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ON TYPICAL COMPACT SUBMANIFOLDS OF EUCLIDEAN SPACE
ПРО ТИПОВІ КОМПАКТНІ ПІДМНОГОВИДИ ЕВКЛІДОВОГО ПРОСТОРУ

We show that typical compact submanifolds of R^n are nowhere differentiable with integer box dimensions.

Показано, що типові компактні підмноговиди простору R^n ніде не диференційовні при цілих розмірностях Мінковського.

1. Introduction. A subset Y of a topological space X is called to be *comeagre*, if there is a countable collection $\{W_i\}$ of open and dense subsets of X such that $\bigcap_i W_i \subset Y$. Complement of a comeagre subset is called a meagre subset. A meagre subset can be considered as a countable union of nowhere dense subsets and they are negligible in some sense. So, we say that some property holds for *typical* elements of X, if it holds on a comeagre subset. Let X be a metric space and $C(X)$ be the set of all compact subsets of X. The Hausdorff metric d_H is defined on $C(X)$ by

$$d_H(E,F) = \max\left\{\sup_{x\in E}\inf_{y\in F} d(x,y), \sup_{y\in F}\inf_{x\in E} d(x,y)\right\}.$$

We will denote by $K(X)$ the set of all connected compact subsets of X. Study of properties of typical elements of X, $C(X)$ and $K(X)$ is a classic and interesting part of mathematics. It is proved in [8] that typical elements of $C(X)$ have zero Hausdorff dimensions. A well known theorem due to Banach states that typical elements of the set of all real valued continuous functions defined on $[0,1]$ are nowhere differentiable. One can see many other interesting results in [2, 3, 5, 8, 10, 11]. It is proved that a typical element of $K(R^n)$ consists of a number of slightly blurred line segments. Typical elements of the set of graphs of all curves in R^n, starting at a fixed point, have Hausdorff dimension 1 (see [5]). It is proved in [3] that if M is a compact differentiable manifold with boundary, imbedded in R^n, and S is the set of all deformations of the boundary of M, then typical elements of S are nowhere differentiable with integer box dimensions. We show in the present paper that similar results are true on a more general case, for the set of all compact topological submanifolds of R^n. Our main results are Theorems 3.1 and 3.2.

2. Preliminaries. The following notations will be used in the proofs:

(1) $\Omega^n = \{M\colon M \textit{ is a compact topological submanifold of } R^n\}$.

(2) $D\Omega^n = \{M \in \Omega^n\colon M \textit{ is differentiable}\}$.

(3) $ND\Omega^n = \{M \in \Omega^n\colon M \textit{ is nowhere differentiable}\}$.

(4) $B_\varepsilon = \{x \in R\colon |x| < \varepsilon\}$, $B^k_{(\varepsilon)} = B_\varepsilon \times \ldots \times B_\varepsilon$ (k times).

(5) $I = [-1,1], I^k = I \times I \times \ldots \times I$ (k times).

(6) If $M \in D\Omega^n$ and U is an open subset of R^n, then

$$C(M,U) = \{f\colon M \to U; f \textit{ is countinuos}\}.$$

(7) $D(M,U) = \{f \in C(M,U)\colon f \textit{ is differentible}\}$.

(8) If $M \in D\Omega^n$, then we will denote by $ND(M,U)$ the set of all nowhere differentiable members of $C(M,U)$.

Let M be a bounded subset of R^n. We denote by $\dim(M)$ the topological dimension of M. For each number $\varepsilon > 0$ put

$$\sharp_\varepsilon(M) = \sup\{\text{card}\,\{Z\}\colon Z \subset M \text{ and for each } x, y \in Z,\ |x-y| > \varepsilon\}.$$

The upper and lower box dimensions of M are defined by

$$\overline{\dim}_B(M) = \limsup_{\varepsilon \to 0} \frac{\log \sharp_\varepsilon(M)}{-\log \varepsilon},$$

$$\underline{\dim}_B(M) = \liminf_{\varepsilon \to 0} \frac{\log \sharp_\varepsilon(M)}{-\log \varepsilon}.$$

If $\overline{\dim}_B(M) = \underline{\dim}_B(M)$, then $\dim_B(M) = \lim_{\varepsilon \to 0} \dfrac{\log \sharp_\varepsilon(M)}{-\log \varepsilon}$ is the box dimension of M.

Notice 2.1. *If M is a differentiable submanifold of R^n and $\dim(M) = m$, then*

(1) $\dim_B(M) = \dim(M) = m$.

(2) *If $g\colon M \to R^n$ is a differentiable map and $M_g = g(M)$, then*

$$\dim_B(M_g) = \dim(M_g) \in \{0, 1, \ldots, m\}.$$

If M is a compact manifold, then $C(M, R^n)$ endowed with the following metric d is a complete metric space

$$d(f,g) = \max_{x \in M} |f(x) - g(x)|.$$

The following theorem due to Banach is well known.

