UDC 517.91

Li Liang (Lanzhou Jiaotong Univ., China),
Dejun Wu (Lanzhou Univ. Technology, China)

ON RESTRICTED PROJECTIVE DIMENSION OF COMPLEXES*

ПРО ОБМЕЖЕНУ ПРОЕКТИВНУ РОЗМІРНІСТЬ КОМПЛЕКСІВ

We study the restricted projective dimension of complexes. We give some new characterizations of the restricted projective dimension. In particular, we show that the restricted projective dimension can be computed in terms of the so-called restricted projective resolutions. As applications, we get some results on the behavior of the restricted projective dimension under change of rings.

Вивчається обмежена проективна розмірність комплексів. Наведено деякі нові властивості обмеженої проективної розмірності. Зокрема, показано, що обмежену проективну розмірність можна обчислити через так звані обмежені проективні резольвенти. Як застосування отримано деякі результати про поведінку обмеженої проективної розмірності при зміні кілець.

Introduction. As is well known, the classical homological dimensions — projective, flat and injective dimensions are defined in terms of resolutions, but they can also be computed in terms of vanishing of appropriate derived functors. For example, the flat dimension of R-module M can be computed as follows:

$$\mathrm{fd}_R(M) = \sup\big\{i \in \mathbb{N}_0 \mid \mathrm{Tor}_i^R(T, M) \neq 0 \ \text{ for some module } \ T\big\}.$$

The restricted flat dimension was defined solely in terms of the vanishing of the derived functor Tor over some classes of test modules that are restricted to assure automatic finiteness over commutative Noetherian rings of finite Krull dimension (see [3]). Accurately, the restricted flat dimension, denoted $\mathrm{Rfd}_R\, M$, of an R-module M is defined as

$$\mathrm{Rfd}_R(M) = \sup\big\{i \in \mathbb{N}_0 \mid \mathrm{Tor}_i^R(T, M) \neq 0 \ \text{ for some module } \ T \ \text{ with } \ \mathrm{fd}_R(T) < \infty\big\}.$$

Christensen, Foxby and Frankild [3] further studied the restricted flat dimension of complexes, and they gave a number of interesting properties. For example, they showed the restricted flat dimension is finite for any homologically bounded complex over commutative Noetherian rings of finite Krull dimension, and it is a refinement of both flat and Gorenstein flat dimensions. Sharif and Yassemi [4] studied the behavior of the restricted flat dimension under change of rings, and generalized some classical results.

Let X be a homologically bounded below complex of R-modules. The restricted projective dimension, denoted $\mathrm{Rpd}_R\, X$, of X was defined by Christensen, Foxby and Frankild in [3]. They showed that this dimension is also finite for any homologically bounded complex over commutative Noetherian rings of finite Krull dimension. In this paper, we give some new characterizations of the restricted projective dimension of complexes as follows, which show that the restricted projective dimension can be computed in terms of the so-called restricted projective resolutions (see Theorem 2.1).

* This work was partially supported by NSF of China (Grant No. 11226059) and the Young Scholars Science Foundation of Lanzhou Jiaotong University (Grant No. 2012020).

Theorem A. *Let X be a homologically bounded below complex and $n \in \mathbb{Z}$. Consider the following conditions*:

(1) $\operatorname{Rpd}_R X \leq n$.

(2) *X is equivalent to a bounded complex P of restricted projective R-modules with* $\sup\{i \in \in \mathbb{Z} \mid P_i \neq 0\} \leq n$; *and P can be chosen such that $P_l = 0$ for $l < \inf X$.*

(3) $\mathrm{H}_i(\mathbf{R}\operatorname{Hom}_R(X, T)) = 0$ *for any $i < -n$ and any R-module T with* $\operatorname{id}_R(T) < \infty$.

(4) $\sup X \leq n$ *and $\mathrm{C}_n(P)$ is a restricted projective R-module whenever P is a bounded below complex of restricted projective R-modules which is equivalent to X.*

(5) $-\inf(\mathbf{R}\operatorname{Hom}(X, U)) + \inf U \leq n$ *for any non-exact complex U with* $\operatorname{id}_R U < \infty$.

Then we have (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Leftarrow$ (5). *If, furthermore, X is homologically degree-wise finite, then all the above statements are equivalent.*

As applications of the above theorem, we get the following result on the behavior of the restricted projective dimension under change of rings (see Propositions 2.4 and 2.5).

Theorem B. *Let $\varphi\colon R \to S$ be a homomorphism of rings and X a homologically bounded below and degree-wise finite complex of S-modules. Then the following statements hold*:

(1) *If Y is a homologically bounded below complex of R-modules with* $\operatorname{fd}_R Y < \infty$, *then we have*

$$\operatorname{Rpd}_R(X \otimes_R^{\mathbf{L}} Y) \leq \operatorname{Rpd}_S X + \operatorname{Rpd}_R Y + \operatorname{Rpd}_R S$$

and

$$\operatorname{Rpd}_S(X \otimes_R^{\mathbf{L}} Y) \leq \operatorname{Rpd}_S X + \operatorname{Rfd}_R S + \sup Y + \dim S.$$

(2) *If Y is a homologically bounded below complex of S-modules with* $\operatorname{fd}_S Y < \infty$, *then we have*

$$\operatorname{Rpd}_R(X \otimes_S^{\mathbf{L}} Y) \leq \operatorname{Rpd}_S X + \operatorname{Rpd}_R Y.$$

