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SEMIDERIVATIONS WITH POWER VALUES ON LIE IDEALS IN PRIME RINGS*

НАПІВПОХІДНІ З СТЕПЕНЕВИМИ ЗНАЧЕННЯМИ НА ІДЕАЛАХ ЛІ У ПРОСТИХ КІЛЬЦЯХ

Let R be a prime ring, L a noncentral Lie ideal, and f a nonzero semiderivation associated with an automorphism σ such that $f(u)^n = 0$ for all $u \in L$, where n is a fixed positive integer. If either $\operatorname{Char} R > n+1$ or $\operatorname{Char} R = 0$, then R satisfies s_4, the standard identity in four variables.

Нехай R — просте кільце, L — нецентральний ідеал Лі та f — ненульова напівпохідна, асоційована з автоморфізмом σ таким, що $f(u)^n = 0$ для всіх $u \in L$, де n — фіксоване натуральне число. Якщо $\operatorname{Char} R > n+1$ або $\operatorname{Char} R = 0$, то R задовольняє стандартну тотожність s_4 у чотирьох змінних.

1. Introduction. The standard identity s_4 in four variables is defined as follows:

$$s_4 = \sum (-1)^{\tau} X_{\tau(1)} X_{\tau(2)} X_{\tau(3)} X_{\tau(4)}$$

where $(-1)^{\tau}$ is the sign of a permutation τ of the symmetric group of degree 4.

In all that follows, unless stated otherwise, R always denotes a prime ring, $Z(R)$ the center of R, Q its Martindale quotient ring. The center of Q, denoted by C, is called the extended centroid of R (we refer the reader to [1] for these objects). It is well-known that C is a field. For any $x, y \in R$, the symbol $[x, y]$ stands for Lie commutator $xy - yx$. An additive subgroup U of R is said to be a Lie ideal of R if $[u, r] \in U$ for all $u \in U$ and $r \in R$. For subsets A, B of R we let $[A, B]$ be the additive subgroup generated by all $[a, b]$ with $a \in A$ and $b \in B$. Recall that a ring R is prime if for any $a, b \in R$, $aRb = (0)$ implies $a = 0$ or $b = 0$, and is semiprime if for any $a \in R$, $aRa = (0)$ implies $a = 0$. In [2], Bergen introduced the notion of a semiderivation: an additive mapping $f\colon R \longrightarrow R$ is called a semiderivation associated with a function $g\colon R \longrightarrow R$ such that $f(xy) = f(x)g(y) + xf(y) = f(x)y + g(x)f(y)$ and $f(g(x)) = g(f(x))$ hold for all $x, y \in R$. In case $g = 1_R$, the identity map on R, f is of course a derivation. Brešar [4] proved that the only semiderivations of prime rings are ordinary derivations and mappings of the form $f(x) = \lambda(x - g(x))$, where $\lambda \in C$ and g is an endomorphism.

This paper is included in a line of investigation in the literature concerning derivations having nilpotent values. The first result is due to Herstein [10] who proved that if R is a prime ring and d is an inner derivation of R satisfying $d(x)^n = 0$ (resp. $d(x)^n \in Z(R)$) for all $x \in R$, where n is a fixed integer, then $d = 0$ (resp. R satisfies s_4). In [8], Carini and Giambruno studied the case when $d(u)^{n(u)} = 0$ for all $u \in L$, a Lie ideal of R and they proved $d(L) = 0$, when R is a prime ring, $\operatorname{Char} R \neq 2$ and R contains no nil right ideals, and they obtained the same conclusion when n is fixed and R is a semiprime ring with $\operatorname{Char} R \neq 2$. Later in [13], Lanski obtained the same results, removing both the bound on the indices of nilpotence and the characteristic assumption on R. In

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[14], Lee extended Herstein's result to the case of generalized derivations. More recently, Chang [5] proved Herstein's above result in the setting of right generalized β-derivations. Motivated by the above results, our purpose here is to obtain some information on the structure of a prime ring R satisfying $f(u)^n = 0$ on a noncentral Lie ideal L, where f is a nonzero semiderivation of R and n is a fixed positive integer.