Theorem 2.1 [1]**.** *Typical elements of $C(I,R)$ are nowhere differentiable.*

It is easy to show that Banach's theorem is also true if we replace $C(I,R)$ by $C(I, B_\varepsilon)$.

The following lemma is a generalization of Banach's theorem.

Lemma 2.1. *If M is a differentiable compact manifold and $\varepsilon > 0$, then typical elements of $C(M, B_\varepsilon^k)$ with the above metric d, are nowhere differentiable.*

Proof. We give the proof in the following steps.

***Step* 1.** *For each $k \in N$, $ND(I^k, B_\varepsilon)$ is a comeagre subset of $C(I^k, B_\varepsilon)$.*

Proof. The claim is true for $k = 1$ (Banach's theorem). Suppose that the claim is true for each natural number m, $m \le k$. We show that it is true for $k+1$. Let $h \in C(I^{k+1}, B_\varepsilon)$ and $t \in I$. Put

$$h_t\colon I^k \to B_\varepsilon,$$

$$h_t(x) = h(x,t)$$

and let

$$\Gamma = \{h \in C(I^{k+1}, B_\varepsilon)\colon \forall t \in I,\ h_t \text{ is nowhere differentiable}\}.$$

We show that Γ is a comeagre subset of $C(I^{k+1}, B_\varepsilon)$.

Consider the set $\prod_{t \in I} C(I^k, B_\varepsilon)_t$, $C(I^k, B_\varepsilon)_t = C(I^k, B_\varepsilon)$ and put

$$\sigma\colon C(I^{k+1}, B_\varepsilon) \to \prod_{t\in I} C(I^k, B_\varepsilon)_t,$$

$$\sigma(h) = \prod_t (h_t),$$

$$W(f, k, \delta) = \{g \in C(I^k, B_\varepsilon)\colon d(g, f) < \delta\},\ \delta > 0,\ f \in C(I^k, B_\varepsilon).$$

Let O be an open subset of $C(I^k, B_\varepsilon)$ and put $U = \prod_t O_t$, $O_t = O$. If $h \in C(I^{k+1}, B_\varepsilon)$, then the function $\alpha\colon I \to O$ defined by $\alpha(t) = h_t$ is continuous. Due to compactness of I, we can find a number $\delta > 0$ such that for all $t \in I$, $W(h_t, k, \delta) \subset O$. Then $W(h, k+1, \delta) \subset \sigma^{-1}(U)$. This means that $\sigma^{-1}(U)$ is open in $C(I^{k+1}, B_\varepsilon)$.

By assumption, $ND(I^k, B_\varepsilon)$ is a comeagre subset of $C(I^k, B_\varepsilon)$. So, there is a countable collection $\{O_m\colon m \in N\}$ of open and dense subsets of $C(I^k, B_\varepsilon)$ such that

$$\bigcap_{m\in N} O_m \subset ND(I^k, B_\varepsilon).$$

Let

$$U_m = \prod_t (O_m)_t, \quad (O_m)_t = O_m.$$

$\sigma^{-1}(U_m)$ is open in $C(I^{k+1}, B_\varepsilon)$ and we have $\bigcap_{m\in N} \sigma^{-1}(U_m) \subset \Gamma$. Also, it is not hard to show that for each $m \in N$, $\sigma^{-1}(U_m)$ is a dense subset of $C(I^{k+1}, B_\varepsilon)$. Now, from the fact that $\Gamma \subset \subset ND(I^{k+1}, B_\varepsilon)$, we get the result.

Step 2. *$ND(M, B_\varepsilon)$ is comeagre in $C(M, B_\varepsilon)$.*

Proof. Let $k = \dim M$ and for each point $p \in M$ consider a chart (O, ψ) around p such that $I^k \subset \psi(O)$. Since M is compact then there is a finite collection of this kind of charts, say $\{(O_1, \psi_1), \ldots, (O_l, \psi_l)\}$, such that $M \subset \psi_1^{-1}(I^k) \cup \ldots \cup \psi_l^{-1}(I^k)$. Put $U_i = \psi_i^{-1}(I^k)$, $1 \le i \le l$, and for each $h \in C(M, B_\varepsilon)$ denote by h_i the restriction of h on U_i, and consider the following function:

$$\varphi_i\colon C(M, B_\varepsilon) \to C(U_i, B_\varepsilon), \quad \varphi_i(h) = h_i.$$

