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## ON ZEROS OF PERIODIC ZETA FUNCTIONS
## ПРО НУЛІ ПЕРІОДИЧНИХ ДЗЕТА-ФУНКЦІЙ

We consider the zeta functions $\zeta(s;\mathfrak{a})$ given by Dirichlet series with multiplicative periodic coefficients and prove that, for some classes of functions $F$, the functions $F(\zeta(s;\mathfrak{a}))$ have infinitely many zeros in the critical strip. For example, this is true for $\sin(\zeta(s;\mathfrak{a}))$.

Розглянуто дзета-функції $\zeta(s;\mathfrak{a})$, що задані рядами Діріхле з мультиплікативними періодичними коефіцієнтами, та доведено, що для деяких класів функцій $F$ функції $F(\zeta(s;\mathfrak{a}))$ мають нескінченну кількість нулів у критичній смузі. Наприклад, це виконується для $\sin(\zeta(s;\mathfrak{a}))$.

**1. Introduction.** The zero distribution of zeta functions is of particular interest in analytic number theory, and, in general, in mathematics. The most important problems are related to the Riemann zeta function $\zeta(s)$, $s=\sigma+it$, which is defined, for $\sigma>1$, by Dirichlet series

$$\zeta(s)=\sum_{m=1}^{\infty}\frac{1}{m^s},$$

and is analytically continued to the whole complex plane, except for a simple pole at the point $s=1$ with residue 1. It is well known that $s=-2m$, $m\in\mathbb{N}$, are so called trivial zeros of $\zeta(s)$. Moreover, $\zeta(s)\neq 0$, for $\sigma\geq 1$, and for $\sigma\leq 0$, $t\neq 0$, however, the function $\zeta(s)$ has infinitely many complex (nontrivial) zeros in the critical strip $\{s\in\mathbb{C}\colon 0<\sigma<1\}$. The famous Riemann hypothesis (RH) says that all nontrivial zeros of $\zeta(s)$ lie on the critical line $\sigma=\frac{1}{2}$, and this is equivalent to the non-vanishing of $\zeta(s)$ in the half-plane $\left\{s\in\mathbb{C}\colon \sigma>\frac{1}{2}\right\}$. The last known result on zero-free regions for $\zeta(s)$ is of the form: there exists an absolute constant $c>0$ such that $\zeta(s)\neq 0$ in the region

$$\left\{s\in\mathbb{C}\colon \sigma\geq 1-\frac{c}{(\log(|t|+2))^{2/3}(\log\log(|t|+2))^{1/3}}\right\}.$$

G. H. Hardy proved [1] that infinitely many nontrivial zeros lie on the critical line. This result was improved by A. Selberg, N. Levinson, B. Conrey. The last result in this direction says [2] that at least 41 percent of all nontrivial zeros of $\zeta(s)$ in the sense of of density are on the critical line. Numerical calculations also support RH: the first $10^{13}$ nontrivial zeros of $\zeta(s)$ lie on the critical line $\sigma=\frac{1}{2}$ [3].

A natural generalization of the function $\zeta(s)$ is the periodic zeta function. Let $\mathfrak{a}=\{a_m\colon m\in\mathbb{N}\}$ be a periodic sequence of complex numbers with minimal period $k\in\mathbb{N}$. The periodic zeta function $\zeta(s;\mathfrak{a})$ is defined, for $\sigma>1$, by the series

$$\zeta(s;\mathfrak{a})=\sum_{m=1}^{\infty}\frac{a_m}{m^s}.$$

Moreover, the function $\zeta(s;\mathfrak{a})$ is analytically continuable to the whole complex plane. Really, let $\zeta(s,\alpha)$ denote the Hurwitz zeta function with parameter $\alpha$, $0<\alpha\leq 1$, given, for $\sigma>1$, by the series

$$\zeta(s,\alpha)=\sum_{m=0}^{\infty}\frac{1}{(m+\alpha)^s},$$

and by analytic continuation elsewhere, except for a simple pole at $s=1$ with residue $1$. Then the periodicity of the sequence $\mathfrak{a}$ implies, for $\sigma>1$, the equality

$$\zeta(s;\mathfrak{a})=\frac{1}{k^s}\sum_{l=1}^{k}a_l\zeta\left(s,\frac{l}{k}\right).$$

Therefore, in virtue of the above remarks, the later equality gives analytic continuation for $\zeta(s;\mathfrak{a})$ to the whole complex plane. If

$$a\overset{\mathrm{df}}{=}\frac{1}{k}\sum_{l=1}^{k}a_l\neq 0,$$

then the function $\zeta(s;\mathfrak{a})$ has a simple pole at $s=1$ with residue $a$, otherwise, the function $\zeta(s;\mathfrak{a})$ is an entire function.

