UDC 512.5

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DERIVATIONS ON PSEUDOQUOTIENTS

ПОХІДНІ НА ПСЕВДОЧАСТКАХ

A space of pseudoquotients, denoted by $\mathcal{B}(X,S)$, is defined as equivalence classes of pairs (x,f), where x is an element of a nonempty set X, f is an element of S, a commutative semigroup of injective maps from X to X, and $(x,f)\sim(y,g)$ if $gx=fy$. If X is a ring and elements of S are ring homomorphosms, then $\mathcal{B}(X,S)$ is a ring. We show that, under natural conditions, a derivation on X has a unique extension to a derivation on $\mathcal{B}(X,S)$. We also consider (α,β)-Jordan derivations, inner derivations, and generalized derivations.

Введено означення простору псевдочасток $\mathcal{B}(X,S)$ як класів еквівалентності пар (x,f), де x — елемент непорожньої множини X, f — елемент комутативної напівгрупи S ін'єктивних відображень із X у X та $(x,f)\sim(y,g)$, якщо $gx=fy$. Якщо X — кільце та елементи S є гомоморфізмами кільця, то $\mathcal{B}(X,S)$ є кільцем. Показано, що за природних умов похідна на X має єдине розширення до похідної на $\mathcal{B}(X,S)$. Також розглянуто (α,β)-жорданові похідні, внутрішні похідні та узагальнені похідні.

1. Introduction. Let X be a ring (or an algebra) with the unit I. An additive (or linear) map δ from X into it self is called a derivation if $\delta(AB)=\delta(A)B+A\delta(B)$ for all $A,B\in X$. Derivations are very important both in theory and applications, and are studied by many mathematicians. An additive (or linear) map δ from a ring (or an algebra) X into itself is called a Jordan derivation if $\delta(A^2)=\delta(A)A+A\delta(A)$ for all $A\in X$.

Let X be any nonempty set and S be a commutative semigroup acting on X injectively. This means that every $\phi\in S$ is an injective map $\phi\colon X\to X$ and $(\phi\psi)x=\phi(\psi x)$ for all ϕ, $\psi\in S$ and $x\in S$ and $x\in X$. For $(x,\phi),(y,\psi)\in X$ x S we write $(x,\phi)\ (y,\psi)$ if $\psi x=\phi y$.

It is easy to check that is an equivalence relation in X x S, finally we define

$B(X,S)=(X\text{x}S)/\!\sim$. The equivalence class of (x,ϕ) will be denoted by $\dfrac{x}{\phi}$. The set of psedo-quotients.

This is a slight absue of notion, but we follow here the tradition of denoting rational numbers by $\dfrac{p}{q}$ even through the same formal problem is present there.

Elements of X can be identified with elements of $B(X,S)$ via the embedding $\iota\colon X\to B(X,S)$ defined by

$$\iota(x)=\frac{\phi x}{\phi},$$

where ϕ is an arbitrary element of S, clearly is well defined that is, it is independent of ϕ. Action of S can be extended to $B(X,S)$ via

$$\phi\frac{x}{\psi}=\frac{\phi x}{\psi}$$

If $\phi\dfrac{x}{\psi}=i(y)$, for some $y\in X$, we will write $\phi\dfrac{x}{\psi}\in X$ and $\phi\dfrac{x}{\psi}=y$, which formally incorrect, but convenient and harmless. For instance, we have $\phi\dfrac{x}{\phi}=x$.

Element of S, when extended to maps on $B(X,S)$, become bijections. The action of ψ^{-1} on $B(X,S)$ can be defined as

$$\psi^{-1}\frac{x}{\phi}=\frac{x}{\phi\psi}.$$

Consequently, S can be extended to a commutative group of bijections acting on $B(X,S)$.

If $(X,\odot)$ is a commutative group and S is a commutative semigroup of injective homomorphsims on X, then $B(X,S)$ is a commutative group with the operation defined as

$$\frac{x}{\phi}\odot\frac{y}{\psi}=\frac{\psi x\odot\phi y}{\phi\psi}.$$

Similarly, if X is a vector space and S is a commutative semigroup of injective linear mapping from X into X, then $B(X,S)$ is a vector space with the operation defined as

$$\frac{x}{\phi}+\frac{y}{\psi}=\frac{\psi x+\phi y}{\phi\psi}\quad\text{and}\quad\lambda\frac{x}{\phi}=\frac{\lambda x}{\phi}.$$

If $\delta\colon X\to X$, if δ extends to a map $\hat{\delta}\colon B(X,S)\to B(X,S)$, it is often important to know what properties of δ are inherited by $\hat{\delta}$. In this section we consider some special situations when an extension is possible, which are important for the particular case studied in this paper .

