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α-SASAKIAN 3-METRIC AS A RICCI SOLITON*

α-САСАКІЄВА 3-МЕТРИКА ЯК СОЛІТОН РІЧЧІ

We prove that if the metric of a 3-dimensional α-Sasakian manifold is a Ricci soliton, then it is either of constant curvature or of constant scalar curvature. We also establish some properties of the potential vector field U of the Ricci soliton. Finally, we give an example of an α-Sasakian 3-metric as a nontrivial Ricci soliton.

Доведено, що якщо метрика тривимірного α-сасакієвого многовиду є солітоном Річчі, то він має або сталу кривину, або сталу скалярну кривину. Встановлено деякі властивості потенціального векторного поля U солітона Річчі. Наведено приклад α-сасакієвої 3-метрики як нетривіального солітона Річчі.

1. Introduction. Over the last few years, Ricci solitons have been the place of concern for many geometers and physicists. The whim of Ricci soliton structure was innovated by R. S. Hamilton (for details we refer to [4]) and there he had excogitated it as a generalisation of an Einstein metric and specified on a Riemannian manifold (M^n, g) together with a vector field U and a constant λ that satisfies

$$\pounds_U g + 2S + 2\lambda g = 0, \tag{1.1}$$

where $\pounds$ stands for the Lie-derivative operators along the complete vector field U and S as the Ricci tensor of the manifold. In case, if λ is positive (respectively zero, respectively negative) then the Ricci soliton is said to be expanding(respectively steady, shrinking). Actually Ricci soliton can be considered as a fixed point of Hamilton's Ricci flow: $\dfrac{\partial}{\partial t} g_{ij} = -2S_{ij}$; viewed as a dynamical system, on the space of Riemannian metrics modulo diffeomorphisms and scalings. In particular if $U = \nabla f$, for some smooth scalar valued function f, then the soliton is said to be a gradient Ricci soliton. In particular, if U is Killing or $U = 0$ the soliton is said to be a trivial Ricci soliton (for details refer to [3, 6]). Also for several classes of these manifolds the existence of nontrivial Ricci solitons is proved. Recently, Sharma and Ghosh [9], proved that if the metric of a 3-dimensional Sasakian manifold is a Ricci soliton then it is homothetic to a standard Heisenberg group nil^3. Since α-Sasakian is a generalization of Sasakian manifold, we are interested to study 3-dimensional α-Sasakian manifold when its metric is a Ricci soliton. We also deduce some properties of the potential vector field U of the Ricci soliton together with an example of an α-Sasakian 3-metric as a nontrivial Ricci soliton.

2. Preliminaries. An odd-dimensional differentiable manifold (M^n, g) is said to admit an almost contact metric structure (ϕ, ξ, η, g) consisting of a Reeb vector field ξ, (1,1)-tensor field ϕ and a Riemannian metric g satisfying

$$\phi^2 = -I + \eta \otimes \xi, \qquad \eta(\xi) = 1, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \quad \forall X, Y \in \chi(M), \tag{2.2}$$

where $\chi(M)$ represents the collection of all smooth vector fields on M.

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Moreover, if the relation
$$d\eta(X,Y) = g(\phi X, Y),$$
holds for arbitrary smooth vector fields X and Y, then we call such a structure a contact metric structure and the manifold with that structure is said to be contact metric manifold. As a consequence of this, the following relations hold:
$$\phi\xi = 0, \qquad \eta \circ \phi = 0, \qquad d\eta(\xi, X) = 0 \quad \forall X \in \chi(M). \tag{2.3}$$
For details we refer to Blair [1].

