the trigonometric analogue see Smith [11], for the Vilenkin system see Gát [1]).

Simon [10] proved that there is an absolute constant c_p, depends only p, such that

$$\sum_{k=1}^{\infty} \frac{\|S_k f\|_p^p}{k^{2-p}} \le c_p \|f\|_{H_p}^p, \tag{1}$$

for all $f \in H_p$, where $0 < p < 1$.

The author [13] proved that sequence $\left\{1/k^{2-p}\right\}_{k=1}^{\infty}$ in inequality (1) is important.

For the two-dimensional Walsh – Fourier series Weisz [16] generalized the result of Simon and proved that if $\alpha \ge 0$ and $f \in H_p(G \times G)$, then

$$\sup_{n,m\ge 2} \left(\frac{1}{\log n \log m}\right)^{[p]} \sum_{2^{-\alpha}\le k/l\le 2^{\alpha},(k,l)\le(n,m)} \frac{\|S_{k,l} f\|_p^p}{(kl)^{2-p}} \le c \|f\|_{H_p}^p,$$

where $0 < p < 1$ and $[p]$ denotes the integer part of p.

Goginava and Gogoladze [5] proved that the following result is true:

Theorem G. *Let $f \in H_1(G \times G)$. Then there exists absolute constant c, such that*

$$\sum_{n=1}^{\infty} \frac{\|S_{n,n} f\|_1}{n \log^2 n} \le c \|f\|_{H_1}.$$

For two-dimensional trigonometric system analogical theorem was proved in [6].

Convergence of quadratical partial sums of two-dimensional Walsh – Fourier series was investigated in details by Weisz [15], Goginava [4], Gát, Goginava, Nagy [2], Gát, Goginava, Tkebuchava [3].

The main aim of this paper is to prove (see Theorem 1) that

$$\sum_{n=1}^{\infty} \frac{\|S_{n,n}f\|_p^p}{n^{3-2p}} \leq c_p \|f\|_{H_p}^p , \tag{2}$$

for all $f \in H_p(G \times G)$, where $0 < p < 1$. We also proved that sequence $\left\{1/n^{3-2p}\right\}_{n=1}^{\infty}$ in inequality (2) is important (see Theorem 2).

2. Definitions and notations. Let $\mathbf{P}$ denote the set of positive integers, $\mathbf{N} := \mathbf{P} \cup \{0\}$. Denote by Z_2 the discrete cyclic group of order 2, that is $Z_2 = \{0,1\}$, where the group operation is the modulo 2 addition and every subset is open. The Haar measure on Z_2 is given such that the measure of a singleton is 1/2. Let G be the complete direct product of the countable infinite copies of the compact groups Z_2. The elements of G are of the form $x = (x_0, x_1, \ldots, x_k, \ldots)$ with $x_k \in \{0,1\}$, $k \in \mathbf{N}$. The group operation on G is the coordinate-wise addition, the measure (denote by μ) and the topology are the product measure and topology. The compact Abelian group G is called the Walsh group. A base for the neighborhoods of G can be given in the following way:

$$I_0(x) := G,$$

$$I_n(x) := I_n(x_0, \ldots, x_{n-1}) :=$$

$$:= \left\{ y \in G : \, y = (x_0, \ldots, x_{n-1}, y_n, y_{n+1}, \ldots) \right\}, \quad x \in G, \quad n \in \mathbf{N}.$$

These sets are called the dyadic intervals. Let $0 = (0 \colon i \in \mathbf{N}) \in G$ denote the null element of G, $I_n := I_n(0)$, $n \in \mathbf{N}$. Set $e_n := (0, \ldots, 0, 1, 0, \ldots) \in G$ the n th coordinate of which is 1 and the rest are zeros $(n \in \mathbf{N})$. Let $\overline{I}_n := G \backslash I_n$.

If $n \in \mathbf{N}$, then $n = \sum_{i=0}^{\infty} n_i 2^i$, where $n_i \in \{0,1\}$, $i \in \mathbf{N}$, i.e., n is expressed in the number system of base 2. Denote $|n| := \max\{j \in \mathbf{N} : n_j \neq 0\}$, that is, $2^{|n|} \leq n < 2^{|n|+1}$.

It is easy to show that for every odd number $n_0 = 1$ and we can write $n = 1 + \sum_{i=1}^{|n|} n_j 2^i$, where $n_j \in \{0,1\}$, $j \in \mathbf{P}$.

For $k \in \mathbf{N}$ and $x \in G$ let as denote by

$$r_k(x) := (-1)^{x_k}, \quad x \in G, \quad k \in \mathbf{N},$$

the k th Rademacher function.