1. Preliminaries. We begin with some notations and terminology for use throughout this paper, which can be found in [2].

1.1. A *complex* $\ldots \longrightarrow X_1 \xrightarrow{\delta_1^X} X_0 \xrightarrow{\delta_0^X} X_{-1} \longrightarrow \ldots$ of R-modules will be denoted by (X, δ^X) or simply X. We frequently (and without warning) identify R-modules with complexes concentrated in degree 0. A complex X is *bounded above* (resp., *bounded below*, *bounded*) if $X_n = 0$ for $n \gg 0$ (resp., $n \ll 0$, $|n| \gg 0$). The nth *boundary* (resp., *cycle*, *homology*) of X is defined as $\operatorname{Im}\delta_{n+1}^X$ (resp., $\operatorname{Ker}\delta_n^X$, $\operatorname{Ker}\delta_n^X / \operatorname{Im}\delta_{n+1}^X$) and it is denoted by $\mathrm{B}_n(X)$ (resp., $\mathrm{Z}_n(X)$, $\mathrm{H}_n(X)$). A complex X is *homologically bounded above* (resp., *homologically bounded below*, *homologically bounded*) if the homology complex $\mathrm{H}(X)$ is bounded above (resp., bounded below, bounded). We use the notation $\mathrm{C}_n(X)$ for the cokernel of the differential δ_{n+1}^X. The *soft truncations* of X at n are the complexes

$$X_{\subset n} \equiv 0 \longrightarrow \mathrm{C}_n(X) \xrightarrow{\overline{\delta_n^X}} X_{n-1} \xrightarrow{\delta_{n-1}^X} X_{n-2} \longrightarrow \ldots$$

and

$$X_{\supset n} \equiv \ldots \longrightarrow X_{n+2} \xrightarrow{\delta_{n+2}^X} X_{n+1} \xrightarrow{\delta_{n+1}^X} \mathrm{Z}_n(X) \longrightarrow 0.$$

The *hard truncations* of X at n are the complexes

$$X_{\leq n} \equiv 0 \longrightarrow X_n \longrightarrow X_{n-1} \longrightarrow X_{n-2} \longrightarrow \ldots$$

and

$$X_{\geq n} \equiv \ldots \longrightarrow X_{n+2} \longrightarrow X_{n+1} \longrightarrow X_n \longrightarrow 0.$$

The *supremum* and *infimum* of X are given by the following formulas:

$$\sup(X) = \sup\big\{i \in \mathbb{Z} \mid \mathrm{H}_i(X) \neq 0\big\} \qquad \text{and} \qquad \inf(X) = \inf\big\{i \in \mathbb{Z} \mid \mathrm{H}_i(X) \neq 0\big\}.$$

For any $m \in \mathbb{Z}$, $\Sigma^m X$ denotes the complex with the degree-n term $(\Sigma^m X)_n = X_{n-m}$ and whose boundary operators are $(-1)^m \delta^X_{n-m}$.

1.2. If X and Y are both complexes, then by a *morphism* $\alpha\colon X \longrightarrow Y$ we mean a sequence $\alpha_n\colon X_n \longrightarrow Y_n$ such that $\alpha_{n-1}\delta^X_n = \delta^Y_n \alpha_n$ for each $n \in \mathbb{Z}$. A *quasiisomorphism*, indicated by the symbol "$\simeq$" next to their arrows, is a morphism of complexes that induces an isomorphism in homology. The *mapping cone* $\mathrm{Cone}(\alpha)$ of α is defined as $\mathrm{Cone}(\alpha)_n = Y_n \oplus X_{n-1}$ with nth boundary operator $\delta^{\mathrm{Cone}(\alpha)}_n = \begin{pmatrix} \delta^N_n \, \alpha_{n-1} \\ 0 - \delta^M_{n-1} \end{pmatrix}$. It is well known that a morphism α is a quasiisomorphism if and only if its mapping cone $\mathrm{Cone}(\alpha)$ is exact. Two complexes X and Y are *equivalent*, we write $X \simeq Y$, if there is the third complex Z and two quasiisomorphisms: $X \xrightarrow{\simeq} Z \xleftarrow{\simeq} Y$.

1.3. Throughout this paper, all rings are assumed to be commutative Noetherian. The category of complexes of R-modules is denoted $\mathcal{C}(R)$, and we use subscripts $\sqsupset$, $\sqsubset$, and $\square$ to denote boundedness conditions, and use subscripts $(\sqsupset)$, $(\sqsubset)$, and $(\square)$ to denote homologically boundedness conditions. For example, $\mathcal{C}_{\sqsupset}(R)$ and $\mathcal{C}_{(\sqsupset)}(R)$ are the full subcategories of $\mathcal{C}(R)$ of bounded below and homologically bounded below complexes, respectively. Superscript "(f)" signifies that the homology is degree-wise finitely generated. Thus, $\mathcal{C}^{(f)}_{(\sqsupset)}$ denotes the full subcategory of $\mathcal{C}(R)$ of homologically bounded below complexes with finitely generated homology modules.

1.4. A *projective* (resp., *flat*) *resolution* of $X \in \mathcal{C}_{(\sqsupset)}(R)$ is a bounded below complex P of projective (resp., flat) R-modules such that $P \simeq X$, and an *injective resolution* of a complex $Y \in \mathcal{C}_{(\sqsubset)}(R)$ is a bounded above complex I of injective R-modules such that $Y \simeq I$. The *projective*, *flat* and *injective dimensions* are defined as follows:

$$\mathrm{pd}_R\, X = \inf\big\{\sup\{l \in \mathbb{Z} \mid P_l \neq 0\} \mid P \text{ is a projective resolution of } X\big\},$$

$$\mathrm{fd}_R\, X = \inf\big\{\sup\{l \in \mathbb{Z} \mid F_l \neq 0\} \mid F \text{ is a flat resolution of } X\big\},$$

and

$$\mathrm{id}_R\, Y = \inf\big\{-\inf\{l \in \mathbb{Z} \mid I_l \neq 0\} \mid I \text{ is an injective resolution of } X\big\}.$$

We use $\mathcal{P}(R)$ (resp., $\mathcal{F}(R)$, $\mathcal{I}(R)$) to denote the full subcategory of $\mathcal{C}_{(\square)}(R)$ of complexes of finite projective (resp., flat, injective) dimension, and use $\mathcal{P}_0(R)$ (resp., $\mathcal{F}_0(R)$, $\mathcal{I}_0(R)$) to denote the full subcategory of R-modules of finite projective (resp., flat, injective) dimension.