If $\delta(fx)=f\delta(x)$ for all $x\in X$ and all $f\in S$, then we say that δ commutes with S.

The following Proposition 1.1 in [1] is use full to prove the following theorems.

Proposition 1.1. *Let $\delta\colon X\to X$. Then*

$$\hat{\delta}\left(\frac{x}{f}\right)=\frac{\delta(x)}{f}$$

is a well-defined extension of δ to $\hat{\delta}\colon B(X,S)\to B(X,S)$ if and only if δ commutes with S.

2. Derivation on pseudoquotients. In this section we study about extension of (α,β)-derivations on $B(X,S)$. And show under certain conditions it commutes with f is an injective ring homomorphisms form set S on X. Where S is a commutative semigroup of injective ring homomorphisms.

2.1. (α,β)-Derivations. Let X be a ring and let α and β be endomorphisms of X. By an (α,β)-derivation on X we mean a map $\delta\colon X\to X$ such that

$$\delta(xy)=\delta(x)\beta(y)+\alpha(x)\delta(y)\quad\text{for all}\quad x,y\in X.$$

A $(1,1)$-derivation, where 1 is the identity map on X is called simply a derivation. That is, by a derivation we mean a map $\delta\colon X\to X$ such that

$$\delta(xy)=\delta(x)y+x\delta(y)\quad\text{for all}\quad x,y\in X.$$

Theorem 2.1. *Let X be a ring and let S be a commutative semigroup of injective ring homomorphisms. Let α and β be homomorphisms from X into itself that commute with S, that is, $\alpha f(x)=f\alpha(x)$ and $\beta f(x)=f\beta(x)$ for every $f\in S$ and $x\in X$. If δ is an (α,β)-derivation on X that commutes with S, then the map $\hat{\delta}\colon B\to B$ defined by*

$$\hat{\delta}\left(\frac{x}{f}\right)=\frac{\delta(x)}{f}\tag{2.1}$$

is an extension of δ to an (α,β)-derivation on B.

Proof. Assume δ is an (α, β)-derivation on X that commutes with S. Then $\hat{\delta}$ is well-defined by Proposition 1.1 in [1]. In order to show that it is an (α, β)-derivation on B, consider $\dfrac{x}{f}, \dfrac{y}{g} \in B(X, G)$. Then

$$\hat{\delta}\left(\frac{x}{f}\frac{y}{g}\right) = \frac{\delta(gxfy)}{fg} = \frac{\delta(gx)\beta(fy) + \alpha(gx)\delta(fy)}{fg} =$$

$$= \frac{\delta x}{f}\frac{\beta(y)}{g} + \frac{\alpha(x)}{f}\frac{\delta y}{g} = \hat{\delta}\left(\frac{x}{f}\right)\beta\left(\frac{y}{g}\right) + \alpha\left(\frac{x}{f}\right)\hat{\delta}\left(\frac{y}{g}\right).$$

Theorem 2.1 is proved.

***Corollary* 2.1.** *Let X be a ring and let S be a commutative semigroup of injective ring homomorphisms. If δ is a derivation on X that commutes with S, then the map $\hat{\delta}\colon B \to B$ defined by*

$$\hat{\delta}\left(\frac{x}{f}\right) = \frac{\delta(x)}{f}$$

is an extension of δ to a derivation on B.

2.2. (α, β)-Jordan derivations. Let α and β be endomorphisms of X. By an (α, β)-Jordan derivation on X we mean a map $\delta\colon X \to X$ such that

$$\delta(x^2) = \delta(x)\beta(x) + \alpha(x)\delta(x) \quad \text{for all} \quad x \in X.$$

A $(1, 1)$-Jordan derivation, where 1 is the identity map on X is called simply a Jordan derivation. That is, by a Jordan derivation on X we mean a map $\delta\colon X \to X$ such that

$$\delta(x^2) = \delta(x)x + x\delta(x) \quad \text{for all} \quad x \in X.$$

Theorem 2.2. *Let X be a ring and let S be a commutative semigroup of injective ring homomorphisms. Let α and β be homomorphisms from X into itself that commute with S, that is, $\alpha f(x) = f\alpha(x)$ and $\beta f(x) = f\beta(x)$ for every $f \in S$ and $x \in X$. If δ is an (α, β)-Jordan derivation on X that commutes with S, then the map $\hat{\delta}\colon B \to B$ defined by*

$$\hat{\delta}\left(\frac{x}{f}\right) = \frac{\delta(x)}{f}$$

is an extension of δ to an (α, β)-Jordan derivation on B.

