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ON SUPPLEMENT SUBMODULES

ПРО ДОПОВНЮЮЧІ ПІДМОДУЛІ

We investigate some properties of supplement submodules. Some relations between lying-above and weak supplement submodules are also studied. Let V be a supplement of a submodule U in M. Then it is possible to define a bijective map between maximal submodules of V and maximal submodules of M that contain U. Let M be an R-module, $U \leq M$, V be a weak supplement of U, and $K \leq V$. In this case, K is a weak supplement of U if and only if V lies above K in M. We prove that an R-module M is amply supplemented if and only if every submodule of M lies above a supplement in M. We also prove that M is semisimple if and only if every submodule of M is a supplement in M.

Досліджено деякі властивості доповнюючих підмодулів. Також вивчено деякі співвідношення між вищерозміщеними та слабкими доповнюючими підмодулями. Нехай V — доповнення підмодуля U в M. Тоді можна означити бієкцію між максимальними підмодулями V та максимальними підмодулями M, що містять U. Нехай M — R-модуль, $U \leq M$, V — слабке доповнення U і $K \leq V$. У цьому випадку K є слабким доповненням U тоді і тільки тоді, коли V лежить вище K у M. Доведено, що R-модуль M є достатньо доповненим тоді і тільки тоді, коли кожен підмодуль модуля M лежить вище доповнення в M. Також доведено, що M є напівпростим тоді і тільки тоді, коли кожен підмодуль модуля M є доповненням у M.

1. Introduction. Throughout this paper R will be an arbitrary ring with identity and all modules are unital left R-modules. Let M be an R-module and V be a submodule of M. If $L = M$ for every submodule L of M such that $V + L = M$ then V is called a *small submodule* of M and written by $V << M$. In this work $\mathrm{Rad}(M)$ will denote the intersection of all maximal submodules of M. If M has no maximal submodule then we define $\mathrm{Rad}(M) = M$. Let M be an R-module. $N \leq M$ will mean N is a submodule of M.

Lemma 1.1 (Modular law)**.** *Let M be an R-module, K, N and H be submodules of M and $H \leq N$. Then $N \cap (H + K) = H + N \cap K$* (see [3]).

Let U be a submodule of $MR-$module. If a submodule V is minimal in the collection of submodules L of M such that $U + L = M$ then V is called *a supplement of U by addition* or simply *a supplement* of U in M. In this case $U + V = M$ is clear. Let V be a supplement of U in M. Then $K = V$ for every $K \leq V$ such that $U + K = M$. The modules whose every submodules have supplements are called *supplemented modules*. If every submodule of the R-module M has at least one supplement that is a direct summand in M, then M is called $\oplus$*-supplemented*. A submodule V of M is called *supplement in M* if V is a supplement of a submodule in M.

We say a submodule U of the R-module M has *ample supplements* in M if for every $V \leq M$ with $U + V = M$, there exists a supplement V' of U with $V' \leq V$. If every submodule of M has ample supplements in M, then we call M *amply supplemented*.

2. Properties of supplement submodules.

Lemma 2.1. *A submodule V of M is a supplement of a submodule U in M if and only if $U + V = M$ and $U \cap V << V$* (see [14]).

Lemma 2.2. *Let $M = U + V$. If a submodule K is a proper submodule of M which contains U and distinct from U, then $K \cap V$ is a proper submodule of V.*

Proof. Because of $U \leq K$, $M = U + V$ and $M \neq K$, then $V \not\subset K$ and $V \cap K \neq V$. By $K = M \cap K = (U + V) \cap K = U + V \cap K$ and $K \neq U$, then $V \cap K \neq 0$. Hence $K \cap V$ is a proper submodule of V.

Lemma 2.3. *Let V be a supplement of a submodule U in M. If U is a maximal submodule, then V is cyclic and $U \cap V$ is the unique maximal submodule of V. In this case $U \cap V = \mathrm{Rad}(V)$* (see [14]).

Lemma 2.4. *Let M be an R-module, U and V be proper submodules of M. If $M = U + V$ and V is simple, then U is a maximal submodule of M.*

Proof. If K is a submodule which contains U and distinct from M and U, then by Lemma 2.2 $K \cap V$ is a proper submodule of V. This contradicts while V is simple. Hence M have no submodules which contains U and distinct from M and U. Thus U is a maximal submodule of M.