We use the standard notations $\mathbf{R}\mathrm{Hom}_R(-,-)$ and $-\otimes^{\mathbf{L}}_R -$ for the derived Hom and derived tensor product of complexes; they are computed by way of the resolutions defined above.

The next two results can be found in [1] ((4.1)) and [2] ((A.4.21) and (A.4.24)).

Lemma 1.1. *Let $\varphi\colon R \longrightarrow S$ be a homomorphism of rings, and let $Z \in \mathcal{C}(S)$ and $X \in \mathcal{C}(R)$. Then*

$$\mathrm{id}_S(\mathbf{R}\mathrm{Hom}_R(Z, Y)) \leq \mathrm{fd}_S(Y) + \mathrm{id}_R(Y).$$

Lemma 1.2. *Assume that* $\varphi\colon R \longrightarrow S$ *is a homomorphism of rings. Then the following statements hold*:

(1) *Let* $Z \in \mathcal{C}_{(\sqsupset)}(S)$, $Y \in \mathcal{C}(S)$ *and* $X \in \mathcal{C}_{(\sqsupset)}(R)$. *Then*

$$\mathbf{R}\operatorname{Hom}_R(Z \otimes^{\mathbf{L}}_R Y, X) = \mathbf{R}\operatorname{Hom}_S(Z, \mathbf{R}\operatorname{Hom}_R(Y, X)).$$

(2) *Let* $Z \in \mathcal{C}^{(f)}_{(\sqsupset)}(S)$, $Y \in \mathcal{C}_{(\square)}(S)$ *and* $X \in \mathcal{C}_{(\sqsubset)}(R)$ *with* $\operatorname{id}_R X < \infty$. *Then*

$$Z \otimes^{\mathbf{L}}_R \mathbf{R}\operatorname{Hom}_R(Y, X) = \mathbf{R}\operatorname{Hom}_R(\mathbf{R}\operatorname{Hom}_S(Z, Y), X).$$

1.5. Following from [3], the *restricted flat dimension*, $\operatorname{Rfd}_R X$, of $X \in \mathcal{C}_{(\sqsupset)}(R)$ is defined as

$$\operatorname{Rfd}_R X = \sup\left\{ \sup(T \otimes^{\mathbf{L}}_R X) \mid T \in \mathcal{F}_0(R)\right\}.$$

There are inequalities $\sup X \leq \operatorname{Rfd}_R X \leq \sup X + \dim R$. In particular, $\operatorname{Rfd}_R X = -\infty$ if and only if $X \simeq 0$, and if $\dim R < \infty$ then $\operatorname{Rfd}_R X < \infty$ if and only if $X \in \mathcal{C}_{(\square)}(R)$.

2. Restricted projective modules and restricted projective dimension. We say that an R-module P is *restricted projective* if $\operatorname{Ext}^i_R(P, T) = 0$ for any R-module T of finite injective dimension and any $i > 0$.

Lemma 2.1. *If* $P \in \mathcal{C}_{\sqsupset}(R)$ *is an exact complex of restricted projective R-modules and* $I \in \mathcal{C}_{\square}(R)$ *is a complex of R-modules in* $\mathcal{I}_0(R)$, *then* $\operatorname{Hom}_R(P, I)$ *is a exact complex.*

Proof. We may assume that I is non-zero, and let $s = \sup\{i \in \mathbb{Z} \mid I_i \neq 0\}$. We proceed by induction on s. Without loss of generality, we assume that $P_l = 0$ and $I_l = 0$ for $l < 0$.

If $s = 0$ then $I \in \mathcal{I}_0(R)$. Note that $P \in \mathcal{C}_{\sqsupset}(R)$ is exact and $\operatorname{Ext}^i_R(P_l, I) = 0$ for all $i > 0$ and $l \in \mathbb{Z}$. One can check, by "Dimension Shift", that $\operatorname{Hom}_R(P, I)$ is exact.

Let $s > 0$ and assume that $\operatorname{Hom}(P, I)$ is exact for any complex $I \in \mathcal{C}_{\square}(R)$ of R-modules in $\mathcal{I}_0(R)$ with $\sup\{i \in \mathbb{Z} \mid I_i \neq 0\} \leq s - 1$. Consider the degree-wise split exact sequence

$$0 \longrightarrow I_{\leq s-1} \longrightarrow I \longrightarrow \Sigma^s I_s \longrightarrow 0$$

of complexes, then it stays exact after application of $\operatorname{Hom}_R(P, -)$. The complex $\operatorname{Hom}_R(P, I_s)$ and $\operatorname{Hom}_R(P, I_{\leq s-1})$ are exact by the induction base and hypothesis, respectively. Thus $\operatorname{Hom}_R(P, I)$ is exact.