Proof. The proof is similar to the proof of Theorem 2.1.

***Corollary* 2.2.** *Let X be a ring and let S be a commutative semigroup of injective ring homomorphisms. If δ is a derivation on X that commutes with S, then the map $\hat{\delta}\colon B \to B$ defined by*

$$\hat{\delta}\left(\frac{x}{f}\right) = \frac{\delta(x)}{f}$$

is an extension of δ to a derivation on B.

In Theorem 2.2 and the above corollary it is necessary to assume that δ commutes with S. The next theorem describes a situation which guarantees that δ commutes with S.

Theorem 2.3. *Let X be an unital Banach algebra and let f be an injective algebra homomorphism. Let α and β be algebra homomorphisms from X into itself that commute with f, if δ is a linear mapping on X such that*

$$\delta(xx^{-1}) = \alpha(x)\delta(x^{-1}) + \delta(x)\beta(x^{-1}) \tag{2.2}$$

for every invertible element $x \in X$, then δ is an (α, β)-Jordan derivation on X and commutes with f.

Proof. It is known that (2.2) implies $\delta(e) = 0$, where e is the identity element in X. Therefore, $\delta(fe) = 0$ for any f injective homomorphism on X. In order to show that linear mapping δ is a Jordan derivation and commutes with S we have to show that $\delta f y^2 = f\delta y^2$. For any T in X. Let n be a positive integer with $n > \|T\| + e$ and $y = ne + T$. We have that y and $e - y$ are invertible in X. Since $\alpha(fx^{-1}) = \alpha(fx)^{-1} = f\alpha(x^{-1}) = f\alpha(x^{-1})$ and $\beta(fx^{-1}) = \beta(fx)^{-1} = f\beta(x^{-1}) = f\beta(x^{-1})$ for any invertible element x in X. Then

$$\delta(fy) = -\alpha(fy)\delta(fy^{-1})\beta(fy) = -\alpha(fy)\delta(fy^{-1}f(e-y)^2 - fy)\beta(fy) =$$

$$= \alpha(fy)\alpha(fy^{-1}f(e-y)^2)\delta(f(e-y)^{-2}fy)\beta(y^{-1}(e-y)^2)\beta(fy) + \alpha(fy)\delta(fy)\beta(fy) =$$

$$= \alpha(fy)\alpha(fy^{-1} - 2fe + fy)\delta(f(e-y)^{-2} - f(e-y)^{-1})\beta(fy^{-1} - 2fe + fy)\beta(fy) +$$

$$+\alpha(fy)\delta(fy)\beta(fy) =$$

$$= (e - 2\alpha(fy) + \alpha f(y)^2)\delta(f(e-y)^{-2} - f(e-y)^{-1})(e - 2\beta(fy) + \beta f(y^2)) +$$

$$+\alpha(fy)\delta(fy)\beta(fy) = \alpha(f(e-y)^2)\delta(f(e-y)^{-2})\beta(f(e-y)^2) -$$

$$-(\alpha f(e-y))^2)\delta((e-y)^{-1})(\beta f(e-y))^2 + \alpha(fy)\delta(fy)\beta(fy) =$$

$$= -\delta(f(e-y)^2) + \alpha f(e-y)\delta f(e-y)\beta f(e-y) + \alpha(fy)\delta(fy)\beta(fy) =$$

$$= 2\delta(fy) - \delta f(y^2) - \delta(fy) + \alpha(fy)\delta(fy) + \delta(fy)\beta(fy) -$$

$$-\alpha(fy)\delta(fy)\beta(fy) + \alpha(fy)\delta(fy)\beta(fy) =$$

$$= \delta(fy) - \delta(fy^2) + \alpha(fy)\delta(fy) + \delta(fy)\beta(fy).$$

Hence $\delta(fy^2) = \delta(fy)\beta(fy) + \alpha(fy)\delta(fy)$. Since $\delta(fe) = 0$ and $fy = f(ne) + ft$, we have that $\delta(ft^2) = \delta(ft)\beta(ft) + \alpha(ft)\delta(ft)$ for any $t \in X$. Similarly we can show for $f\delta x$.

Theorem 2.3 is proved.