2. Main results.

Theorem 2.1. *Let R be a prime ring, L a noncentral Lie ideal and f a nonzero semiderivation associated with an automorphism σ such that $f(u)^n = 0$ for all $u \in L$, where n is a fixed positive integer. If either $\operatorname{Char} R > n+1$ or $\operatorname{Char} R = 0$, then R satisfies s_4, the standard identity in four variables.*

Proof. If $\sigma = 1_R$, then f is a derivation, and we are done by a result of Bergen and Carini [3]. So we assume next that $\sigma \neq 1_R$. In this case, it is well-known that there exists a nonzero two-sided ideal I of R such that $0 \neq [I, R] \subseteq L$. In particular, $[I, I] \subseteq L$, hence without loss of generality we may assume that $L = [I, I] \subseteq L$. In view of Brešar [4] (Theorem), $f(x) = \lambda(x - \sigma(x))$ for all $x \in R$, where $0 \neq \lambda \in C$. We are given that $(\lambda[x, y] - \lambda\sigma[x, y])^n = 0$, which implies that

$$(\sigma[x, y] - [x, y])^n = ([\sigma(x), \sigma(y)] - [x, y])^n = 0 \quad \text{for all} \quad x, y \in Q. \tag{2.1}$$

By Kharchenko's theorem [12], we divide the proof into two cases.

Case 1. Let σ be Q-outer. Since either $\operatorname{Char} R > n+1$ or $\operatorname{Char} R = 0$, by Chuang [7] (Main theorem), we see that $([u, v] - [x, y])^n = 0$ for all $u, v, x, y \in I$, in particular, letting $x = 0$ then $[u, v]^n = 0$ for all $u, v \in I$. Then by Herstein [10] (Theorem 2) R is commutative, a contradiction.

Case 2. Suppose now that σ is Q-inner, then there exists an invertible element $b \in Q - C$ such that $\sigma(x) = b^{-1}xb$ for all $x \in R$. By Chuang [6] (Theorem 2), I, R and Q satisfy the same generalized polynomial identities (or GPIs in brief), from (2.1) we have

$$([b^{-1}xb, b^{-1}yb] - [x, y])^n = 0 \quad \text{for all} \quad x, y \in Q. \tag{2.2}$$

In case the center C of Q is infinite, we have $([b^{-1}xb, b^{-1}yb] - [x, y])^n = 0$ for all $x, y \in Q \bigotimes_C \overline{C}$, where $\overline{C}$ is the algebraic closure of C. Since both Q and $Q \bigotimes_C \overline{C}$ are prime and centrally closed [9] (Theorem 2.5 and Theorem 3.5), we may replace R by Q or $Q \bigotimes_C \overline{C}$ according as C is finite or infinite. Thus we may assume that R is centrally closed over C (i.e., $RC = R$) which is either finite or algebraically closed and $([b^{-1}xb, b^{-1}yb] - [x, y])^n = 0$ for all $x, y \in R$. By Martindale [15] (Theorem 3), RC (and so R) is a strongly primitive ring. In light of Jacobson's theorem [11, p. 75], R is isomorphic to a dense ring of linear transformations of a vector space V. Let ${}_RV$ be a faithful irreducible left R-module with commuting division $D = \operatorname{End}({}_RV)$. Since C is either finite or algebraically closed, we know that D must coincide with C. By the density theorem, R acts densely on V_D. For any given $v \in V$, we want to show that v and bv are linearly D-dependent. If $bv = 0$ then v and bv are D-dependent and we are done in this case. Suppose that $bv \neq 0$, v and bv are D-independent. We consider the following two cases.