An almost contact structure on M is said to be an α-Sasakian manifold, where α is a non-zero constant, if
$$(\nabla_X \phi)Y = \alpha\big(g(X,Y)\xi - \eta(Y)X\big) \quad \forall X, Y \in \chi(M) \tag{2.4}$$
holds. As a consequence, it follows that:
$$\nabla_X \xi = -\alpha\phi X, \tag{2.5}$$
$$(\nabla_X \eta)Y = -\alpha g(\phi X, Y) \quad \forall X, Y \in \chi(M). \tag{2.6}$$
If $\alpha = 1$, then the α-Sasakian structure reduces to Sasakian manifold, thus α-Sasakian structure may be considered as a generalization of Sasakian one. In other words, Sasakian manifold is a particular case of α-Sasakian manifold. Also in a 3-dimensional α-Sasakian manifold the following relations are true:
$$R(X,Y)\xi = \alpha^2\{\eta(Y)X - \eta(X)Y\}, \tag{2.7}$$
$$S(X,\xi) = 2\alpha^2\eta(X), \tag{2.8}$$
$$Q\xi = 2\alpha^2\xi \quad \forall X, Y \in \chi(M), \tag{2.9}$$
where R is the Riemannian curvature tensor and Q is the Ricci operator associated with the $(0,2)$ Ricci tensor S. For details we refer to [5].

Definition 2.1 [1]. *In an almost contact Riemannian manifold, if an infinitesimal transformation U satisfies*
$$(\pounds_U \eta)(X) = \sigma\eta(X), \tag{2.10}$$
for a scalar function σ, then we call it an infinitesimal contact transformation. If σ vanishes identically, then it is called an infinitesimal strict transformation.

3. α-Sasakian 3-metric as a Ricci soliton. Before proceeding towards the main results we state the following lemma.

Lemma 3.1. *In an α-Sasakian 3-metric, the Ricci tensor S is given by*
$$S = \left(\frac{r}{2} - \alpha^2\right) g + \left(3\alpha^2 - \frac{r}{2}\right) \eta \otimes \eta. \tag{3.1}$$

Proof. We recall that the Riemannian curvature tensor in a 3-dimensional Riemannian manifold is given by

$$R(X,Y)Z = g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y - \frac{r}{2}\{g(Y,Z)X - g(X,Z)Y\}, \tag{3.2}$$

where r is the Ricci scalar curvature and $X, Y, Z \in \chi(M)$.

Replacing Z with ξ in (3.2) and recalling (2.8),

$$\eta(Y)QX - \eta(X)QY + \left(\alpha^2 - \frac{r}{2}\right)\{\eta(Y)X - \eta(X)Y\} = 0.$$

Again, replacing Y with ξ and thereby using (2.9), we get the desired result.

Theorem 3.1. *If the metric of a 3-dimensional α-Sasakian manifold is a nontrivial Ricci soliton, then it is of constant scalar curvature $-2\alpha^2$ and the soliton is expanding.*

Proof. Combining (1.1) and (3.1) yields,

$$(\pounds_U g)(X,Y) = \left(2\alpha^2 - 2\lambda - r\right) g(X,Y) + \left(r - 6\alpha^2\right) \eta(X)\eta(Y). \tag{3.3}$$

The identity

$$(\nabla_X \pounds_U g)(Y,Z) = g((\pounds_U \nabla)(X,Y),Z) + g((\pounds_U \nabla)(X,Z),Y), \tag{3.4}$$

can be deduced from the formula [10],

$$(\pounds_U \nabla_X g - \nabla_X \pounds_U g - \nabla_{[U,X]} g)(Y,Z) = -g((\pounds_U \nabla)(X,Y),Z) - g((\pounds_U \nabla)(X,Z),Y). \tag{3.5}$$

Differentiating covariantly (1.1) with respect to the vector field Z, we obtain

$$(\nabla_Z \pounds_U g)(X,Y) + 2(\nabla_Z S)(X,Y) = 0. \tag{3.6}$$

Again, differentiating (3.1) covariantly with respect to Z and using (2.6), we have

$$(\nabla_Z S)(X,Y) = \frac{1}{2} dr(Z)\{g(X,Y) - \eta(X)\eta(Y)\} -$$

$$-\alpha\left(3\alpha^2 - \frac{r}{2}\right)\{g(\phi Z, X)\eta(Y) + g(\phi Z, Y)\eta(X)\}. \tag{3.7}$$