***Corollary* 2.1.** *Let V be a supplement of U in M. Then U is a maximal submodule of M if and only if V or $V/U \cap V$ is simple.*

Lemma 2.5. *Let V be a supplement in M and K be a submodule of V. Then $K << M$ if and only if $K << V$* (see [4]).

The following lemma is in [4] (Exercise 20.39). We prove this lemma as follows.

Lemma 2.6. *Let V be a supplement of U in M, K and T be submodules of V. Then T is a supplement of K in V if and only if T is a supplement of $U + K$ in M.*

Proof. ($\Rightarrow$) Let T be a supplement of K in V. Let $U + K + L = M$ for $L \leq T$. Then $K + L \leq V$ and because V is a supplement of U, $K + L = V$. Since $L \leq T$ and T is a supplement of K in V, $L = T$. Hence T is a supplement of $U + K$ in M.

($\Leftarrow$) Let T be a supplement of $U + K$ in M. Then by Lemma 2.1 $U + K + T = M$ and $(U + K) \cap T << T$. Since $U + K + T = M$ and $K + T \leq V$, then we can have $K + T = V$. Since $K \cap T \leq (U + K) \cap T << T$, $K \cap T << T$. Then by Lemma 2.1 T is a supplement of K in V.

***Corollary* 2.2.** *Let $M = U \oplus V$, K and T be submodules of V. Then T is a supplement of K in V if and only if T is a supplement of $U + K$ in M.*

***Corollary* 2.3.** *Let U and V be mutual supplements in M, L be a supplement of S in U and T be a supplement of K in V. Then $L + T$ is a supplement of $K + S$ in M.*

Proof. Since $U = S + L$ and V is a supplement of U then by Lemma 2.6 T is a supplement of $S + L + K$ in M and then $(S + L + K) \cap T << T$. Since $V = K + T$ and U is a supplement of V, then by Lemma 2.6 L is a supplement of $S + K + T$ in M and then $(S + K + T) \cap L << L$. Because $U = S + L$, $V = K + T$ and $M = U + V$, then we have $M = S + L + K + T = S + K + L + T$. We can also have $(S + K) \cap (L + T) \leq L \cap (S + K + T) + T \cap (S + K + L) << L + T$. Hence $L + T$ is a supplement of $K + S$ in M.

***Corollary* 2.4.** *Let $M = U \oplus V$, L be a supplement of S in U and T be a supplement of K in V. Then $L + T$ is a supplement of $K + S$ in M.*

Lemma 2.7. *Let V be a supplement of U in M and K be a maximal submodule of V. Then $U + K$ is a maximal submodule of M. In this case $K = (U + K) \cap V$.*

Proof. Because K is a maximal submodule of V, $K \neq V$. Since V is a supplement of U, $U + K \neq M$. Since $U \cap V << V$ and K is a maximal submodule of V, we have $U \cap V \leq K$ and $K = U \cap V + K = (U + K) \cap V$. Then by $M/(U + K) = (U + K + V)/(U + K) \cong V/V \cap (U + K) = = V/K$, we have $M/(U + K)$ is simple and $U + K$ is a maximal submodule of M.

Lemma 2.8. *Let M be an R-module and V be a submodule of M. If K is a maximal submodule of M and $V \not\subset K$, then $V \cap K$ is a maximal submodule of V.*

Proof. Because of $V \not\subset K$, $V \cap K \neq V$. Let $v \in V \setminus (V \cap K)$. Then $v \notin K$ and $K + Rv = M$. We get intersection by V in two side, by using Modular law we have $K \cap V + Rv = V \cap M = V$ and then $V \cap K$ is obtained to be maximal in V.

Theorem 2.1. *Let V be a supplement of a submodule U in M. Then it is possible to define a bijective map between maximal submodules of V and maximal submodules of M which contain U.*

Proof. Let $\Gamma = \{K \mid U \leq K, K \; is \; maximal \; in \; M\}$, $\Lambda = \{T \mid T \; is \; maximal \; in \; V\}$. We can define a map $f : \Gamma \to \Lambda$, $K \to f(K) = K \cap V$. Since $U \leq K$ and K is maximal in M for every $K \in \Gamma$, $V \not\subset K$ and then by Lemma 2.8 $K \cap V$ is a maximal submodule of V. That is, f is a function.

Let $T \in \Lambda$. Since T is maximal in V, then by Lemma 2.7 $U + T \in \Gamma$ and $f(U + T) = (U + T) \cap V = T$. Thus f is surjective.