Lemma 2.2. *If* $X \simeq P$ *and* $U \simeq I$, *where* $P \in \mathcal{C}_{\sqsupset}(R)$ *is a complex of restricted projective R-modules and* $I \in \mathcal{C}_{\square}(R)$ *is a complex of R-modules in* $\mathcal{I}_0(R)$, *then* $\mathbf{R}\operatorname{Hom}_R(X, U)$ *is represented by* $\operatorname{Hom}_R(P, I)$.

Proof. Take a projective resolution $X \xleftarrow{\simeq} Q \in \mathcal{C}_{\sqsupset}(R)$, then $\mathbf{R}\operatorname{Hom}_R(X, U)$ is represented by $\operatorname{Hom}_R(Q, U)$. Since $Q \simeq P$, there exists a quasiisomorphism $\alpha\colon Q \longrightarrow P$ by [1, (1.4.P)], and hence we have a morphism

$$\operatorname{Hom}_R(\alpha, I)\colon \operatorname{Hom}_R(P, I) \longrightarrow \operatorname{Hom}_R(Q, I).$$

Since $\operatorname{Cone}(\alpha) \in \mathcal{C}_{\sqsupset}(R)$ is an exact complex of restricted projective R-modules, we have $\operatorname{Cone}(\operatorname{Hom}_R(\alpha, I)) \cong \Sigma^1 \operatorname{Hom}_R(\operatorname{Cone}(\alpha), I)$ is exact by Lemma 2.1, and hence $\operatorname{Hom}_R(\alpha, I)$ is a quasiisomorphism. Thus $\operatorname{Hom}_R(Q, U) \simeq \operatorname{Hom}_R(P, I)$. This implies that $\mathbf{R}\operatorname{Hom}_R(X, U)$ is represented by $\operatorname{Hom}_R(P, I)$.

Lemma 2.3. *Let $P \in \mathcal{C}_{\sqsupset}(R)$ be a complex of restricted projective R-modules and $T \in \mathcal{I}_0(R)$, and let X be a complex of R-modules such that $\sup X \leq n < \infty$ and $X \simeq P$. Then, for any $i > 0$, we have*

$$\mathrm{Ext}_R^i(\mathrm{C}_n(P), T) = \mathrm{H}_{-(i+n)}(\mathbf{R}\mathrm{Hom}_R(X, T)).$$

Proof. Since $\sup P = \sup X \leq n$, we have $P_{\geq n} \simeq \Sigma^n \mathrm{C}_n(P)$, and hence $\mathrm{C}_n(P) \simeq \Sigma^{-n} P_{\geq n}$. Thus by Lemma 2.2, for each $i > 0$, we have

$$\mathrm{Ext}_R^i(\mathrm{C}_n(P), T) = \mathrm{H}_{-i}(\mathbf{R}\mathrm{Hom}_R(\mathrm{C}_n(P), T)) =$$

$$= \mathrm{H}_{-i}(\mathrm{Hom}_R(\Sigma^{-n} P_{\geq n}, T)) = \mathrm{H}_{-i}(\Sigma^n \mathrm{Hom}_R(P_{\geq n}, T)) =$$

$$= \mathrm{H}_{-(i+n)}(\mathrm{Hom}_R(P_{\geq n}, T)) = \mathrm{H}_{-(i+n)}(\mathrm{Hom}_R(P, T)) =$$

$$= \mathrm{H}_{-(i+n)}(\mathbf{R}\mathrm{Hom}_R(X, T)).$$

Following from [3], the *restricted projective dimension*, $\mathrm{Rpd}_R X$, of $X \in \mathcal{C}_{(\sqsupset)}(R)$ is defined as

$$\mathrm{Rpd}_R X = \sup\left\{-\inf(\mathbf{R}\mathrm{Hom}_R(X, T)) \mid T \in \mathcal{I}_0(R)\right\}.$$

It can be checked easily that, for $X \in \mathcal{C}_{(\sqsupset)}(R)$, $\mathrm{Rfd}_R X \leq \mathrm{Rpd}_R X$, and there are inequalities

$$\sup X \leq \mathrm{Rpd}_R X \leq \sup X + \dim R.$$

In particular, $\mathrm{Rpd}_R X = -\infty$ if and only if $X \simeq 0$, and if $\dim R < \infty$ then $\mathrm{Rpd}_R X < \infty$ if and only if $X \in \mathcal{C}_{(\sqsubset\sqsupset)}(R)$.

The next theorem gives some new characterizations of the restricted projective dimension of complexes.

Theorem 2.1. *Let $X \in \mathcal{C}_{(\sqsupset)}(R)$ and $n \in \mathbb{Z}$. Consider the following conditions*:

(1) $\mathrm{Rpd}_R X \leq n$.

(2) *X is equivalent to a bounded complex P of restricted projective R-modules with* $\sup\{i \in \in \mathbb{Z} \mid P_i \neq 0\} \leq n$; *and P can be chosen such that $P_l = 0$ for $l < \inf X$.*

(3) $\mathrm{H}_i(\mathbf{R}\mathrm{Hom}_R(X, T)) = 0$ *for any $i < -n$ and any $T \in \mathcal{I}_0(R)$.*

(4) $\sup X \leq n$ *and $\mathrm{C}_n(P)$ is a restricted projective R-module whenever P is a bounded below complex of restricted projective R-modules which is equivalent to X.*

(5) $-\inf(\mathbf{R}\mathrm{Hom}(X, U)) + \inf U \leq n$ *for any nonexact complex $U \in \mathcal{I}(R)$.*

Then we have (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Leftarrow$ (5). *If, furthermore, $X \in \mathcal{C}^{(f)}_{(\sqsupset)}(R)$, then all the above statements are equivalent.*

Proof. (1) $\Rightarrow$ (4). Obviously, $\sup X \leq \mathrm{Rpd}_R X \leq n$. Let $P \in \mathcal{C}_{\sqsupset}(R)$ be a complex of restricted projective R-modules such that $P \simeq X$, and let $T \in \mathcal{I}_0(R)$. Then, by Lemma 2.3, $\mathrm{Ext}_R^i(\mathrm{C}_n(P), T) = \mathrm{H}_{-(i+n)}(\mathbf{R}\mathrm{Hom}_R(X, T)) = 0$ for any $i > 0$ since $\mathrm{Rpd}_R X \leq n$, and so $\mathrm{C}_n(P)$ is an restricted projective R-module.