Combining (3.6) with (3.7), one obtains

$$(\nabla_Z \pounds_U g)(X,Y) + dr(Z)\{g(X,Y) - \eta(X)\eta(Y)\} -$$

$$-2\alpha\left(3\alpha^2 - \frac{r}{2}\right)\{g(\phi Z, X)\eta(Y) + g(\phi Z, Y)\eta(X)\} = 0. \tag{3.8}$$

Using (3.4) in (3.8) one obtains

$$g((\pounds_U \nabla)(Z,X),Y) + g((\pounds_U \nabla)(Z,Y),X) + dr(Z)\{g(X,Y) - \eta(X)\eta(Y)\} -$$

$$-2\alpha\left(3\alpha^2 - \frac{r}{2}\right)\{g(\phi Z, X)\eta(Y) + g(\phi Z, Y)\eta(X)\} = 0. \tag{3.9}$$

Permuting X, Y, Z and then by combinatorial combination we find,

$$2(\pounds_U \nabla)(Y,Z) + \big\{dr(Y)(Z - \eta(Z)\xi) + dr(Z)(Y - \eta(Y)\xi) -$$

$$-Dr(g(Y,Z)-\eta(Y)\eta(Z))\}-$$

$$-2\alpha\left(3\alpha^2-\frac{r}{2}\right)\{\eta(Z)\phi Y+\eta(Y)\phi Z\}=0, \tag{3.10}$$

for all vector field X and D is the gradient operator of g. Now from [10], we have the following identity:

$$(\pounds_U R)(X,Y)Z=(\nabla_X \pounds_U \nabla)(Y,Z)-(\nabla_Y \pounds_U \nabla)(X,Z). \tag{3.11}$$

Now the use of (2.5) and (3.10) in the identity (3.11), we obtain on taking $Z=\xi$

$$(\pounds_U R)(X,Y)\xi=\frac{\alpha}{2}\big[dr(Y)\phi X-dr(X)\phi Y\big]+2\alpha^2(6\alpha^2-r)\{\eta(X)Y-\eta(Y)X\}+$$

$$+\alpha g(\phi X,Y)Dr+\frac{\alpha}{2}\big[dr(\phi Y)(X-\eta(X)\xi)-dr(\phi X)(Y-\eta(Y)\xi)\big]. \tag{3.12}$$

Taking the Lie-derivative of (2.7) along the direction of U and using (3.3), one obtains

$$(\pounds_U R)(X,Y)\xi=-R(X,Y)\pounds_U\xi+2\alpha^2(\lambda+2\alpha^2)\{\eta(X)Y-\eta(Y)X\}+$$

$$+\alpha^2\{g(Y,\pounds_U\xi)X-g(X,\pounds_U\xi)Y\}. \tag{3.13}$$

Equating (3.12) and (3.13), it follows that

$$2\alpha^2\{g(Y,\pounds_U\xi)X-g(X,\pounds_U\xi)Y\}=2R(X,Y)\pounds_U\xi-4\alpha^2(4\alpha^2-\lambda-r)\{\eta(X)Y-\eta(Y)X\}+$$

$$+2\alpha g(\phi X,Y)Dr+\alpha\big[dr(Y)\phi X-dr(X)\phi Y\big]+$$

$$+\alpha\big[dr(\phi Y)(X-\eta(X)\xi)-dr(\phi X)(Y-\eta(Y)\xi)\big].$$

Contracting the above equation over Y and thereby using (3.1), we find

$$2\alpha dr(\phi X)=(r-6\alpha^2)[g(\pounds_U\xi,X)-\eta(\pounds_U\xi)\eta(X)]+8\alpha^2(r-4\alpha^2+\lambda)\eta(X). \tag{3.14}$$

Substituting $X=\xi$, yields

$$r=4\alpha^2-\lambda, \quad \text{since} \quad \alpha\neq 0. \tag{3.15}$$

Now, the integrability condition of the Ricci soliton (for details refer to [2, 8]) is given by

$$\pounds_U r=-\mathrm{div}.Dr+2\lambda r+2|S|^2.$$

By using (3.15) and (3.1), one obtains from above

$$r^2-4\alpha^2 r-12\alpha^4=0, \quad \text{which implies} \quad r=6\alpha^2 \quad \text{or} \quad r=-2\alpha^2.$$

Theorem 3.1 is proved.