Let $f(K) = f(L)$ for $K, L \in \Gamma$. Then $K \cap V = L \cap V$. Since $U \leq K$ and $U \leq L$, then by Modular law $K = M \cap K = (U+V) \cap K = U + V \cap K = U + V \cap L = (U+V) \cap L = M \cap L = L$.

Hence f is bijective.

The Theorems 2.2 and 2.3 are in [14]. We prove these theorems by different ways.

Theorem 2.2. *Let U be a submodule which has a supplement in M which is distinct from zero, and* $\mathrm{Rad}(M) << M$. *Then U is contained in a maximal submodule of M.*

Proof. Let V be a supplement of U which distinct from zero in M. If V is contained in all maximal submodules of M, because $U + V = M$, $U + \mathrm{Rad}(M) = M$ and then because $\mathrm{Rad}(M) << M$, we get $U = M$. This contradicts $V \neq 0$. Hence there exists a maximal submodule K of M which doesn't contain V. By Lemma 2.8 $V \cap K$ is a maximal submodule of V. Then by Lemma 2.7 $U + V \cap K$ is a maximal submodule of M which contains U.

Theorem 2.3. *Let V be a supplement submodule in M. Then* $\mathrm{Rad}(V) = V \cap \mathrm{Rad}(M)$.

Proof. Let V be a supplement of U in M. If $V \leq \mathrm{Rad}(M)$, then V has no maximal submodules, because if K were a maximal submodule of V then $U + K$ would be a maximal submodule of M and by $V \leq U + K$, $M = U + V \leq U + K \leq M$ and then $K = V$. Hence if $V \leq \mathrm{Rad}(M)$, then V has no maximal submodules. In this case $\mathrm{Rad}(V) = V = V \cap \mathrm{Rad}(M)$.

Let $V \not\subset \mathrm{Rad}(M)$. This case clearly we can prove that V has at least one maximal submodule. Clearly we can see that $\mathrm{Rad}(V) = \cap\{K \mid K \; is \; maximal \; in \; V\} = \cap\{V \cap (U + K) \mid K \; is \; maximal \; in \; V\} = V \cap [\cap\{(U + K) \mid K \; is \; maximal \; in \; V\}]$. At the end of this equality because $U + K$ is maximal in M (by Lemma 2.7), by definition of $\mathrm{Rad}(M)$, $\mathrm{Rad}(M) = \cap\{N \mid N \; is \; maximal \; in \; M\} \leq \cap\{(U + K) \mid K \; is \; maximal \; in \; V\}$. Thus $V \cap \mathrm{Rad}(M) \leq \mathrm{Rad}(V)$.

At the end of the equality $V \cap \mathrm{Rad}(M) = V \cap [\cap\{N \mid N \; is \; maximal \; in \; M\}] = \cap\{V \cap N \mid N \; is \; maximal \; in \; M\}$, because N is maximal in M, by Lemma 2.8 $V \cap N = V$ or $V \cap N$ is maximal in V. Thus $\mathrm{Rad}(V) \leq V \cap \mathrm{Rad}(M)$. Since $V \cap \mathrm{Rad}(M) \leq \mathrm{Rad}(V)$ and $\mathrm{Rad}(V) \leq V \cap \mathrm{Rad}(M)$, $\mathrm{Rad}(V) = V \cap \mathrm{Rad}(M)$.

A submodule U of M *has a weak supplement* V in M if $U + V = M$ and $U \cap V << M$. M is called *weakly supplemented* if every submodule of M has a weak supplement in M. A submodule V of M is called *weak supplement in* M if V is a weak supplement of a submodule of M.

A submodule L of M is *said to lie above* a submodule N of M if $N \leq L$ and $L/N << M/N$.

Some properties of weakly supplemented modules are investigated in [10]. Some properties of *lying above* are in [11]. We investigate some relations between *lying above* and weak supplement submodules.

Lemma 2.9. *Let L and N are submodules of M and $N \leq L$. Then L lies above N if and only if $N + T = M$ for every submodule T of M such that $L + T = M$.*

Proof. See [4].

Lemma 2.10. *Let $M = U + V$ and $M = T + U \cap V$. Then $M = U + T \cap V = V + T \cap U$.*

Proof. See [4].