Since $\psi(U_i) = I^k$ then we get from Step 1, that $ND(U_i, B_\varepsilon)$ is a comeagre subset of $C(U_i, B_\varepsilon)$. So there is a countable collection $\{W_m^i\colon m \in N\}$ of open and dense subsets of $C(U_i, B_\varepsilon)$ such that

$$\bigcap_m W_m^i \subset ND(U_i, B_\varepsilon).$$

We show that for each $i, m \in N$, $\varphi_i^{-1}(W_m^i)$ is a dense subset of $C(M, B_\varepsilon)$. Suppose $h \in C(M, B_\varepsilon)$ and let $\delta > 0$. Since W_m^i is dense in $C(U_i, B_\varepsilon)$, then there is a function $f \in W_m^i$ such that

$$d(h_i, f) < \frac{\delta}{2}. \tag{2.1}$$

Let $\hat{f}\colon M \to B_\varepsilon$ be a continuous extension of f on M. Since h and $\hat{f}$ are continuous, then by (2.1), there is an open subset B of M such that $U_i \subset B$ and

$$x \in B \Rightarrow d(h(x), \quad \hat{f}(x)) < \delta. \tag{2.2}$$

Now let $\eta\colon M \to [0,1]$ be a continuous function such that

$$\eta(x) = 1 \text{ for } x \in U_i \text{ and } \eta(x) = 0 \text{ for } x \in M - B.$$

Put

$$\tau(x) = h(x) + \eta(x)(\hat{f}(x) - h(x)). \tag{2.3}$$

Then

$$|h(x) - \tau(x)| = |\eta(x)|\hat{f}(x) - h(x)| < \delta.$$

Since the image of h is compact and included in B_ε then for sufficiently small δ, the image of τ will be included in B_ε, so $\tau \in C(M, B_\varepsilon)$. If $x \in U_i$, then $\tau(x) = f(x)$, so $\varphi_i(\tau) = f$. Thus $\tau \in \varphi_i^{-1}(W_m^i)$. This means that $\varphi_i^{-1}(W_m^i)$ is dense in $C(M, B_\varepsilon)$. It is easy to show that

$$\bigcap_{m\in N} \bigcap_{1\le i\le l} \varphi_i^{-1}(W_m^i) \subset ND(M, B_\varepsilon).$$

Therefore, $ND(M, B_\varepsilon)$ is a comeagre subset of $C(M, B_\varepsilon)$.

***Step* 3.** *Proof of the lemma.*

For each $h \in C(M, B_\varepsilon^k)$ we have $h = (h_1, \ldots, h_k)$ such that $h_i \in C(M, B_\varepsilon)$. Consider the map

$$\psi\colon C(M, B_\varepsilon^k) \to C(M, B_\varepsilon) \times \ldots \times C(M, B_\varepsilon) \quad (k \text{ times}),$$

$$\psi(h) = (h_1, \ldots, k_k),$$

ψ is a homeomorphism and

$$\psi^{-1}[ND(M, B_\varepsilon) \times \ldots \times ND(M, B_\varepsilon)] \subset ND(M, B_\varepsilon^k). \tag{2.4}$$

Since by Step 2, $ND(M, B_\varepsilon)$ is comeagre in $C(M, B_\varepsilon)$, then $ND(M, B_\varepsilon) \times \ldots \times ND(M, B_\varepsilon)$ is comeagre in $C(M, B_\varepsilon) \times \ldots \times C(M, B_\varepsilon)$. Thus $\psi^{-1}[ND(M, B_\varepsilon) \times \ldots \times ND(M, B_\varepsilon)]$ must be comeagre in $C(M, B_\varepsilon^k)$. Now, we get the result by (2.4).

3. Main results.

Theorem 3.1. *Typical elements of the set of compact submanifolds of R^n are nowhere differentiable.*

Proof. Let M be a differentiable compact submanifold of R^n. If $k = n - \dim M$ and $p \in M$, then R^k can be considered as the set of all vectors perpendicular to M at p. For each $v \in R^k$ denote by v_p the corresponding vector in $T_pM^\perp$. Since M is compact then there is an $\varepsilon > 0$ such that the following map ψ, is a diffeomorphism from $M \times B_\varepsilon^k$ onto an open neighborhood of M in R^n:

$$\psi\colon M \times B_\varepsilon^k \to R^n, \quad \psi(p, v) = p + v_p.$$

For each $g \in C(M, B_\varepsilon^k)$ let $M_g = \{(\psi(x, g(x))\colon x \in M)\}$ and put

$$\lambda(M) = \{M_g\colon g \in C(M, B_\varepsilon)\},$$

$$ND(\lambda(M)) = \{M_g \in \lambda(M)\colon g \text{ is nowhere differentiable}\}.$$