***Corollary* 2.3.** *Let X be an unital Banach algebra and let f be an injective algebra homomorphism on X. Let α and β be algebra homomorphisms from X onto itself that commute with f. If δ is a linear mapping on X such that*

$$\delta(xx^{-1}) = \alpha(x)\delta(x^{-1}) + \delta(x)\beta(x^{-1})$$

for every invertible element $x \in X$, then δ is an (α, β)-Jordan derivation on X that commutes with $S = \{f^n \colon n = 1, 2, 3, \ldots\}$ and the map $\hat{\delta} \colon B(X, S) \to B(X, S)$, defined by

$$\hat{\delta}\left(\frac{x}{f}\right) = \frac{\delta(x)}{f},$$

is an extension of δ to an (α, β)-Jordan derivation on $B(X, S)$.

2.3. Idempotent. An idempotent element of a ring is an element which is idempotent with respect to the ring's multiplication, that is, $r^2 = r$. A ring in which all elements are idempotent is called a Boolean ring.

Lemma 2.1. *Let X be a ring and let S be a commutative semigroup of injective ring homomorphisms. $\frac{x}{f}$ is idempotent in $B(X, S)$ if and only if x is idempotent in X.*

Proof. If $\frac{x}{f}$ is idempotent in $B(X, S)$, then

$$\frac{x}{f} = \frac{x}{f}\frac{x}{f} = \frac{fxfx}{f^2} = \frac{f(x^2)}{f^2} = \frac{x^2}{f}.$$

Consequently, $x = x^2$. The proof in the other direction follows from the above.

Theorem 2.4. *Let X be a commutative ring and let S be a commutative semigroup of injective ring homomorphisms and let δ be a derivation on X that commutes with S. If $\hat{\delta}$ is the extension of δ onto $B(X, S)$ as defined by* (2.1) *and $\frac{x}{f} \in B(X, S)$ is idempotent, then*

(i) $\hat{\delta}\left(\frac{x}{f}\right) = 0,$

(ii) $\hat{\delta}\left(\frac{y}{g}\frac{x}{f}\right) = \hat{\delta}\left(\frac{y}{g}\right)\frac{x}{f}$ *for any* $\frac{y}{g} \in B(X, S),$

(iii) $\hat{\delta}\left(\frac{x}{f}\frac{y}{g}\right) = \frac{x}{f}\hat{\delta}\left(\frac{y}{g}\right)$ *for any* $\frac{y}{g} \in B(X, S).$

Proof. For any idempotent $\frac{x}{f} \in B(X, S)$, we have

$$\hat{\delta}\left(\frac{x}{f}\right) = \hat{\delta}\left(\frac{x}{f}\frac{x}{f}\right) = \hat{\delta}\left(\frac{x}{f}\right)\frac{x}{f} + \frac{x}{f}\hat{\delta}\left(\frac{x}{f}\right).$$

As X is a commutative ring,

$$= \hat{\delta}\left(\frac{x}{f}\right)\frac{x}{f} + \hat{\delta}\left(\frac{x}{f}\right)\frac{x}{f}$$

and consequently

$$\hat{\delta}\left(\frac{x}{f}\right)\frac{x}{f} = \hat{\delta}\left(\frac{x}{f}\right)\frac{x}{f} + \hat{\delta}\left(\frac{x}{f}\right)\frac{x}{f}.$$

This shows that $\hat{\delta}\left(\frac{x}{f}\right) = 0.$

$$\text{(ii)} \quad \hat{\delta}\left(\frac{y}{g}\frac{x}{f}\right) = \hat{\delta}\left(\frac{y}{g}\right)\frac{x}{f} + \frac{g}{y}\hat{\delta}\left(\frac{x}{f}\right) = \frac{g}{y}\hat{\delta}\left(\frac{x}{f}\right).$$

Similarly we can show $\hat{\delta}\left(\frac{x}{f}\frac{y}{g}\right) = \frac{x}{f}\hat{\delta}\left(\frac{y}{g}\right).$

Theorem 2.4 is proved.