Theorem 2.4. *Let $U \leq M$, $L \leq U$ and U lies above L. If U and L have weak supplements in M, then they have the same weak supplements in M.*

Proof. Let V be a weak supplement of U in M. Then $U+V = M$ and by Lemma 2.9 $L+V = M$. Since V is a weak supplement of U and $L \leq U$, $L \cap V \leq U \cap V << M$. Thus V is a weak supplement of L.

Let T be a weak supplement of L in M. Then $L + T = M$ and by $L \leq U$, $U + T = M$. Let $U \cap T + S = M$. Then by Lemma 2.10 $U + T \cap S = M$ and by Lemma 2.9 $L + T \cap S = M$. By also Lemma 2.10 $L \cap T + S = M$ and because $L \cap T << M$, $S = M$. Thus $U \cap T << M$ and T is a weak supplement of U in M.

Theorem 2.5. *Let $U \leq M$, $L \leq U$ and U lies above L. If U and L have supplements in M then they have the same supplements in M.*

Proof. Let V be a supplement of U in M. Then $U + V = M$ and by Lemma 2.9 $L + V = M$. Since V is a supplement of U and $L \leq U$, $L \cap V \leq U \cap V << V$. Thus V is a supplement of L.

Let T be a supplement of L in M. Then $L+T = M$ and by $L \leq U$, $U+T = M$. Let $U+S = M$ for some $S \leq T$. Then by Lemma 2.9 $L + S = M$ and since T is a supplement of L in M, $S = T$. Thus T is a supplement of U in M.

Lemma 2.11. *Let M be an R-module, $U \leq M$, V be a weak supplement of U and $K \leq V$. Then K is a weak supplement of U if and only if V lies above K in M.*

Proof. ($\Rightarrow$) Let K be a weak supplement of U. Then by definition $U+K = M$ and $U \cap K << M$. Since $K \leq V$, by Modular law $V = V \cap M = V \cap (U + K) = K + U \cap V$. Let $V + T = M$ for some submodule T of M. Then $K + U \cap V + T = M$ and since $U \cap V << M$, $K + T = M$. Thus by Lemma 2.9 V lies above K.

($\Leftarrow$) Because V lies above K and $M = U + V$, then by Lemma 2.9 $M = U + K$. Since $M = U + K$ and $U \cap K \leq U \cap V << M$, K is a weak supplement of U in M.

Lemma 2.12. *Let M be an R-module, $T \leq U \leq M$ and V be a weak supplement of T in M. Then V is a weak supplement of U if and only if U lies above T in M.*

Proof. ($\Rightarrow$) Let V be a weak supplement of U in M. Then U is a weak supplement of V in M. Since T is a weak supplement of V in M and $T \leq U$, then by Lemma 2.11 U lies above T.

($\Leftarrow$) Since V is a weak supplement of T in M, then $M = T + V$ and $T \cap V << M$. Since $T \leq U$, then $M = U + V$. Let S be any submodule of M such that $U \cap V + S = M$. Then by Lemma 2.10 $U + S \cap V = M$ and since U lies above $T, T + S \cap V = M$. Since $V + S = M$ and $T + S \cap V = M$, $T \cap V + S = M$. Then by $T \cap V << M$ we obtain $S = M$. Thus $U \cap V << M$ and V is a weak supplement of U in M.

***Corollary* 2.5.** *Let M be a weakly supplemented module and $L \leq U \leq M$. Then U and L have the same weak supplements in M if and only if U lies above L.*

***Corollary* 2.6.** *Let V be a supplement of U in M and $L \leq U$. Then V is a supplement of L in M if and only if U lies above L.*

***Corollary* 2.7.** *Let V be a weak supplement of U in M. Then V is a supplement of U if and only if V lies above no proper submodule.*

***Corollary* 2.8.** *Let M be an R-module. If every submodule of M has a weak supplement that is a direct summand of M, then M is $\oplus$-supplemented.*

Proof. Let U has a weak supplement V in M and let $M = V \oplus X$. Then V is a supplement of X and by Corollary 2.7 V lies above no proper submodule. Then also by Corollary 2.7 V is a supplement of U. Thus M is $\oplus$-supplemented.

Theorem 2.6. *An R-module M is weakly supplemented if and only if every submodule of M lies above a weak supplement in M.*

Proof. ($\Rightarrow$) Since M is weakly supplemented, every submodule of M is a weak supplement in M. Since every submodule of M lies above itself, every submodule of M lies above a weak supplement in M.