For $r=6\alpha^2$ we see that the manifold is Einstein and being of dimension 3 it becomes a space of constant curvature α^2. Hence we have the following corollary.

***Corollary* 3.1.** *If the metric of an α-Sasakian manifold is Ricci soliton then it is either a space of constant curvature α^2 or of constant scalar curvature.*

We now deduce some properties of the potential vector field U related to the underlying contact structure of the α-Sasakian 3-metric as a Ricci soliton: Putting $r = -2\alpha^2$ and $\lambda = 6\alpha^2$ in (3.1) and using the result in (1.1), one obtains

$$\pounds_U g = -8\alpha^2(g + \eta \otimes \eta),$$

which implies U is homothetic on the distribution $D = \ker(\eta)$. Replacing X, Y with ξ, one obtains from (3.10)

$$(\pounds_U \nabla)(\xi, \xi) = 0, \quad \text{since} \quad \xi \quad \text{is killing}. \tag{3.16}$$

Substituting $X = Y = \xi$ in (3.5) provides,

$$(\pounds_U \nabla)(X, Y) = \nabla_X \nabla_Y U - \nabla_{\nabla_X Y} U + R(U, X)Y. \tag{3.17}$$

Thereby using (3.16) in (3.17) together with $X = Y = \xi$, we have

$$\nabla_\xi \nabla_\xi U + R(U, \xi)\xi = 0. \tag{3.18}$$

Hence from (3.18), it is quite evident that U is a Jacobi along geodesics of ξ. Again, using (3.7) in (3.14), we find

$$(r - 6\alpha^2)\{\pounds_U \xi - \eta(\pounds_U \xi)\xi\} = 0.$$

If g is a nontrivial Ricci soliton, the above equation yields

$$\pounds_U \xi = \sigma\xi, \quad \text{where} \quad \sigma = \eta(\pounds_U \xi).$$

Then the use of (1.1), (3.1) and the above equation, one obtains

$$\pounds_U \eta = (\sigma - 16\alpha^2)\eta,$$

which proves that U is an infinitesimal contact transformation. Setting $X = Y = \xi$ in (1.1) and in view of (2.8), $\pounds_U \xi = \sigma\xi$, we get

$$\sigma = 8\alpha^2,$$

from which we see that

$$\sigma - 16\alpha^2 = -8\alpha^2 (\neq 0).$$

This implies the infinitesimal contact transformation is nonstrict. Summing up all these results we can state as follows:

Theorem 3.2. *If an α-Sasakian* 3*-metric admits a nontrivial Ricci soliton together with the potential vector field U, then the following statements hold*:

(1) *U is homothetic on the distribution $D = \ker(\eta)$.*

(2) *U is a Jacobi vector field along geodesics of ξ.*

(3) *U is an infinitesimal contact transformation.*

Thus the ϕ-sectional curvature of α-Sasakian manifold (of 3-dimension) admitting nontrivial Ricci soliton is given by α^2. Hence, we can state as follows:

Theorem 3.3. *For a 3-dimensional α-Sasakian manifold the ϕ-sectional curvature* (*sectional curvature with respect to a plane orthogonal to ξ*) *is constant and equals to α^2.*

4. Example of an α-Sasakian 3-metric as a Ricci soliton. Let us consider the 3-dimensional Riemannian manifold $M = \mathbb{R}^3$ with a rectangular cartesian coordinate system (x_i).