Obviously, if $a_1=1$ and $k=1$, then $\zeta(s;\mathfrak{a})=\zeta(s)$.

We use the notation

$$a_m^{\pm}=\frac{1}{k}\sum_{l=1}^{k}a_l\exp\left\{\pm 2\pi il\frac{m}{k}\right\}$$

and $\mathfrak{a}^{\pm}=\{a_m^{\pm}\colon m\in\mathbb{N}\}$. Then the sequences of complex numbers $\mathfrak{a}^{\pm}$ are also periodic with period $k$. In [4], it was proved that the function $\zeta(s;\mathfrak{a})$ satisfies the functional equation

$$\zeta(1-s;\mathfrak{a})=\left(\frac{k}{2\pi}\right)^s\Gamma(s)\left(\exp\left\{\frac{\pi is}{2}\right\}\zeta(s;\mathfrak{a}^-)+\exp\left\{-\frac{\pi is}{2}\right\}\zeta(s;\mathfrak{a}^+)\right),$$

where $\Gamma(s)$, as usual, stands for the Euler gamma function.

In [5], J. Steuding began to study the zero distribution of the function $\zeta(s;\mathfrak{a})$. Denote the zeros of $\zeta(s;\mathfrak{a})$ by $\rho=\beta+i\gamma$. Moreover, let $c_{\mathfrak{a}}=\max(|a_m|\colon 1\leq m\leq k)$, $m_{\mathfrak{a}}=\min\{1\leq m\leq k\colon a_m\neq$ $\neq 0\}$, and

$$A(\mathfrak{a})=\frac{m_{\mathfrak{a}}c_{\mathfrak{a}}}{|a_{m_{\mathfrak{a}}}|}.$$

Then it was established in [5] that $\zeta(s;\mathfrak{a})\neq 0$ for $\sigma>1+A(\mathfrak{a})$.

Now let

$$\hat{a}_m^{\pm}=\frac{1}{\sqrt{k}}\sum_{l=1}^{k}a_l\exp\left\{\pm 2\pi il\frac{m}{k}\right\},$$

$\hat{\mathfrak{a}}^{\pm}=\{\hat{a}_m^{\pm}\colon m\in\mathbb{N}\}$ and $B(\mathfrak{a})=\max\{A(\hat{\mathfrak{a}}^{\pm})\}$. Then it was obtained in [5] that the function $\zeta(s;\mathfrak{a})$, for $\sigma<-B(\mathfrak{a})$, can have only zeros close to the negative real axis if $m_{\hat{\mathfrak{a}}^+}=m_{\hat{\mathfrak{a}}^-}$, and close to the line

$$\sigma = 1 + \frac{\pi t}{\log \frac{m_{\hat{\mathfrak{a}}-}}{m_{\hat{\mathfrak{a}}+}}}$$

if $m_{\hat{\mathfrak{a}}+} \neq m_{\hat{\mathfrak{a}}-}$. The zeros $\rho$ of $\zeta(s;\mathfrak{a})$ with $\beta < -B(\mathfrak{a})$ are called trivial, and other zeros of $\zeta(s;\mathfrak{a})$ are nontrivial. So, nontrivial zeros lie in the strip $-B(\mathfrak{a}) \leq \sigma \leq 1 + A(\mathfrak{a})$.

In [5], an asymptotic formula for the number of nontrivial zeros $\rho$ of $\zeta(s;\mathfrak{a})$ with $|\gamma| \leq T$ also was obtained, and proved that the nontrivial zeros of $\zeta(s;\mathfrak{a})$ are clustered around the critical line.

Suppose that $k > 2$, $a_m$ is not a multiple of a Dirichlet character $\mathrm{mod} k$, and $a_m = 0$ for $(m,k) > 1$. Then it was observed in [6, p. 223] that $\zeta(s;\mathfrak{a})$ has infinitely many zeros in the strip $D = \left\{ s \in \mathbb{C} \colon \frac{1}{2} < \sigma < 1 \right\}$. Note that, in this case, the sequence $\mathfrak{a}$ is non multiplicative (we recall that $\mathfrak{a}$ is multiplicative if $a_1 = 1$ and $a_{mn} = a_m a_n$ for all $m, n \in \mathbb{N}$, $(m,n) = 1$), and the function $\zeta(s;\mathfrak{a})$ has no the Euler product over primes.