(4) $\Rightarrow$ (2). Take a projective resolution $X \xleftarrow{\simeq} P \in \mathcal{C}_{\sqsupset}(R)$ of X with $P_l = 0$ for $l < \inf X$. Since $\sup P = \sup X \leq n$, we have $X \simeq P \simeq P_{\subset n}$. Obviously, $P_{\subset n}$ is a bounded complex of restricted projective R-modules.

(2) $\Rightarrow$ (3). Let $T \in \mathcal{I}_0(R)$. By Lemma 2.2, we have $\mathrm{H}_i(\mathbf{R}\mathrm{Hom}_R(X, T)) = \mathrm{H}_i(\mathrm{Hom}_R(P, T))$. For $i < -n$, $\mathrm{Hom}_R(P, T)_i = \prod_{t \in \mathbb{Z}} \mathrm{Hom}_R(P_t, T_{t+i}) = 0$ since $P_t = 0$ for $t > n$, and so $\mathrm{H}_i(\mathbf{R}\mathrm{Hom}_R(X, T)) = 0$.

(3) $\Rightarrow$ (1) and (5) $\Rightarrow$ (3) are trivial.

Finally, we let $X \in \mathcal{C}^{(f)}_{(\sqsupset)}(R)$, and let E be a faithfully injective R-module. Then, for any non-exact complex $U \in \mathcal{I}(R)$, we have

$$-\inf(\mathbf{R}\operatorname{Hom}_R(X,U)) = \sup(\mathbf{R}\operatorname{Hom}_R(\mathbf{R}\operatorname{Hom}_R(X,U),E)) =$$

$$= \sup(X \otimes^{\mathbf{L}}_R \mathbf{R}\operatorname{Hom}_R(U,E)) \leq \operatorname{Rfd}_R X + \sup(\mathbf{R}\operatorname{Hom}_R(U,E)) \leq$$

$$\leq \operatorname{Rpd}_R X - \inf U,$$

where the second equality holds by Lemma 1.2(2), the third inequality by [3] (Theorem 2.4(a)) and the last by [3] (Lemma 5.6). Thus the implication (1) $\Rightarrow$ (5) holds.

Recall, from [5], that an R-module M is *strongly torsion free* if $\operatorname{Tor}^R_1(T,M) = 0$ for any $T \in \mathcal{F}_0(R)$. One can check easily that M is strongly torsion free if and only if $\operatorname{Tor}^R_i(T,M) = 0$ for any $T \in \mathcal{F}_0(R)$ and any $i > 0$. Using a similar method as proved in Theorem 2.1, we get the next result.

Proposition 2.1. *Let $X \in \mathcal{C}_{(\sqsupset)}(R)$ and $n \in \mathbb{Z}$. Then the following statements are equivalent*:

(1) $\operatorname{Rfd}_R X \leq n$.

(2) *X is equivalent to a bounded complex F of strongly torsion free R-modules with* $\sup\{i \in \in \mathbb{Z} \mid F_i \neq 0\} \leq n$; *and F can be chosen such that $F_l = 0$ for $l < \inf X$.*

(3) $\operatorname{H}_i(T \otimes^{\mathbf{L}}_R X) = 0$ *for any $i > n$ and any $T \in \mathcal{F}_0(R)$.*

(4) $\sup X \leq n$ *and $\operatorname{C}_n(F)$ is a strongly torsion free R-module whenever F is a bounded below complex of strongly torsion free R-modules which is equivalent to X.*

Let $X \in \mathcal{C}_{(\sqsupset)}(R)$. We say that P is a *restricted projective resolution* of X if P is a bounded below complex of restricted projective R-modules such that $P \simeq X$. A restricted projective resolution of an R-module M is a sequence

$$\ldots \longrightarrow P_l \longrightarrow \ldots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow 0$$

of restricted projective R-modules which is exact at P_i for $i > 0$ and satisfies

$$P_0/\operatorname{Im}(P_1 \longrightarrow P_0) \cong M.$$

That is, the sequence

$$\ldots \longrightarrow P_l \longrightarrow \ldots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

is exact.

The next two corollaries are immediate by Theorem 2.1.

***Corollary* 2.1.** *If $X \in \mathcal{C}_{(\sqsupset)}(R)$, then*

$$\operatorname{Rpd}_R X = \inf\big\{\sup\{l \in \mathbb{Z} \mid P_l \neq 0\} \mid P \text{ is a restricted projective resolution of } X\big\}.$$

***Corollary* 2.2.** *If $X \in \mathcal{C}^{(f)}_{(\sqsupset)}(R)$, then*

$$\operatorname{Rpd}_R X = \sup\big\{\inf U - \inf(\mathbf{R}\operatorname{Hom}_R(X,U)) \mid U \in \mathcal{I}(R) \wedge U \not\simeq 0\big\}.$$

In particular, $-\inf(\mathbf{R}\operatorname{Hom}_R(X,U)) \leq \operatorname{Rpd}_R X - \inf U$ *for any $X \in \mathcal{C}^{(f)}_{(\sqsupset)}(R)$ and any $U \in \mathcal{I}(R)$.*

The next lemma can be proved easily.