The Walsh – Paley system is defined as the sequence of Walsh – Paley functions:

$$w_n(x) := \prod_{k=0}^{\infty} (r_k(x))^{n_k} = r_{|n|}(x) (-1)^{\sum_{k=0}^{|n|-1} n_k x_k}, \quad x \in G, \quad n \in \mathbf{P}.$$

The Walsh – Dirichlet kernel is defined by

$$D_n(x) = \sum_{k=0}^{n-1} w_k(x).$$

Recall that (see [8, p. 7])

$$D_{2^n}(x) = \begin{cases} 2^n, & x \in I_n, \\ 0, & x \in \overline{I}_n. \end{cases} \tag{3}$$

Furthermore, the following representation holds for the D_n's. Let $n \in \mathbf{N}$ and $n = \sum_{i=0}^{\infty} n_i 2^i$. Then

$$D_n(x) = w_n(x) \sum_{j=0}^{\infty} n_j w_{2^j}(x) D_{2^j}(x). \tag{4}$$

The rectangular partial sums of the 2-dimensional Walsh – Fourier series of function $f \in L_2(G \times \times G)$ are defined as follows:

$$S_{M,N} f(x,y) := \sum_{i=0}^{M-1} \sum_{j=0}^{N-1} \widehat{f}(i,j) w_i(x) w_j(y),$$

where the numbers

$$\widehat{f}(i,j) = \int_{G \times G} f(x,y) w_i(x) w_j(y) \, d\mu(x,y)$$

is said to be the (i,j) th Walsh – Fourier coefficient of the function f.

Denote

$$S_M^{(1)} f(x,y) := \int_G f(s,y) D_M(x+s) \, d\mu(s)$$

and

$$S_N^{(2)} f(x,y) := \int_G f(x,t) D_N(y+t) \, d\mu(t).$$

The norm (or quasinorm) of the space $L_p(G \times G)$ is defined by

$$\|f\|_p := \left(\int_{G \times G} |f|^p \, d\mu \right)^{1/p}, \qquad 0 < p < \infty.$$

The space weak $- L_p(G \times G)$ consists of all measurable functions f for which

$$\|f\|_{\text{weak}-L_p(G \times G)} := \sup_{\lambda > 0} \lambda \mu (f > \lambda)^{1/p} < +\infty.$$

The σ-algebra generated by the dyadic 2-dimensional $I_n(x) \times I_n(y)$ square of measure $2^{-n} \times 2^{-n}$ will be denoted by $F_{n,n}$, $n \in \mathbf{N}$. Denote by $f = (f_{n,n},\ n \in \mathbf{N})$ one-parameter martingale with respect to $F_{n,n}$, $n \in \mathbf{N}$.

The expectation operator and the conditional expectation operator relative to the $F_{n,n}$, $n \in \mathbf{N}$, are denoted by E and $E_{n,n}$, respectively.

The maximal function of a martingale f is defined by

$$f^* = \sup_{n \in \mathbf{N}} |f_{n,n}|.$$

Let $f \in L_1(G \times G)$. Then the dyadic maximal function is given by

$$f^{*}(x,y)=\sup_{n\in\mathbf{N}}\frac{1}{\mu\left(I_n(x)\times I_n(y)\right)}\left|\int\limits_{I_n(x)\times I_n(y)} f\left(s,t\right)d\mu\left(s,t\right)\right|,\quad (x,y)\in G\times G.$$

The dyadic Hardy space $H_p(G\times G)$ $(0<p<\infty)$ consists of all functions for which

$$\|f\|_{H_p}:=\|f^{*}\|_p<\infty.$$

If $f\in L_1(G\times G)$, then it is easy to show that the sequence $(S_{2^n,2^n}(f)\colon n\in\mathbf{N})$ is a martingale. If $f=(f_{n,n},n\in\mathbf{N})$ is a martingale, then the Walsh – Fourier coefficients must be defined in a slightly different manner:

$$\widehat{f}\left(i,j\right):=\lim_{k\to\infty}\int\limits_{G} f_{k,k}(x,y)w_i\left(x\right)w_j\left(y\right)d\mu(x,y).$$

It is known [12] that that Fourier coefficients of $f\in H_p(G\times G)$ are not bounded when $0<p<1$.

The Walsh – Fourier coefficients of $f\in L_1(G\times G)$ are the same as those of the martingale $(S_{2^n,2^n}f\colon n\in\mathbf{N})$ obtained from f.

A bounded measurable function a is a p-atom, if there exists a dyadic 2-dimensional cube $I\times I$, such that

a) $\int\limits_{I\times I} a d\mu=0,$

b) $\|a\|_{\infty}\leq\mu(I\times I)^{-1/p}$,

c) $\operatorname{supp}(a)\subset I\times I$.