Our aim is to consider the case of a multiplicative sequence $\mathfrak{a}$, and to prove that the function $F(\zeta(s;\mathfrak{a}))$ with certain $F$ has infinitely many zeros in the strip $D$. In other words, we will construct composite functions of zeta functions with Euler product for which RH is not true. This is motivated by a better understanding of the RH problem.

Let $G$ be a region on the complex plane. Denote by $H(G)$ the space of analytic functions on $G$ equipped with the topology of uniform convergence on compacta. Define some classes of functions $F \colon H(G) \to H(G)$ for certain regions $G$. Let $V > 0$ be an arbitrary fixed number, $D_V = \left\{ s \in \mathbb{C} \colon \frac{1}{2} < \sigma < 1,\ |t| < V \right\}$, and $S_V = \{ g \in H(D_V) \colon g(s) \neq 0 \text{ or } g(s) \equiv 0 \}$. Denote by $U_V$ the class of continuous functions $F \colon H(D_V) \to H(D_V)$ such that, for each polynomial $p = p(s)$, the set $(F^{-1}\{p\}) \cap S_V$ is nonempty.

It is easily seen that the function

$$F(g) = \sum_{k=1}^{r} c_k g^{(k)}, \quad g \in H(D_V), \quad c_1, \ldots, c_r \in \mathbb{C} \setminus \{0\},$$

where $g^{(k)}$ stands for the $k$th derivative of $g$, is an element of the class $U_V$. Really, for arbitrary polynomial $p(s)$ of degree $k$, there exists a polynomial $\hat{p}(s)$ of degree $k+1$, $\hat{p}(s) \neq 0$ for $s \in D_V$, such that $F(\hat{p}) = p$.

Let $S = \{ g \in H(D) \colon g(s) \neq 0 \text{ or } g(s) \equiv 0 \}$. Now we introduce a class of functions $F$ for which the image $F(S)$ is a certain subset of $H(D)$. For $a_1, \ldots, a_r \in \mathbb{C}$, denote by $U_{a_1,\ldots,a_r}$ the class of continuous functions $F \colon H(D) \to H(D)$ such that $F(S) \supset H_{a_1,\ldots,a_r;F(0)}(D)$, where

$$H_{a_1,\ldots,a_r;F(0)}(D) = \{ g \in H(D) \colon g(s) \neq a_j,\ j = 1, \ldots, r \} \cup \{F(0)\}.$$

For example, the functions $F(g) = \sin g$, $F(g) = \cos g$, $F(g) = \sinh g$ and $F(g) = \cosh g$ belong to the class $U_{-1,1}$. To see this, it suffices to solve the equation $F(g) = f$ in $g \in S$. In the case of $F(g) = \cos g$, we have that

$$\frac{\mathrm{e}^{ig} + \mathrm{e}^{-ig}}{2} = f.$$

Hence, we find that

$$g_{\pm} = \frac{1}{i} \log \left( f \pm \sqrt{f^2 - 1} \right).$$

Thus, if $f \in H_{-1,1;1}(D)$, then we can choose, say, the solution $g_+$ which belongs to $S$. Therefore, $F \in U_{-1,1}$.

Our last class is very simple. We say that a continuous function $F\colon H(D) \to H(D)$ belongs to the class $U$, if $s - a \in F(S)$ for all $a \in \left(\frac{1}{2}, 1\right)$.

It is easily seen that the function $F(g) = gg'$, $g \in H(D)$, belongs to the class $U$. Really, solving the equation $gg' = s - a$, we find that $g = \pm\sqrt{s^2 - 2as + C}$ with arbitrary constant $C$. We can choose $C$ such that $s^2 - 2as + C \neq 0$ for $s \in D$. Thus, there exists $g \in S$ satisfying $F(g) = s - a$.

Now we are ready to state the theorems on zeros of the function $F(\zeta(s; \mathfrak{a}))$. In the notation used in Introduction, we suppose that $c_{\mathfrak{a}} < \sqrt{2} - 1$. Note that this inequality implies, for all primes $p$, the inequality

$$\sum_{\alpha=1}^{\infty} \frac{|a_{p^\alpha}|}{p^{\alpha/2}} \leq c < 1. \tag{1}$$

**Theorem 1.** *Suppose that the sequence $\mathfrak{a}$ is multiplicative such that the inequality $c_{\mathfrak{a}} < \sqrt{2} - 1$ is satisfied, and that $F$ belongs to at least one of the classes $U_V$ and $U$ with sufficiently large $V$. Then, for every $\sigma_1$, $\sigma_2$, $\frac{1}{2} < \sigma_1 < \sigma_2 < 1$, there exists a constant $c = c(\sigma_1, \sigma_2, \mathfrak{a}, F) > 0$ such that, for sufficiently large $T$ (in the case of the class $U_V$, we suppose that $T < V$), the function $F(\zeta(s; \mathfrak{a}))$ has more than $cT$ zeros in the rectangle $\{s \in \mathbb{C}\colon \sigma_1 < \sigma < \sigma_2,\ 0 < t < T\}$.*