By induction, it is easy to show that for any idempotents $\dfrac{x_1}{f}, \dfrac{x_2}{f}, \ldots, \dfrac{x_n}{f} \in B$ and any $\dfrac{y}{g} \in B$,

$$\hat{\delta}\left(\frac{x_1}{f}\frac{x_2}{f}\ldots\frac{x_n}{f}\frac{y}{g}\right) = \frac{x_1}{f}\frac{x_2}{f}\ldots\frac{x_n}{f}\hat{\delta}\left(\frac{y}{g}\right).$$

2.4. Inner derivations. An inner derivation on X is a map $\delta\colon X \to X$ such that

$$\delta(x) = xy - yx \quad \text{for each} \quad y \in X.$$

Let X be a ring and let S be a commutative semigroup of injective ring homomorphisms. $\delta\colon X \to X$ is an inner derivation for each $x \in X$ and for each $f \in S$

$$\delta(x) = xf - fx.$$

Theorem 2.5. *Let X be a ring and let S be a commutative semigroup of injective ring homomorphisms. If δ is a inner derivation on X, then the map $\hat{\delta}\colon B \to B$ defined by $\hat{\delta}\left(\dfrac{x}{f}\right) = \dfrac{2\delta(x)}{f} - \dfrac{\delta(fx)}{f^2}$ is an extension of δ to a inner derivation on B if $xf - fx$ commutes with S for every $f \in S$.*

2.5. Generalized derivation. $\delta\colon X \to X$ is a map on X is called a generalized derivation if there exists a derivation $d\colon X \to X$ such that

$$\delta(xy) = \delta(x)y + xd(y) \quad \text{for all} \quad x, y \in X.$$

Theorem 2.6. *Let X be a ring and let S be a commutative semigroup of injective ring homomorphisms. If δ is a generalized derivation on X, then the map $\hat{\delta}\colon B \to B$ defined by*

$$\hat{\delta}\left(\frac{x}{f}\right) = \frac{\delta(x)}{f}$$

is an extension of δ to a generalized derivation on B.

Proof. Assume that δ and d commutes with S. In order to show that it is an δ is a generalized derivation on B, consider $\dfrac{x}{f}, \dfrac{y}{g} \in B(X, G)$:

$$\hat{\delta}\left(\frac{x}{f}\frac{y}{g}\right) = \frac{\delta(gxfy)}{fg} = \frac{\delta(gx)(fy) + (gx)d(fy)}{fg} =$$

$$= \frac{\delta x}{f}\frac{y}{g} + \frac{x}{f}\frac{dy}{g} = \hat{\delta}\left(\frac{x}{f}\right)\left(\frac{y}{g}\right) + \alpha\left(\frac{x}{f}\right)\hat{d}\left(\frac{y}{g}\right).$$

Theorem 2.6 is proved.

***Example* 2.1.** Let R be a commutative ring and let δ be a derivation on R. For an element $x \in R$ we denote by M_x the homomorphism defined by $M_x(y) = xy$. Let

$$S = \{M_x\colon x \in R, M_x \ \text{ is injective, and } \ \delta(x) = 0\}.$$

Since

$$\delta(M_x(y)) = \delta(xy) = \delta(x)y + x\delta(y) = x\delta(y) = M_x(\delta y)$$

for every $M_x \in S$, δ can has a unique extension to a derivation on $\mathcal{B}(R, S)$.

For a simple example we can take for R the ring of polynomials in x and y and $\delta = \dfrac{\partial}{\partial y}$. Then S is not trivial and, since it contains homomorphism that are not surjective, $\mathcal{B}(R, S)$ is a nontrivial extension of R.

***Example* 2.2.** Let $\mathcal{N}$ be a nest algebra and S be a commutative semigroup acting on $\mathcal{N}$ generated by finite rank operators. δ is a derivation on $\mathcal{N}$ with $\delta(\phi) = 0$.

Let for any arbitrary n from $\mathcal{N}$ and ϕ from G. From [4] every finite rank operator in $\mathcal{N}$ represented as a sum of rank one operators. From [3] Every rank one operator in $\mathcal{N}$ denoted as linear combination of at most four idempotents.

Hence we have $\delta(\phi n) = \phi\delta(n) + n\delta(\phi)$. Where $\delta(\phi) = 0$, for every rank one operator from S.

1. *Atanasiu D., Mikusiński P., Nemzer D.* An algebraic approach to tempered distributions // J. Math. Anal. and Appl. – 2011. – **384**. – P. 307 – 319.
2. *JianKui Li., Jiren Zhou.* Characterization of Jordan derivations and Jordan homomorphisms // Linear and Multilinear Algebra. – 2011. – **52**, № 2. – P. 193 – 204.
3. *Erdos J. A.* Operator of finite rank in nest algebras // London Math.Soc. – 1968. – **43**. – P. 391 – 397.
4. *Hadwin L. B.* Local multiplications on algebras spanned by idempotents // Linear and Multilinear Algebras. – 1994. – **37**. – P. 259 – 263.

Received 23.04.12