Lemma 2.4. *Let* $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ *be an exact sequence of* R*-modules. Then the following statements hold*:

(1) *If* M'' *is restricted projective, then* M *is restricted projective if and only if* M' *is so.*

(2) *If the sequence splits, then* M *is restricted projective if and only if both* M' *and* M'' *are so.*

Lemma 2.5. *Let* M *be an* R*-module, and let* $P \in \mathcal{C}_{\sqsupset}(R)$ *be a complex of restricted projective* R*-modules such that* $P \simeq M$. *Then the soft truncated complex* $P_{\supset 0}$ *is a restricted projective resolution of* M.

Proof. Since $P \simeq M$, we have $\inf P = 0$, and hence $P_{\supset 0} \simeq P \simeq M$. Thus we have an exact sequence

$$\ldots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow \mathrm{Z}_0(P) \longrightarrow M \longrightarrow 0$$

of R-modules. In the following we show that $\mathrm{Z}_0(P)$ is restricted projective. Let $i = \inf\{l \in \mathbb{Z} \mid P_l \neq \neq 0\}$, then the sequence

$$0 \longrightarrow \mathrm{Z}_0(P) \longrightarrow P_0 \longrightarrow \ldots \longrightarrow P_{i+1} \longrightarrow P_i \longrightarrow 0$$

of R-modules is exact, and so $Z_0(P)$ is restricted projective by Lemma 2.4.

***Corollary* 2.3.** *Let* $M \neq 0$ *be an* R*-module. Then* M *is restricted projective if and only if* $\mathrm{Rpd}_R\, M = 0$.

Proof. Immediately by Corollary 2.1 and Lemma 2.5.

***Corollary* 2.4.** *Let* M *be an* R*-module and* $n \in \mathbb{N}_0$. *Then the following statements are equivalent*:

(1) $\mathrm{Rpd}_R\, M \leq n$.

(2) *There is an exact sequence* $0 \longrightarrow P_n \longrightarrow \ldots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$ *of* R*-modules with* P_i *restricted projective for each* $0 \leq i \leq n$.

(3) $\mathrm{Ext}^i_R(M, T) = 0$ *for any* $i > n$ *and any* $T \in \mathcal{I}_0(R)$.

(4) *For any restricted projective resolution* $\ldots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$ *of* M, $K_n = = \mathrm{Ker}(P_{n-1} \longrightarrow P_{n-2})$ *is a restricted projective* R*-module, where* $K_0 = M$ *and* $K_1 = = \mathrm{Ker}(P_0 \longrightarrow M)$.

Proof. We notice that if the sequence $\ldots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$ is exact, then M is equivalent to the complex $P = \ldots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow 0$, and $\mathrm{C}_0(P) \cong M$, $\mathrm{C}_1(P) \cong \cong \mathrm{Ker}(P_0 \longrightarrow M)$ and $\mathrm{C}_l(P) \cong \mathrm{Z}_{l-1}(P) = \mathrm{Ker}(P_{l-1} \longrightarrow P_{l-2})$ for $l \geq 2$. In view of Lemma 2.5, the equivalence of the four conditions now follows from Theorem 2.1.

Similarly, by Proposition 2.1, we get the following result.

***Corollary* 2.5.** *Let* M *be an* R*-module and* $n \in \mathbb{N}_0$. *Then the following statements are equivalent*:

(1) $\mathrm{Rfd}_R\, M \leq n$.

(2) *There is an exact sequence* $0 \longrightarrow F_n \longrightarrow \ldots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$ *of* R*-modules with* F_i *strongly torsion free for each* $0 \leq i \leq n$.

(3) $\mathrm{Tor}^R_i(T, M) = 0$ *for all* $i > n$ *and all* $T \in \mathcal{F}_0(R)$.

(4) *For strongly torsion free resolution* $\ldots \longrightarrow F_l \longrightarrow \ldots \longrightarrow F_0 \longrightarrow M \longrightarrow 0$ *of* M, $K_n = = \mathrm{Ker}(F_{n-1} \longrightarrow F_{n-2})$ *is a strongly torsion free* R*-module, where* $K_0 = M$ *and* $K_1 = = \mathrm{Ker}(F_0 \longrightarrow M)$.

Recall that a finite R-module M belongs to the *G-class* $\mathrm{G}(R)$ if $\mathrm{Ext}^i_R(M,R) = 0 = \mathrm{Ext}^i_R(\mathrm{Hom}_R(M,R),R)$ for $i > 0$ and the biduality map

$$\delta_M \colon M \longrightarrow \mathrm{Hom}_R(\mathrm{Hom}_R(M,R),R),$$

defined by $\delta_M(x)(\psi) = \psi(x)$ for $\psi \in \mathrm{Hom}_R(M,R)$ and $x \in M$, is an isomorphism. A complex G is said to be a *G-resolution* of $X \in \mathcal{C}^{(f)}_{(\sqsupset)}(R)$ if G is a bounded below complex of R-modules in $\mathrm{G}(R)$ such that $G \simeq X$. The *G-dimension*, $\text{G-dim}_R X$, of X is defined as

$$\text{G-dim}_R X = \inf\big\{\sup\{l \in \mathbb{Z} \mid G_l \neq 0\} \mid G \text{ is a G-resolution of } X\big\}.$$

The next lemma shows that the restricted projective dimension is a refinement of the G-dimension.