3. Formulation of main results.

Theorem 1. *Let* $0<p<1$ *and* $f\in H_p(G\times G)$. *Then*

$$\sum_{n=1}^{\infty}\frac{\|S_{n,n}f\|_p^p}{n^{3-2p}}\leq c_p\|f\|_{H_p}^p.$$

Theorem 2. *Let* $0<p<1$ *and* $\Phi\colon\mathbf{N}\to[1,\infty)$ *is any nondecreasing function, satisfying the condition* $\lim_{n\to\infty}\Phi(n)=+\infty$. *Then there exists a martingale* $f\in H_p(G\times G)$ *such that*

$$\sum_{n=1}^{\infty}\frac{\|S_{n,n}f\|_{\text{weak}-L_p}^p\,\Phi(n)}{n^{3-2p}}=\infty.$$

4. Auxiliary propositions.

Lemma 1 [14]. *A martingale* $f\in L_p(G\times G)$ *is in* $H_p(G\times G)$, $0<p\leq1$, *if and only if there exist a sequence* $(a_k,\ k\in\mathbf{N})$ *of p-atoms and a sequence* $(\mu_k,\ k\in\mathbf{N})$ *of a real numbers such that*

$$\sum_{k=0}^{\infty}\mu_k E_{n,n}a_k=f_{n,n} \tag{5}$$

and

$$\sum_{k=0}^{\infty}|\mu_k|^p<\infty.$$

Moreover, $\|f\|_{H_p}\backsim\inf\left(\sum_{k=0}^{\infty}|\mu_k|^p\right)^{1/p}$, where the infimum is taken over all decomposition of f of the form (5).

5. Proof of the theorems. ***Proof of Theorem* 1.** If we apply Lemma 1 we only have to prove that

$$\sum_{n=1}^{\infty}\frac{\|S_{n,n}a\|_p^p}{n^{3-2p}}\leq c_p<\infty, \tag{6}$$

for every p atom a.

Let a be an arbitrary p-atom with support $I_N(z')\times I_N(z'')$ and $\mu(I_N)=\mu(I_N)=2^{-N}$. We can suppose that $z'=z''=0$.

Let $(x,y)\in\overline{I}_N\times\overline{I}_N$. In this case $D_{2^i}(x+s)1_{I_N}(s)=0$ and $D_{2^i}(y+t)1_{I_N}(t)=0$ for $i\geq N$. Recall that $w_{2^j}(x+t)=w_{2^j}(x)$ for $t\in I_N$ and $j<N$. Consequently, from (4) we obtain

$$S_{n,n}a(x,y)=$$

$$=\int\limits_{G\times G}a(s,t)\,D_n(x+s)\,D_n(y+t)\,d\mu(s,t)=$$

$$=\int\limits_{I_N\times I_N}a(s,t)\,D_n(x+s)\,D_n(y+t)\,d\mu(s,t)=$$

$$=\int\limits_{I_N\times I_N}a(s,t)\,w_n(x+s+y+t)\sum_{i=0}^{N-1}n_iw_{2^i}(x+s)\,D_{2^i}(x+s)\times$$

$$\times\sum_{j=0}^{N-1}n_jw_{2^j}(y+t)\,D_{2^j}(y+t)\,d\mu(s,t)=$$

$$=w_n(x)\sum_{i=0}^{N-1}n_iw_{2^i}(x)\,D_{2^i}(x)\,w_n(y)\sum_{j=0}^{N-1}n_jw_{2^j}(y)\,D_{2^j}(y)\times$$

$$\times\int\limits_{I_N\times I_N}a(s,t)\,w_n(s+t)\,d\mu(s,t)=$$

$$=w_n(x+y)\sum_{i=0}^{N-1}n_iw_{2^i}(x)\,D_{2^i}(x)\sum_{j=0}^{N-1}n_jw_{2^j}(y)\,D_{2^j}(y)\times$$

$$\times\int\limits_{I_N}\left(\int\limits_{I_N}a(t+\tau,t)\,d\mu(t)\right)w_n(\tau)\,d\mu(\tau)=$$

$$=w_n(x+y)\sum_{i=0}^{N-1}n_iw_{2^i}(x)\,D_{2^i}(x)\sum_{j=0}^{N-1}n_jw_{2^j}(y)\,D_{2^j}(y)\int\limits_{I_N}\Phi(\tau)\,w_n(\tau)\,d\mu(\tau)=$$

$$= w_n(x+y)\sum_{i=0}^{N-1} n_i w_{2^i}(x) D_{2^i}(x) \sum_{j=0}^{N-1} n_j w_{2^j}(y) D_{2^j}(y)\widehat{\Phi}(n),$$