Let us choose the vector fields $\{E_1, E_2, E_3\}$ as

$$E_1 = \frac{\partial}{\partial x_1}, \qquad E_2 = -2\alpha\frac{\partial}{\partial x_2}, \qquad E_3 = x_2\frac{\partial}{\partial x_1} + x_3\frac{\partial}{\partial x_3}, \qquad \alpha \text{ being a nonzero constant.}$$

Thus, $\{E_1, E_2, E_3\}$ forms a basis of $\chi(M) = \chi(\mathbb{R}^3)$.

Let g be the Riemannian metric on $\chi(\mathbb{R}^3)$ defined by

$$g(E_1, E_1) = g(E_2, E_2) = g(E_3, E_3) = 1,$$

$$g(E_1, E_2) = g(E_1, E_3) = g(E_2, E_3) = 0.$$

Let $\xi = E_1$ be the vector field associated with the 1-form η. The (1,1)-tensor field ϕ be defined by,

$$\phi(E_1) = 0, \qquad \phi(E_2) = -E_3, \qquad \phi(E_3) = E_2.$$

Since, $\{E_1, E_2, E_3\}$ is a basis, any vector fields X and Y in M can be uniquely expressed as

$$X = X^1E_1 + X^2E_2 + X^3E_3 \qquad \text{and} \qquad Y = Y^1E_1 + Y^2E_2 + Y^3E_3,$$

where X^i, Y^i, $i = 1, 2, 3$, are smooth functions over M.

Now using the linearity of ϕ and g, and taking $\xi = E_1$ we have

$$\eta(\xi) = 1, \qquad \phi^2X = -X + \eta(X)\xi, \qquad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$$

for any vector fields X and Y in M. Thus (ϕ, ξ, η, g) defines an almost contact metric structure on M.

Let ∇ be the Levi–Civita connection with respect to the Riemannian metric g. Then we obtain

$$[E_2, E_3] = -2\alpha e_1, \qquad [E_1, E_2] = 0, \qquad [E_1, E_3] = 0.$$

By using Koszul's formulae (see [7]), we have

$$\nabla_{E_1}E_3 = -\alpha E_2, \qquad \nabla_{E_1}E_2 = \alpha E_3, \qquad \nabla_{E_1}E_1 = 0,$$

$$\nabla_{E_2}E_3 = -\alpha E_1, \qquad \nabla_{E_2}E_1 = \alpha E_3, \qquad \nabla_{E_2}E_2 = 0,$$

$$\nabla_{E_3}E_1 = -\alpha E_2, \qquad \nabla_{E_3}E_2 = \alpha E_1, \qquad \nabla_{E_3}E_3 = 0.$$

Also, the Riemannian curvature tensor R is given by,

$$R(X, Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z.$$

Then

$$R(E_1, E_2)E_2 = \alpha^2E_1, \qquad R(E_1, E_3)E_3 = \alpha^2E_1, \qquad R(E_2, E_1)E_1 = \alpha^2E_2,$$

$$R(E_2, E_3)E_3 = -3\alpha^2E_2, \qquad R(E_3, E_1)E_1 = \alpha^2E_3, \qquad R(E_3, E_2)E_2 = -3\alpha^2E_3,$$

$$R(E_1, E_2)E_3 = 0, \qquad R(E_2, E_3)E_1 = 0, \qquad R(E_3, E_1)E_2 = 0.$$

Then, the Ricci tensor S is given by

$$S(E_1, E_1) = 2\alpha^2, \qquad S(E_2, E_2) = -2\alpha^2, \qquad S(E_3, E_3) = -2\alpha^2,$$

$$S(E_1, E_2) = 0, \qquad S(E_1, E_3) = 0, \qquad S(E_2, E_3) = 0.$$

Thus the scalar curvature $r = -2\alpha^2$ is constant. The conditions (2.4) to (2.9) hold for any smooth vector fields X and Y in M. Taking the potential vector field

$$U = f_1 E_1 + f_2 E_2 + f_3 E_3,$$

where f_1, f_2 and f_3 are smooth functions on M, it can be easily shown that it satisfies the soliton equation and the soliton is expanding in nature.

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