Lemma 2.6. *If* $X \in \mathcal{C}^{(f)}_{(\sqsupset)}(R)$, *then* $\mathrm{Rpd}_R X \leq \text{G-dim}_R X$, *and the equality hold if* $\text{G-dim}_R X < \infty$.

Proof. If $\text{G-dim}_R X = \infty$, then the inequality is trivial. If $\text{G-dim}_R X < \infty$, then, by [2] ((2.4.7)), $\text{G-dim}_R X = \sup\{-\inf(\mathbf{R}\mathrm{Hom}_R(X,T)) \mid T \in \mathcal{I}_0(R)\} = \mathrm{Rpd}_R X$.

By [3] ((5.17)), $\mathrm{Rpd}_R X \leq \mathrm{pd}_R X$ for any $X \in \mathcal{C}_{(\sqsupset)}(R)$, and if R is local, $X \in \mathcal{C}^{(f)}_{(\sqsupset)}(R)$ and $\mathrm{pd}_R X < \infty$, then $\mathrm{Rpd}_R X = \mathrm{pd}_R X$. In the following we see that the condition "R is local" is superfluous.

Proposition 2.2. *If* $X \in \mathcal{C}_{(\sqsupset)}(R)$, *then* $\mathrm{Rpd}_R X \leq \mathrm{pd}_R X$, *and the equality hold if* $X \in \mathcal{C}^{(f)}_{(\sqsupset)}(R)$ *and* $\mathrm{pd}_R X < \infty$.

Proof. Note that $\text{G-dim}_R X \leq \mathrm{pd}_R X$ for $X \in \mathcal{C}^{(f)}_{(\sqsupset)}(R)$ and the equality holds if $\mathrm{pd}_R X < \infty$ (see [2] (2.3.10)), then we get the desired result by Lemma 2.6.

A complex P is said to be a *Gorenstein projective resolution* of $X \in \mathcal{C}_{(\sqsupset)}(R)$, if P is a bounded below complex of Gorenstein projective R-modules such that $P \simeq X$. The *Gorenstein projective dimension*, $\mathrm{Gpd}_R X$, of X is defined as

$$\mathrm{Gpd}_R X = \inf\big\{\sup\{l \in \mathbb{Z} \mid P_l \neq 0\} \mid P \text{ is a Gorenstein projective resolution of } X\big\}.$$

Proposition 2.3. *If R is a Gorenstein local ring, then* $\mathrm{Rpd}_R X = \mathrm{Gpd}_R X$ *for any* $X \in \mathcal{C}_{(\sqsupset)}(R)$.

Proof. We first prove $\mathrm{Rpd}_R X \leq \mathrm{Gpd}_R X$. If $\mathrm{Gpd}_R X = \infty$ then the inequality is trivial. Now we assume that $\mathrm{Gpd}_R X < \infty$, then we have

$$\mathrm{Gpd}_R X = \sup\{-\inf(\mathbf{R}\mathrm{Hom}_R(X,T)) \mid T \in \mathcal{F}_0(R)\} =$$

$$= \sup\{-\inf(\mathbf{R}\mathrm{Hom}_R(X,T)) \mid T \in \mathcal{I}_0(R)\} = \mathrm{Rpd}_R X,$$

where the first equality holds by [2] ((4.4.5)), and the second by [2] ((3.3.4)).

Next we show that $\mathrm{Gpd}_R X \leq \mathrm{Rpd}_R X$. If $\mathrm{Rpd}_R X = \infty$ then the inequality is trivial. Now we assume that $\mathrm{Rpd}_R X < \infty$, then $X \in \mathcal{C}_{(\sqsupset\sqsubset)}(R)$. Thus $\mathrm{Gpd}_R X < \infty$ by [2] ((4.4.8)), and so $\mathrm{Gpd}_R X = \mathrm{Rpd}_R X$ as proved above.

Proposition 2.4. *Let* $\varphi\colon R \longrightarrow S$ *be a homomorphism of rings,* $X \in \mathcal{C}^{(f)}_{(\sqsupset)}(S)$ *and* $Y \in \mathcal{F}(R)$. *Then we have the following inequalities*:

(1) $\mathrm{Rpd}_R(X \otimes^{\mathbf{L}}_R Y) \leq \mathrm{Rpd}_S X + \mathrm{Rpd}_R Y + \mathrm{Rpd}_R S$.

(2) $\mathrm{Rpd}_S(X \otimes^{\mathbf{L}}_R Y) \leq \mathrm{Rpd}_S X + \mathrm{Rfd}_R S + \sup Y + \dim S$.

Proof. (1) Choose $T \in \mathcal{I}_0(R)$ such that

$$\operatorname{Rpd}_R(X \otimes^{\mathbf{L}}_R Y) = -\inf(\mathbf{R}\operatorname{Hom}_R(X \otimes^{\mathbf{L}}_R Y, T)) =$$

$$= -\inf(\mathbf{R}\operatorname{Hom}_R((X \otimes^{\mathbf{L}}_S S) \otimes^{\mathbf{L}}_R Y, T)) =$$

$$= -\inf(\mathbf{R}\operatorname{Hom}_R(X \otimes^{\mathbf{L}}_S (S \otimes^{\mathbf{L}}_R Y), T)) =$$

$$= -\inf(\mathbf{R}\operatorname{Hom}_S(X, \mathbf{R}\operatorname{Hom}_R(S \otimes^{\mathbf{L}}_R Y, T))) \leq$$

$$\leq \operatorname{Rpd}_S X - \inf(\mathbf{R}\operatorname{Hom}_R(S \otimes^{\mathbf{L}}_R Y, T)) =$$

$$= \operatorname{Rpd}_S X - \inf(\mathbf{R}\operatorname{Hom}_R(S, \mathbf{R}\operatorname{Hom}_R(Y, T))) \leq$$

$$\leq \operatorname{Rpd}_S X + \operatorname{Rpd}_R S - \inf(\mathbf{R}\operatorname{Hom}_R(Y, T)) \leq$$

$$\leq \operatorname{Rpd}_S X + \operatorname{Rpd}_R S + \operatorname{Rpd}_R Y.$$