where

$$\Phi(\tau) = \int_{I_N} a(t+\tau,t)\,d\mu(t).$$

Let $x \in I_s \backslash I_{s+1}$. Using (3) we get

$$\sum_{i=0}^{N-1} D_{2^i}(x) \leq c2^s.$$

Since

$$\overline{I_N} = \bigcup_{s=0}^{N-1} I_s \backslash I_{s+1}$$

we obtain

$$\int_{\overline{I}_N} \left(\sum_{i=0}^{N-1} D_{2^i}(x)\right)^p d\mu(x) \leq c_p \sum_{s=0}^{N-1} \int_{I_s\backslash I_{s+1}} 2^{ps} d\mu(x) \leq$$

$$\leq c_p \sum_{s=0}^{\infty} 2^{(p-1)s} < c_p < \infty, \quad 0<p<1, \tag{7}$$

applying (7) we can write

$$\sum_{n=1}^{\infty} \frac{1}{n^{3-2p}} \int_{\overline{I}_N \times \overline{I}_N} |S_{n,n}a(x,y)|^p \, d\mu(x,y) \leq$$

$$\leq \sum_{n=1}^{\infty} \frac{\left|\widehat{\Phi}(n)\right|^p}{n^{3-2p}} \left(\int_{\overline{I}_N} \left(\sum_{i=0}^{N-1} D_{2^i}(x)\right)^p d\mu(x)\right)^2 \leq$$

$$\leq c_p \sum_{n=1}^{\infty} \frac{\left|\widehat{\Phi}(n)\right|^p}{n^{3-2p}}.$$

Let $n < 2^N$. Since $w_n(\tau) = 1$, for $\tau \in I_N$ we have

$$\widehat{\Phi}(n) = \int_{I_N} \Phi(\tau) w_n(\tau)\, d\mu(\tau) =$$

$$= \int_{I_N} \left(\int_{I_N} a(t+\tau,t)\, d\mu(t)\right) w_n(\tau)\, d\mu(\tau) =$$

$$= \int_{I_N \times I_N} a(s,t)\, d\mu(s,t) = 0.$$

Hence, we can suppose that $n \geq 2^N$. By Hölder inequality we obtain

$$\sum_{n=1}^{\infty} \frac{\left|\widehat{\Phi}(n)\right|^p}{n^{3-2p}} \leq \left(\sum_{n=2^N}^{\infty} \left|\widehat{\Phi}(n)\right|^2 \right)^{p/2} \left(\sum_{n=2^N}^{\infty} \frac{1}{n^{(3-2p)\cdot(2/(2-p))}} \right)^{(2-p)/2} \leq$$

$$\leq \left(\frac{1}{2^{N(2(3-2p)/(2-p)-1)}} \right)^{(2-p)/2} \left(\int_G |\Phi(\tau)|^2 d\mu(\tau) \right)^{p/2} \leq$$

$$\leq \frac{c_p}{2^{N(4-3p)/2}} \left(\int_{I_N} \left| \int_{I_N} a(t+\tau, t)\, d\mu(t) \right|^2 d\mu(\tau) \right)^{p/2} \leq$$

$$\leq \frac{c_p}{2^{N(4-3p)/2}} \|a\|_\infty^p \frac{1}{2^{Np/2}} \frac{1}{2^{Np}} \leq$$

$$\leq \frac{c_p}{2^{N(4-3p)/2}} 2^{2N} \frac{1}{2^{3pN/2}} < c_p < \infty. \tag{8}$$

Let $(x,y) \in \overline{I}_N \times I_N$. Then we have

$$S_{n,n}a(x,y) = w_n(x) \sum_{j=0}^{N-1} n_j w_{2^j}(x) D_{2^j}(x) \times$$

$$\times \int_{G\times G} a(s,t)\, w_n(s) D_n(y+t)\, d\mu(s,t) =$$

$$= w_n(x) \sum_{j=0}^{N-1} n_j w_{2^j}(x) D_{2^j}(x) \int_G S_n^{(2)} a(s,y)\, w_n(s)\, d\mu(s) =$$

$$= w_n(x) \sum_{j=0}^{N-1} n_j w_{2^j}(x) D_{2^j}(x) \widehat{S}_n^{(2)} a(n,y).$$