**Theorem 2.** *Suppose that the sequence $\mathfrak{a}$ is the same as in Theorem 1, and $F \in U_{a_1,\ldots,a_r}$, where $\mathrm{Re}a_j \notin \left(-\frac{1}{2}, \frac{1}{2}\right)$, $j = 1, \ldots, r$. Then the same assertion as in Theorem 1 is true.*

For the proof of Theorems 1 and 2, we apply the universality property of the function $\zeta(s; \mathfrak{a})$.

**2. Universality.** The universality property of zeta functions was discovered by S. M. Voronin in 1975. In [8], he proved that the Riemann zeta function $\zeta(s)$ is universal in the sense that its shifts $\zeta(s + i\tau)$, $\tau \in \mathbb{R}$, approximate a wide class of analytic functions. The last version of the Voronin theorem is contained in the following theorem, see, for example, [9, p. 225]. Denote by $\mathrm{meas}\{A\}$ the Lebesgue measure of a measurable set $A \subset \mathbb{R}$.

**Theorem 3.** *Let $K \subset D$ be a compact set with connected complement, and let $f(s)$ be a continuous nonvanishing function on $K$ which is analytic in the interior of $K$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \mathrm{meas} \left\{ \tau \in [0, T]\colon \sup_{s\in K} |\zeta(s + i\tau) - f(s)| < \varepsilon \right\} > 0.$$

The first result on the universality of the function $\zeta(s; \mathfrak{a})$ was obtained in [10, p. 145], see also [11]. We state a more general case given in [6, p. 219].

**Theorem 4.** *Suppose that $k > 2$, $a_m$ is not a multiple of a Dirichlet character $\mathrm{mod}k$, and $a_m = 0$ for $(m, k) > 1$. Let $K \subset D$ be a compact set with connected complement, and let $f(s)$ be a continuous function on $K$ which is analytic in the interior of $K$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \mathrm{meas} \left\{ \tau \in [0, T]\colon \sup_{s\in K} |\zeta(s + i\tau; \mathfrak{a}) - f(s)| < \varepsilon \right\} > 0.$$

Note that the sequence $\mathfrak{a}$ in Theorem 4 is not multiplicative. The general case was discussed in [12] and [13]. We recall an universality theorem for $\zeta(s;\mathfrak{a})$ with multiplicative sequence $\mathfrak{a}$ [14].

**Theorem 5.** *Suppose that the sequence* $\mathfrak{a}$ *is multiplicative and inequality* (1) *holds. Let* $K$ *and* $f(s)$ *be the same as in Theorem* 3. *Then the same assertion as in Theorem* 4 *is true.*

Since $f(s)$ is nonvanishing on $K$, Theorem 5 does not give any information on zeros of the function $\zeta(s;\mathfrak{a})$.

In [7], the first author began to study the universality of $F(\zeta(s;\mathfrak{a}))$ for some classes of functions $F$, and in theorems obtained the shifts $F(\zeta(s+i\tau;\mathfrak{a}))$ approximate not necessarily nonvanishing analytic functions. Therefore, the theorems of such a kind provide an information on the zero-distribution of the function $F(\zeta(s;\mathfrak{a}))$. For the proof of Theorems 1 and 2, we use the following universality statements.

**Lemma 1.** *Suppose that the sequence* $\mathfrak{a}$ *is multiplicative and inequality* (1) *holds,* $K\subset D$ *is a compact set with connected complement, and* $f(s)$ *is a continuous function on* $K$ *and is analytic in the interior of* $K$. *Let* $V>0$ *be such that* $K\subset D_V$, *and* $F\in U_V$. *Then, for every* $\varepsilon>0$,

$$\liminf_{T\to\infty}\frac{1}{T}\mathrm{meas}\left\{\tau\in[0,T]\colon \sup_{s\in K}|F(\zeta(s+i\tau;\mathfrak{a}))-f(s)|<\varepsilon\right\}>0.$$

Proof of the lemma is given in [7].

Now we state an universality theorem for the functions from the class $U_{a_1,\ldots,a_r}$.