($\Leftarrow$) Let $U \leq M$. Then by hypothesis U lies above a weak supplement T in M. Since T is a weak supplement in M, there exists a submodule V of M such that T is a weak supplement of V in M. Since U lies above T, then by Lemma 2.12 V is also a weak supplement of U in M.

Theorem 2.7. *An R-module M is amply supplemented if and only if every submodule of M lies above a supplement in M.*

Proof. ($\Rightarrow$) Let $U \leq M$. Since M is amply supplemented, then M is supplemented and U has a supplement V in M. Since V is a supplement of U in M, then $M = U + V$. Since M is amply supplemented, then V has a supplement T in M such that $T \leq U$. Since T is a supplement of V in M, then V is a weak supplement of T in M. Since V is a supplement of U in M, then V is a weak supplement of U in M. Thus by Lemma 2.12 U lies above T. Hence U lies above a supplement in M.

($\Leftarrow$) Let every submodule of M be lie above a supplement in M. Let $U \leq M$ and $M = U + V$. Then by hypothesis $U \cap V$ lies above a supplement submodule T in M. Let T be a supplement of K in M. Then K is a weak supplement of T in M. Since $U \cap V$ lies above T then by Lemma 2.12 K is a weak supplement of $U \cap V$ in M and then $U \cap V \cap K << M$. Since $M = U \cap V + K$ then by Modular law $V = V \cap M = V \cap (U \cap V + K) = U \cap V + V \cap K$. Hence $M = U + V = = U + U \cap V + V \cap K = U + V \cap K$. Since $U \cap V \cap K << M$, $V \cap K$ is a weak supplement of U in M. By hypothesis $V \cap K$ lies above a supplement submodule S in M. Since $V \cap K$ is a weak supplement of U in M then by Lemma 2.11 S is a weak supplement of U in M. Hence $M = U + S$ and $U \cap S << M$. Since S is a supplement in M and $U \cap S << M$ then by Lemma 2.5 $U \cap S << S$ and then S is a supplement of U in M with $S \leq V$. Thus every submodule of M has ample supplements in M and M is amply supplemented.

Theorem 2.8. *Let M be an R-module. Then the following statements are equivalent*:

(a) *Every submodule of M lies above a direct summand of M.*

(b) *M is amply supplemented and every supplement submodule of M is a direct summand.*

(c) *For every submodules U and V of M such that $U + V = M$, there is a supplement X of U in M such that $X \leq V$ and X is a direct summand of M.*

Proof. (a) $\Leftrightarrow$ (b) is proved in [12].

(b) $\Rightarrow$ (c) Clear.

(c) $\Rightarrow$ (a) Let $U \leq M$. By hypothesis U has a supplement V in M. Then U is a weak supplement of V in M. Also by hypothesis V has a supplement X in M such that $X \leq U$ and X is a direct summand of M. Because U and X are weak supplements of V and $X \leq U$, then by Lemma 2.11 U lies above X. Thus every submodule of M lies above a direct summand of M.

Lemma 2.13. *Let M be an R-module. Then the following statements are equivalent*:

(a) *M is semisimple.*

(b) *Every submodule of M is a direct summand of M.*

(c) *Every submodule of M is a supplement in M.*

Proof. (a) $\Leftrightarrow$ (b) is proved in [6].

(b) $\Rightarrow$ (c) Clear, because every direct summand of M is a supplement in M.

(c) $\Rightarrow$ (b) Let $U \leq M$. Then by hypothesis U is a supplement in M. Let U be a supplement of X in M. Then $X + U = M$ and $X \cap U << U$. Also by hypothesis $X \cap U$ is a supplement in M. Let $X \cap U$ be a supplement of T in M. Then $X \cap U + T = M$. And then by $X \cap U << M$, $T = M$. Thus $U \cap X$ is a supplement of M in M. Hence $U \cap X = 0$ and $M = U \oplus X$.

Theorem 2.9. *Let M be a weakly supplemented module. Then every weak supplement is a supplement in M if and only if M is semisimple.*

Proof. ($\Rightarrow$) Let $U \leq M$. By hypothesis U has a weak supplement V in M. Then U is a weak supplement of V in M. By hypothesis U is a supplement in M. Thus every submodule of M is a supplement in M. Then by Lemma 2.13 M is semisimple.

($\Leftarrow$) Since M is semisimple, every submodule of M is a supplement in M. Thus every weak supplement is a supplement in M.

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