Using (7) we get

$$\sum_{n=1}^{\infty} \frac{1}{n^{3-2p}} \int_{\overline{I}_N \times I_N} |S_{n,n}a(x,y)|^p d\mu(x,y) \leq$$

$$\leq \sum_{n=1}^{\infty} \frac{1}{n^{3-2p}} \int_{\overline{I}_N \times I_N} \left(\sum_{j=0}^{N-1} D_{2^j}(x) \left| \widehat{S}_n^{(2)} a(n,y) \right| \right)^p d\mu(x,y) \leq$$

$$\leq \sum_{n=1}^{\infty} \frac{1}{n^{3-2p}} \int_{\overline{I}_N} \left(\sum_{i=0}^{N-1} D_{2^i}(x) \right)^p d\mu(x) \cdot \int\limits_{I_N} \left| \widehat{S}_n^{(2)} a(n,y) \right|^p d\mu(y) \leq$$

$$\leq \sum_{n=1}^{\infty} \frac{1}{n^{3-2p}} \int\limits_{I_N} \left| \widehat{S}_n^{(2)} a(n,y) \right|^p d\mu(y).$$

Let $n < 2^N$. Then by the definition of the atom we have

$$\widehat{S}_n^{(2)} a(n,y) = \int\limits_G \left(\int\limits_G a(s,t) D_n(y+t) d\mu(t) \right) w_n(s) d\mu(s) =$$

$$= D_n(y) \int\limits_{I_N \times I_N} a(s,t) d\mu(s,t) = 0.$$

Therefore, we can suppose that $n \geq 2^N$. Hence

$$\sum_{n=1}^{\infty} \frac{1}{n^{3-2p}} \int\limits_{\overline{I}_N \times I_N} |S_{n,n} a(x,y)|^p d\mu(x,y) \leq$$

$$\leq \sum_{n=2^N}^{\infty} \frac{1}{n^{3-2p}} \int\limits_{I_N} \left| \widehat{S}_n^{(2)} a(n,y) \right|^p d\mu(y).$$

Since

$$\left\| S_n^{(2)} a(n,y) \right\|_2 \leq c \|a\|_2$$

from Hölder inequality we can write

$$\int\limits_{I_N} \left| \widehat{S}_n^{(2)} a(n,y) \right|^p d\mu(y) \leq \frac{c_p}{2^{N(1-p)}} \left(\int\limits_{I_N} \left| \widehat{S}_n^{(2)} a(n,y) \right| d\mu(y) \right)^p =$$

$$= \frac{c_p}{2^{N(1-p)}} \left(\int\limits_{I_N} \left| \int\limits_{I_N} S_n^{(2)} a(s,y) w_n(s) d\mu(s) \right| d\mu(y) \right)^p =$$

$$= \frac{c_p}{2^{N(1-p)}} \left(\int\limits_{I_N} \left| \int\limits_{I_N} \left(\int\limits_{I_N} a(s,t) D_n(y+t) d\mu(t) \right) w_n(s) d\mu(s) \right| d\mu(y) \right)^p \leq$$

$$\leq \frac{c_p}{2^{N(1-p)}} \left(\int\limits_{I_N} \left(\int\limits_{I_N} \left| \int\limits_{I_N} a(s,t) D_n(y+t) d\mu(t) \right| d\mu(y) \right) d\mu(s) \right)^p \leq$$

$$\leq \frac{c_p}{2^{N(1-p)}}\left(\frac{1}{2^{N/2}}\int\limits_{I_N}\left(\int\limits_{I_N}\left|\int\limits_{I_N} a\left(s,t\right)D_n\left(y+t\right)d\mu\left(t\right)\right|^2 d\mu\left(y\right)\right)^{1/2} d\mu\left(s\right)\right)^p \leq$$

$$\leq \frac{c_p}{2^{N(1-p)}}\left(\frac{1}{2^{N/2}}\int\limits_{I_N}\left(\int\limits_{I_N}\left|a\left(s,t\right)\right|^2 d\mu\left(t\right)\right)^{1/2} d\mu\left(s\right)\right)^p \leq$$

$$\leq \frac{c_p}{2^{N(1-p)}}\left(\frac{\|a\|_\infty}{2^{N/2}}\frac{1}{2^N}\frac{1}{2^{N/2}}\right)^p \leq \frac{c_p}{2^{N(1-p)}}\left(\frac{2^{2N/p}}{2^{2N}}\right)^p \leq c_p 2^{N(1-p)}.$$

Consequently,

$$\sum_{n=1}^{\infty}\frac{1}{n^{3-2p}}\int\limits_{\overline{I}_N\times I_N}\left|S_{n,n}a(x,y)\right|d\mu(x,y) \leq$$

$$\leq c_p\sum_{n=2^N}^{\infty}\frac{1}{n^{3-2p}}2^{N(1-p)} \leq \frac{c_p}{2^{N(1-p)}} \leq c_p < \infty. \tag{9}$$