Consider the following metric d on $\lambda(M)$:

$$d(M_g, M_h) = \sup_{x \in M} |g(x) - h(x)|.$$

By using of Lemma 2.1, we get that typical elements of $\lambda(M)$ with the metric d are nowhere differentiable. Then it is easy to show that typical elements of $\lambda(M)$ with the Hausdorff metric are also nowhere differentiable. Now consider the following subspace of $C(R^n)$:

$$\Lambda(R^n) = \bigcup_{M \in D\Omega^n} \lambda(M).$$

We show that *typical elements of* $\Lambda(R^n)$ (*with the Hausdorff metric*) *are nowhere differentiable.*

Since for each differentiable submanifold M of R^n, typical elements of $\lambda(M)$ are nowhere differentiable then there is a collection $\{O_{(M,i)} \colon i \in N\}$ of open and dense subsets of $\lambda(M)$ such that

$$\bigcap_{i \in N} O_{(M,i)} \subset ND(\lambda(M)). \tag{3.1}$$

Since $\lambda(M)$ is a subspace of Ω^n, for each $i \in N$ there is a countable collection $\{U_{(M,i,j)} \colon j \in N\}$ of open subsets of Ω^n such that $O_{(M,i)} = U_{(M,i,j)} \bigcap \lambda(M)$ and

$$\sup \{d_H(M_g, M_h) \in O_{(M,i)} \times U_{(M,i,j)}\} < \frac{1}{j}. \tag{3.2}$$

Now put

$$W_{(M,i,j)} = U_{(M,i,j)} - \{x \colon x \text{ is a boundary point of } O_{(M,i)} \text{ in } \Omega^n\}. \tag{3.3}$$

We get from (3.2) and (3.3) that

$$\bigcap_j W_{(M,i,j)} = O_{(M,i)}. \tag{3.4}$$

Let

$$W_{i,j} = \bigcup_{M \in D\Omega^n} W_{(M,i,j)}.$$

If $ND(\Lambda R^n) = \{M \in \Lambda R^n \colon M \text{ is nowhere differentiable}\}$, then by (3.1) and (3.4)

$$\bigcap_{i,j \in N} W_{i,j} \subset ND(\Lambda(R^n)).$$

Since for each i, $O_{(M,i)}$ is dense in $\lambda(M)$, then for each i, j, $W_{i,j} \cap \Lambda(R^n)$ is dense in $\Lambda(R^n)$. Also the set of differentiable submanifolds of R^n is dense in Ω^n, so $W_{i,j}$ is dense in Ω^n. Therefore, $ND(\Lambda(R^n))$ is a comeagre subset of $\Lambda(R^n)$. Now we get the result from the fact that $ND(\Lambda(R^n)) \subset$ $\subset ND\Omega^n$.

Theorem 3.2. *Typical elements of the set of compact submanifolds of R^n have integer box dimensions.*

Proof. Suppose $M, N \in \Omega^n$ and $d_H(M,N) < \varepsilon$. Let $O_1, \ldots, O_{\sharp_\varepsilon(N)}$ be balls with radius ε such that

$$N \subset \bigcup_i O_i.$$

For each $1 \leq i \leq \sharp_\varepsilon(N)$, let $\widehat{O_i}$ be the ball of radius 2ε with the same center as O_i. Each $\widehat{O_i}$ can be covered by 4^n balls with radius ε. Thus

$$\sharp_\varepsilon(M) \leq 4^n \sharp_\varepsilon(N).$$

In a similar way, we can show that $\sharp_\varepsilon(N) \leq 4^n \sharp_\varepsilon(M)$. Then

$$4^{-n}\sharp_\varepsilon(M) \leq \sharp_\varepsilon(N) \leq 4^n \sharp_\varepsilon(M).$$

Therefore,

$$\frac{-n\log 4 + \log \sharp_\varepsilon(M)}{-\log\varepsilon} \leq \frac{\log \sharp_\varepsilon(N)}{-\log\varepsilon} \leq \frac{n\log 4 + \log \sharp_\varepsilon(M)}{-\log\varepsilon}.$$

If M is differentiable then $\dim_B(M)$ is an integer $\leq n$. Thus $\lim_{\varepsilon\to\infty} \dfrac{\log \sharp_\varepsilon(M)}{-\log\varepsilon} = \dim M \in$ $\in \{0,1,\ldots,n\}$. Then for each $k \in N$ there is an open neighborhood $U_{k,M}$ of M in Ω^n such that for each $N \in U_{k,M}$

$$\dim M - \frac{1}{k} \leq \frac{\log \sharp_\varepsilon(N)}{-\log\varepsilon} \leq \dim M + \frac{1}{k}.$$

Put $W_k = \bigcup_{M\in D\Omega^n} U_{k,M}$. Since $D\Omega^n$ is dense in Ω^n then for any $k \in N$, W_k is dense in Ω^n. Now put

$$W = \bigcap_k W_k.$$

W is comeagre in Ω^n and for each $N \in W$, $\dim_B N$ is an integer number.

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