Where the fourth equality holds by Lemma 1.2(1). Since

$$\operatorname{id}_S\left(\mathbf{R}\operatorname{Hom}_R(S \otimes^{\mathbf{L}}_R Y, T)\right) \leq \operatorname{fd}_S(S \otimes^{\mathbf{L}}_R Y) + \operatorname{id}_R T \leq \operatorname{fd}_R Y + \operatorname{id}_R T < \infty$$

by [1] ((4.1)), the fifth inequality follows from Corollary 2.2. The sixth equality comes from Lemma 1.2(1), and the seventh inequality holds by Corollary 2.2 since $\operatorname{id}_R(\mathbf{R}\operatorname{Hom}_R(Y, T)) \leq \leq \operatorname{fd}_R Y + \operatorname{id}_R T < \infty$ by Lemma 1.1.

(2) Choose $T \in \mathcal{I}_0(S)$ such that

$$\operatorname{Rpd}_S(X \otimes^{\mathbf{L}}_R Y) = -\inf(\mathbf{R}\operatorname{Hom}_S(X \otimes^{\mathbf{L}}_R Y, T)) =$$

$$= -\inf(\mathbf{R}\operatorname{Hom}_S((X \otimes^{\mathbf{L}}_S (S \otimes^{\mathbf{L}}_R Y), T)) =$$

$$= -\inf(\mathbf{R}\operatorname{Hom}_S(X, \mathbf{R}\operatorname{Hom}_S(S \otimes^{\mathbf{L}}_R Y, T))) \leq$$

$$\leq \operatorname{Rpd}_S X - \inf(\mathbf{R}\operatorname{Hom}_S(S \otimes^{\mathbf{L}}_R Y, T)) \leq$$

$$\leq \operatorname{Rpd}_S X + \sup(Y \otimes^{\mathbf{L}}_R S) + \operatorname{id}_S T \leq$$

$$\leq \operatorname{Rpd}_S X + \operatorname{Rfd}_R S + \sup Y + \dim S,$$

where the third equality holds by Lemma 1.2(1), the fourth inequality by Corollary 2.2 since $\operatorname{id}_S(\mathbf{R}\operatorname{Hom}_S(S \otimes^{\mathbf{L}}_R Y, T)) \leq \operatorname{fd}_S(S \otimes^{\mathbf{L}}_R Y) + \operatorname{id}_S T \leq \operatorname{fd}_R Y + \operatorname{id}_S T < \infty$ by Lemma 1.1, the fifth by [2] ((A.5.2)), and the last by [3] ((2.4(1))).

Proposition 2.5. *Let $\varphi\colon R \longrightarrow S$ be a homomorphism of rings, $X \in \mathcal{C}^{(f)}_{(\sqsupset)}(S)$ and $Y \in \mathcal{F}(S)$. Then*

$$\operatorname{Rpd}_R(X \otimes^{\mathbf{L}}_S Y) \leq \operatorname{Rpd}_S X + \operatorname{Rpd}_R Y.$$

Proof. Choose $T \in \mathcal{I}_0(R)$ such that

$$\operatorname{Rpd}_R(X \otimes_S^{\mathbf{L}} Y) = -\inf(\mathbf{R}\operatorname{Hom}_R(X \otimes_S^{\mathbf{L}} Y, T)) =$$

$$= -\inf(\mathbf{R}\operatorname{Hom}_S(X, \mathbf{R}\operatorname{Hom}_R(Y, T))) \leq$$

$$\leq \operatorname{Rpd}_S X - \inf(\mathbf{R}\operatorname{Hom}_R(Y, T)) \leq \operatorname{Rpd}_S X + \operatorname{Rpd}_R Y,$$

where the second equality holds by Lemma 1.2(1), and the third inequality by Corollary 2.2 since $\operatorname{id}_S(\mathbf{R}\operatorname{Hom}_R(Y, T)) \leq \operatorname{fd}_S Y + \operatorname{id}_R T < \infty$ by Lemma 1.1.

***Corollary* 2.6.** *Let $\varphi\colon R \longrightarrow S$ be a homomorphism of rings and $X \in \mathcal{C}_{(\sqsupset)}^{(f)}(S)$. Then*

$$\operatorname{Rpd}_R X \leq \operatorname{Rpd}_S X + \operatorname{Rpd}_R S.$$

Proof. Immediately by Proposition 2.4(1) or 2.5.

1. *Avramov L. L., Foxby H.-B.* Homological dimensions of unbounded complexes // J. Pure and Appl. Algebra. – 1991. – **71**. – P. 129 – 155.
2. *Christensen L. W.* Gorenstein dimensions // Lect. Notes Math. – 2000. – **1747**.
3. *Christensen L. W., Foxby H.-B., Frankild A.* Restricted homological dimensions and Cohen – Macaulayness // J. Algebra. – 2002. – **251**. – P. 479 – 502.
4. *Sharif T., Yassemi S.* Depth formulas, restricted Tor-dimension under base change // Rocky Mountain J. Math. – 2004. – **34**. – P. 1131 – 1146.
5. *Xu J.* Flat covers of modules // Lect. Notes Math. – 1996. – **1634**.

Received 28.05.12