Analogously, we can prove that

$$\sum_{n=1}^{\infty}\frac{1}{n^{3-2p}}\int\limits_{I_N\times\overline{I}_N}\left|S_{n,n}a(x,y)\right|^p d\mu\left(x,y\right) \leq c_p < \infty. \tag{10}$$

Let $(x,y)\in I_N\times I_N$. Then by the definition of the atom we can write

$$\int\limits_{I_N\times I_N}\left|S_{n,n}a(x,y)\right|^p d\mu(x,y) \leq$$

$$\leq \frac{1}{2^{N(2-p)}}\left(\int\limits_{I_N\times I_N}\left|S_{n,n}a(x,y)\right|^2 d\mu(x,y)\right)^{p/2} \leq$$

$$\leq \frac{1}{2^{N(2-p)}}\left(\int\limits_{I_N\times I_N}\left|a(x,y)\right|^2 d\mu(x,y)\right)^{p/2} \leq$$

$$\leq \frac{\|a\|_\infty^p}{2^{N(2-p)}}\frac{1}{2^{Np}} \leq c_p\frac{1}{2^{N(2-p)}}2^{2N}\frac{1}{2^{Np}} \leq c_p < \infty.$$

It follows that

$$\sum_{n=1}^{\infty}\frac{1}{n^{3-2p}}\int\limits_{I_N\times I_N}|S_{n,n}a(x,y)|\,d\mu(x,y)\leq c_p\sum_{n=1}^{\infty}\frac{1}{n^{3-2p}}\leq c_p<\infty. \tag{11}$$

Combining (6) – (11) we complete the proof of Theorem 1.

***Proof of Theorem* 2.** Let $0<p<1$ and $\Phi(n)$ is any nondecreasing, nonnegative function, satisfying condition

$$\lim_{n\to\infty}\Phi(n)=\infty.$$

For this function $\Phi(n)$, there exists an increasing sequence of the positive integers $\{\alpha_k : k\geq 0\}$ such that:

$$\alpha_0\geq 2$$

and

$$\sum_{k=0}^{\infty}\frac{1}{\Phi^{p/4}\left(2^{\alpha_k}\right)}<\infty. \tag{12}$$

Let

$$f_{A,A}(x,y)=\sum_{\{k;\alpha_k<A\}}\lambda_k a_k,$$

where

$$\lambda_k=\frac{1}{\Phi^{1/4}\left(2^{\alpha_k}\right)}$$

and

$$a_k(x,y)=2^{\alpha_k(2/p-2)}\left(D_{2^{\alpha_k+1}}(x)-D_{2^{\alpha_k}}(x)\right)\left(D_{2^{\alpha_k+1}}(y)-D_{2^{\alpha_k}}(y)\right).$$

It is easy to show that the martingale $f=(f_{1,1},f_{2,2},\ldots,f_{A,A},\ldots)\in H_p$.

Indeed, since

$$S_{2^A}a_k(x,y)=\begin{cases}a_k(x,y), & \alpha_k<A,\\ 0, & \alpha_k\geq A,\end{cases} \tag{13}$$

$$\operatorname{supp}(a_k)=I_{\alpha_k},$$

$$\int\limits_{I_{\alpha_k}}a_k d\mu=0$$

and

$$\|a_k\|_\infty\leq 2^{\alpha_k(2/p-2)}2^{2\alpha_k}\leq 2^{2\alpha_k/p}=(\operatorname{supp}a_k)^{-1/p}$$

from Lemma 1 and (12) we conclude that $f\in H_p$.

It is easy to show that

$$\widehat{f}(i,j)=\begin{cases}\dfrac{2^{\alpha_k(2/p-2)}}{\Phi^{1/4}\left(2^{\alpha_k}\right)}, & \text{if } (i,j)\in\left\{2^{\alpha_k},\ldots,\ 2^{\alpha_k+1}-1\right\}\times\\ & \qquad\times\left\{2^{\alpha_k},\ldots,\ 2^{\alpha_k+1}-1\right\},\quad k=0,1,2\ldots,\\ 0, & \text{if } (i,j)\notin\bigcup\limits_{k=1}^{\infty}\left\{2^{\alpha_k},\ldots,\ 2^{\alpha_k+1}-1\right\}\times\left\{2^{\alpha_k},\ldots,\ 2^{\alpha_k+1}-1\right\}.\end{cases} \tag{14}$$