**Lemma 2.** *Suppose that* $\mathfrak{a}$ *is the same as in Lemma* 1, *and* $F\in U_{a_1,\ldots,a_r}$. *For* $r=1$, *let* $K\subset D$ *be a compact set with connected complement, and* $f(s)$ *be a continuous and* $\neq a_1$ *function on* $K$ *and analytic in the interior of* $K$. *For* $r\geq 2$, *let* $K\subset D$ *be an arbitrary compact set, and* $f\in H_{a_1,\ldots,a_r;F(0)}(D)$. *Then the same assertion as in Lemma* 1 *is true.*

Note that in [7], the universality of $F(\zeta(s;\mathfrak{a}))$ with $F$ satisfying a stronger condition $F(S)= = H_{a_1,\ldots,a_r;F(0)}(D)$ has been considered.

**Lemma 3.** *Suppose that* $\mathfrak{a}$ *is the same as in Lemma* 1, $K\subset D$ *is a compact subset, and* $f\in F(S)$. *Then the same assertion as in Lemma* 1 *is true.*

Lemmas 2 and 3 are deduced from a limit theorem on the weak convergence of probability measures in the space $H(D)$ [14] as well as from the Mergelyan theorem on the approximation of analytic functions by polynomials [15], see also [16, p. 436].

**3. Remarks on Theorems 1 and 2.** Theorems 1 and 2 are consequences of the classical Rouché theorem, see, for example, [17, p. 246] and Lemmas 1, 3, and 2, respectively.

1. *Hardy G. H.* Sur les zéros de la function $\zeta(s)$ de Riemann // C. R. Acad. Sci. Paris. – 1914. – **158**. – P. 1012 – 1014.
2. *Bui H. M., Conrey B., Young M. P.* More than 41% of the zeros of the zeta function are on the critical line // Acta Arith. – 2011. – **150**. – P. 35 – 64.
3. *Gourdon X.* The $10^{13}$ first zeros of the Riemann zeta function and zeros computation at very large height / http://numbers.computation.free.fr (2004).
4. *Schnee W.* Die Funktionalgleichung der Zetafunction und der Dirichletschen Reichen mit Periodishen Koeffizienten // Math. Z. – 1930. – **31**. – S. 378 – 390.
5. *Steuding J.* On Dirichlet series with periodic coefficients // Ramanujan J. – 2002. – **6**. – P. 295 – 306.
6. *Steuding J.* Value Distribution of $L$-Functions // Lect. Notes Math. – 2007. – **1877**.
7. *Laurinčikas A.* Universality of composite functions of periodic zeta functions (in Russian) // Mat. Sb. – 2012. – **203**, № 11. – P. 105 – 120 (English transl.: Sb. Math. – 2012. – **203**, № 11. – P. 1631 – 1646).
8. *Voronin S. M.* Theorem on the "universality"of the Riemann zeta function in Russian // Izv. Akad. Nauk SSSR, Ser. Mat. – 1975. – **39**. – P. 475 – 486 (English transl.: Math. USSR Izv. – 1975. – **9**. – S. 443 – 453).
9. *Laurinčikas A.* Limit theorems for the Riemann zeta function. – Dordrecht etc.: Kluwer Acad. Publ., 1996.

10. *Bagchi B.* The statistical behaviour and universality properties of the Riemann zeta function and other allied Dirichlet series: Ph. D. Thesis. – Calcuta, 1981.
11. *Bagchi B.* A joint universality theorem for Dirichlet $L$-functions // Math. Z. – 1982. – **181**. – S. 319 – 334.
12. *Sander J., Steuding J.* Joint universality for sums and products of Dirichlet $L$-functions // Analysis. – 2006. – **26**. – P. 285 – 312.
13. *Kaczorowski J.* Some remarks on the universality of periodic $L$-functions // New Dirrections in Value Distribution Theory of Zeta and $L$-Functions / Eds R. Steuding, J. Steuding. – Aachen: Shaker Verlag, 2009. – P. 113 – 120.
14. *Laurinčikas A., Šiaučiūnas D.* Remarks on the universality of periodic zeta functions // Mat. Zametki. – 2006. – **80**, No. 4. – S. 561 – 568 (English transl.: Math. Notes. – 2006. – **80**, № 3-4. – P. 532 – 538).
15. *Mergelyan S. N.* Uniform approximations to functions of complex variable // Usp. Mat. Nauk. – 1954. – **7**. – S. 31 – 122 (English transl.: Amer. Math. Soc. Trans. – 1954. – **101**. – P. 99).
16. *Walsh J. L.* Interpolation and approximation by rational functions in the complex domain (in Russian). – Moscow: Izd. Inost. Lit., 1961.
17. *Privalov I. I.* Introduction to the theory of functions of complex variable (in Russian). – Moscow: Nauka, 1967.

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