Subcase 1. Assume that v, bv, $b^{-1}v$ are D-independent. Then by the density of R, there exist $x, y \in R$ such that

$$xv = bv, \qquad xbv = v, \qquad yv = bv, \qquad ybv = 0.$$

Application of (2.2) yields that

$$0 = \big([b^{-1}xb, b^{-1}yb] - [x, y]\big)^n v = (-2)^n v \neq 0$$

a contradiction.

Subcase 2. Otherwise, $v, bv, b^{-1}v$ are D-dependent. Since v and bv are D-independent, then $b^{-1}v = vd_1 + bvd_2$ for some $d_1, 0 \neq d_2 \in D$. By the density of R, there exist $x, y \in R$ such that

$$xv = 0, \qquad xbv = v, \qquad yv = b^{-1}v = vd_1 + bvd_2, \qquad ybv = bvd_1.$$

In view of (2.2), we have

$$0 = \big([b^{-1}xb, b^{-1}yb] - [x, y]\big)^n v = 2^n v d_2^n \neq 0$$

a contradiction. From the above we have proven that $bv = v\alpha_v$ for all $v \in V$, where $\alpha_v \in D$ depends on $v \in V$. In fact, it is easy to check that α_v is independent of the choice of $v \in V$. Indeed, for any $v, w \in V$, by the above arguments, there exist $\alpha_v, \alpha_w, \alpha_{v+w} \in D$ such that $bv = v\alpha_v$, $bw = w\alpha_w$, $b(v + w) = (v + w)\alpha_{v+w}$ and so $v\alpha_v + w\alpha_w = b(v + w) = (v + w)\alpha_{v+w}$. Hence $v(\alpha_v - \alpha_{v+w}) + w(\alpha_w - \alpha_{v+w}) = 0$. If v and w are D-independent, then $\alpha_v = \alpha_{v+w} = \alpha_w$ and we are done. Otherwise, v and w are D-dependent, say $v = \lambda w$ for some $\lambda \in D$. Thus $v\alpha_v = bv = b\lambda w = \lambda bw = \lambda w\alpha_w = v\alpha_w$, that is $V(\alpha_v - \alpha_w) = 0$. Since V is faithful, hence $\alpha_v = \alpha_w$. So we conclude that there exists $\delta \in D$ such that $bv = v\delta$ for all $v \in V$. We claim that $\delta \in Z(D)$, the center of D. Indeed, for any $\beta \in D$, we have $b(v\beta) = (v\beta)\delta = v(\beta\delta)$ and on the other hand $b(v\beta) = (bv)\beta = (v\delta)\beta = v(\delta\beta)$. Therefore $V(\beta\delta - \delta\beta) = 0$ and hence $\beta\delta = \delta\beta$, which implies that $\delta \in Z(D)$. So $b \in C$, a contradiction.

Theorem 2.1 is proved.

Proceeding on same lines with necessary variations, we can prove the following theorem.

Theorem 2.2. *Let R be a prime ring, I a nonzero ideal and f a nonzero semiderivation associated with an automorphism σ such that $f([x, y])^n = 0$ for all $x, y \in I$, where n is a fixed positive integer. If either* $\operatorname{Char} R > n + 1$ *or* $\operatorname{Char} R = 0$, *then R is commutative.*

The following example demonstrates that R to be prime is essential in Theorem 2.2.

***Example* 2.1.** Let Z be the ring of all integers. Set

$$R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} | a, b, c \in Z \right\} \qquad \text{and} \qquad I = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} | a, b \in Z \right\}.$$

Next, let us define a mapping $f \colon R \longrightarrow R$ given by

$$f \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2a & 0 \\ 0 & 0 & 2c \\ 0 & 0 & 0 \end{pmatrix}.$$

The fact $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq 0$ implies that

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0,$$

proving R is not prime. Then, it is straightforward to check that I is a nonzero ideal of R and T is a nonzero semiderivation of R. And it is easy to find that $\big(f([x, y])\big)^n = 0$ for all $x, y \in I$. However R is not commutative.

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