Let $2^{\alpha_k} < n < 2^{\alpha_k+1}$. From (14) we have

$$S_{n,n}f(x,y) = \sum_{i=0}^{2^{\alpha_{k-1}+1}-1} \sum_{j=0}^{2^{\alpha_{k-1}+1}-1} \widehat{f}(i,j)w_i(x)\,w_j(y) +$$

$$+\sum_{i=2^{\alpha_k}}^{n-1} \sum_{j=2^{\alpha_k}}^{n-1} \widehat{f}(i,j)w_i(x)\,w_j(y) =$$

$$=\sum_{\eta=0}^{k-1} \sum_{i=2^{\alpha_\eta}}^{2^{\alpha_\eta+1}-1} \sum_{j=2^{\alpha_\eta}}^{2^{\alpha_\eta+1}-1} \widehat{f}(i,j)w_i(x)\,w_j(y) +$$

$$+\sum_{i=2^{\alpha_k}}^{n-1} \sum_{j=2^{\alpha_k}}^{n-1} \widehat{f}(i,j)w_i(x)\,w_j(y) =$$

$$=\sum_{\eta=0}^{k-1} \sum_{i=2^{\alpha_\eta}}^{2^{\alpha_\eta+1}-1} \sum_{j=2^{\alpha_\eta}}^{2^{\alpha_\eta+1}-1} \frac{2^{\alpha_\eta(2/p-2)}}{\Phi^{1/4}(2^{\alpha_\eta})} w_i(x)\,w_j(y) +$$

$$+\sum_{i=2^{\alpha_k}}^{n-1} \sum_{j=2^{\alpha_k}}^{n-1} \frac{2^{\alpha_k(2/p-2)}}{\Phi^{1/4}(2^{\alpha_k})} w_i(x)\,w_j(y) =$$

$$=\sum_{\eta=0}^{k-1} \frac{2^{\alpha_\eta(2/p-2)}}{\Phi^{1/4}(2^{\alpha_\eta})}\left(D_{2^{\alpha_\eta+1}}(x) - D_{2^{\alpha_\eta}}(x)\right)\left(D_{2^{\alpha_\eta+1}}(y) - D_{2^{\alpha_\eta}}(y)\right) +$$

$$+\frac{2^{\alpha_k(2/p-2)}}{\Phi^{1/4}(2^{\alpha_k})}\left(D_n(x) - D_{2^{\alpha_k}}(x)\right)\left(D_n(y) - D_{2^{\alpha_k}}(y)\right) =$$

$$= I + II. \tag{15}$$

Let $(x,y) \in (G\backslash I_1) \times (G\backslash I_1)$ and n is odd number. Since $n - 2^{\alpha_k}$ is odd number too and

$$D_{n+2^{\alpha_k}}(x) = D_{2^{\alpha_k}}(x) + w_{2^{\alpha_k}}(x)\,D_n(x), \quad \text{when} \quad n < 2^{\alpha_k},$$

from (3) and (4) we can write

$$|II| = \frac{2^{\alpha_k(2/p-2)}}{\Phi^{1/4}(2^{\alpha_k})}\left|w_{2^{\alpha_k}}(x)\,D_{n-2^{\alpha_k}}(x)\,w_{2^{\alpha_k}}(y)\,D_{n-2^{\alpha_k}}(y)\right| =$$

$$= \frac{2^{\alpha_k(2/p-2)}}{\Phi^{1/4}(2^{\alpha_k})}\left|w_{2^{\alpha_k}}(x)\,w_{n-2^{\alpha_k}}(x)\,D_1(x)\,w_{2^{\alpha_k}}(y)\,w_{n-2^{\alpha_k}}(y)\,D_1(y)\right| =$$

$$= \frac{2^{\alpha_k(2/p-2)}}{\Phi^{1/4}(2^{\alpha_k})}. \tag{16}$$

Applying (3) and condition $\alpha_n \geq 2$ $(n \in \mathbf{N})$ for I we have

$$I = \sum_{\eta=0}^{k-1} \frac{2^{\alpha_k(2/p-2)}}{\Phi^{1/4}(2^{\alpha_\eta})} (D_{2^{\alpha_\eta+1}}(x) - D_{2^{\alpha_\eta}}(x))(D_{2^{\alpha_\eta+1}}(y) - D_{2^{\alpha_\eta}}(y)) = 0. \tag{17}$$

Hence

$$\|S_{n,n}f(x,y)\|_{\text{weak}-L_p} \geq$$

$$\geq \frac{2^{\alpha_k(2/p-2)}}{2\Phi^{1/4}(2^{\alpha_k})} \left(\mu \left\{ (x,y) \in (G \backslash I_1) \times (G \backslash I_1) : |S_{n,n}f(x,y)| \geq \frac{2^{\alpha_k(2/p-2)}}{2\Phi^{1/4}(2^{\alpha_k})} \right\} \right)^{1/p} \geq$$

$$\geq \frac{2^{\alpha_k(2/p-2)}}{2\Phi^{1/4}(2^{\alpha_k})} |(G \backslash I_1) \times (G \backslash I_1)| \geq \frac{c_p 2^{\alpha_k(2/p-2)}}{\Phi^{1/4}(2^{\alpha_k})}. \tag{18}$$

Using (18) we have

$$\sum_{n=1}^{2^{\alpha_k+1}-1} \frac{\|S_{n,n}f\|^p_{\text{weak}-L_p} \Phi(n)}{n^{3-2p}} \geq$$

$$\geq \sum_{n=2^{\alpha_k}+1}^{2^{\alpha_k+1}-1} \frac{\|S_{n,n}f\|^p_{\text{weak}-L_p} \Phi(n)}{n^{3-2p}} \geq$$

$$\geq c_p \Phi(2^{\alpha_k}) \sum_{n=2^{\alpha_k-1}+1}^{2^{\alpha_k}-1} \frac{\|S_{2n+1,2n+1}f\|^p_{\text{weak}-L_p}}{(2n+1)^{3-2p}} \geq$$

$$\geq c_p \Phi(2^{\alpha_k}) \frac{2^{2\alpha_k(1-p)}}{\Phi^{1/4}(2^{\alpha_k})} \sum_{n=2^{\alpha_k-1}+1}^{2^{\alpha_k}-1} \frac{1}{(2n+1)^{3-2p}} \geq$$

$$\geq c_p \Phi^{3/4}(2^{\alpha_k}) \to \infty, \quad \text{when} \quad k \to \infty. \tag{19}$$

Combining (12) – (19) we complete the proof of Theorem 2.

1. *Gát G.* Inverstigations of certain operators with respect to the Vilenkin system // Acta math. hung. – 1993. – **61**. – P. 131 – 149.
2. *Gát G., Goginava U., Nagy K.* On the Marcinkiewicz – Fejár means of double Fourier series with respect to the Walsh – Kaczmarz system // Stud. sci. math. hung. – 2009. – **46**, № 3. – P. 399 – 421.
3. *Gát G., Goginava U., Tkebuchava G.* Convergence in measure of logarithmic means of quadratical partial sums of double Walsh – Fourier series // J. Math. Anal. and Appl. – 2006. – **323**, № 1. – P. 535 – 549.
4. *Goginava U.* The weak type inequality for the maximal operator of the Marcinkiewicz – Fejer means of the two-dimensional Walsh – Fourier series // J. Approxim. Theory. – 2008. – **154**, № 2. – P. 161 – 180.
5. *Goginava U., Gogoladze L. D.* Strong convergence of cubic partial sums of two-dimensional Walsh – Fourier series // Constructive theory of functions (Sozopol, 2010): In memory of Borislav Bojanov. – Sofia: Acad. Publ. House, 2012. – P. 108 – 117.

6. *Gogoladze L. D.* On the strong summability of Fourier series // Bull Acad. Sci. Georg. SSR. – 1968. – **52**, № 2. – P. 287 – 292.
7. *Golubov B., Efimov A., Skvortsov V.* Walsh series and transformations. – Dordrecht etc.: Kluwer Acad. Publ., 1991.
8. *Schipp F., Wade W. R., Simon P., Pál J.* Walsh series // Introduction to Dyadic Harmonic Analysis. – Bristol; New York: Adam Hilger, 1990.
9. *Simon P.* Strong convergence of certain means with respect to the Walsh – Fourier series // Acta math. hung. – 1987. – **49**. – P. 425 – 431.
10. *Simon P.* Strong convergence theorem for Vilenkin – Fourier series // J. Math. Anal. and Appl. – 2000. – **245**. – P. 52 – 68.
11. *Smith B.* A strong convergence theorem for $H_1(T)$ // Lect. Notes Math. – 1994. – **995**. – P. 169 – 173.
12. *Tephnadze G.* On the Vilenkin – Fourier coefficients // Georg. Math. J. (to appear).
13. *Tephnadze G.* A note of the Fourier coefficients and partial sums of Vilenkin – Fourier series // Acta Math. Acad. Ped. Nyiregyhaziensis (to appear).
14. *Weisz F.* Martingale Hardy spaces and their applications in Fourier analysis. – Berlin etc.: Springer, 1994.
15. *Weisz F.* Summability of multi-dimensional Fourier series and Hardy space. – Dordrecht etc.: Kluwer Acad. Publ., 2002.
16. *Weisz F.* Strong convergence theorems for two-parameter Walsh – Fourier and trigonometric-Fourier series // Stud. Math. – 1996. – **117**, № 2. – P. 173 – 194.

Received 18.04.12,
after revision — 14.11.12