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EVALUATION FIBRATIONS AND PATH-COMPONENTS OF THE MAPPING SPACE $M(\mathbb{S}^{n+k}, \mathbb{S}^n)$ FOR $8 \leq k \leq 13$*

ОЦІНОЧНІ РОЗШАРУВАННЯ І КОМПОНЕНТИ ЛІНІЙНОЇ ЗВ'ЯЗНОСТІ ПРОСТОРУ ВІДОБРАЖЕНЬ $M(\mathbb{S}^{n+k}, \mathbb{S}^n)$ ПРИ $8 \leq k \leq 13$

Let $M(\mathbb{S}^m, \mathbb{S}^n)$ be the space of maps from the m-sphere $\mathbb{S}^m$ into the n-sphere $\mathbb{S}^n$ with $m, n \geq 1$. We estimate the number of homotopy types of path-components $M_\alpha(\mathbb{S}^{n+k}, \mathbb{S}^n)$ and fiber homotopy types of the evaluation fibrations $\omega_\alpha\colon M_\alpha(\mathbb{S}^{n+k}, \mathbb{S}^n) \to \mathbb{S}^n$ for $8 \leq k \leq 13$ and $\alpha \in \pi_{n+k}(\mathbb{S}^n)$ extending the results of [Mat. Stud. – 2009. – **31**, № 2. – P. 189 – 194]. Further, the number of strong homotopy types of $\omega_\alpha\colon M_\alpha(\mathbb{S}^{n+k}, \mathbb{S}^n) \to \mathbb{S}^n$ for $8 \leq k \leq 13$ is determined and some improvements of the results from [Mat. Stud. – 2009. – **31**, № 2. – P. 189 – 194] are obtained.

Нехай $M(\mathbb{S}^m, \mathbb{S}^n)$ — простір відображень із m-сфери $\mathbb{S}^m$ в n-сферу $\mathbb{S}^n$ з $m, n \geq 1$. Ми оцінюємо число типів гомотопії для компонент лінійної зв'язності $M_\alpha(\mathbb{S}^{n+k}, \mathbb{S}^n)$ та типів гомотопій шарів для оціночних розшарувань $\omega_\alpha\colon M_\alpha(\mathbb{S}^{n+k}, \mathbb{S}^n) \to \mathbb{S}^n$ при $8 \leq k \leq 13$ та $\alpha \in \pi_{n+k}(\mathbb{S}^n)$, узагальнюючи результати з [Mat. Stud. – 2009. – **31**, № 2. – P. 189 – 194]. Крім того, визначаємо число типів сильних гомотопій $\omega_\alpha\colon M_\alpha(\mathbb{S}^{n+k}, \mathbb{S}^n) \to \mathbb{S}^n$ при $8 \leq k \leq 13$ та отримуємо деякі покращення результатів з [Mat. Stud. – 2009. – **31**, № 2. – P. 189 – 194].

1. Introduction. Given spaces X and Y, let $M(X, Y)$ be the mapping space of all continuous maps of X into Y equipped with the compact-open topology. The space $M(X, Y)$ is generally disconnected and its path-components are in one-to-one correspondence with the set $[X, Y]$ of (free) homotopy classes of maps of X into Y.

Given $x_0 \in X$, consider the evaluation map

$$\omega\colon M(X, Y) \to Y$$

defined by $\omega(f) = f(x_0)$ for $f \in M(X, Y)$. Let $M_\alpha(X, Y)$ be the path-component of $M(X, Y)$ which contains all maps in $\alpha \in [X, Y]$. By [13, p. 83] (Theorem III.13.1), the evaluation map $\omega_\alpha\colon M_\alpha(X, Y) \to Y$ obtained by restricting ω to $M_\alpha(X, Y)$ is a Hurewicz fibration provided X is locally compact. Then, the natural classification problems arise:

(1) divide the set of path-components of $M(X, Y)$ into homotopy types;

(2) divide the set of evaluation fibrations $\omega_\alpha\colon M_\alpha(X, Y) \to Y$ into fibre- and strong fibre-homotopy types for $\alpha \in [X, Y]$.

Conditions for when two path-components of $M(X, Y)$ are homotopy equivalent are presented in [16] provided that spaces X and Y are connected and countable CW-complexes.

Let now $\mathbb{S}^n$ be the n-sphere. To study coincidences of fiberwise maps between sphere bundles over $\mathbb{S}^1$, the set of fiberwise homotopy classes of those maps has been considered in [7]. But, the set

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of fiberwise maps between the trivial bundles $\mathbb{S}^1 \times \mathbb{S}^m$ and $\mathbb{S}^1 \times \mathbb{S}^n$ over $\mathbb{S}^1$ coincides with the free loop space $\mathcal{L}M(\mathbb{S}^m, \mathbb{S}^n) = M(\mathbb{S}^1, M(\mathbb{S}^m, \mathbb{S}^n))$.

Certainly, for the space $M(\mathbb{S}^m, \mathbb{S}^n)$ with $m, n \geq 1$, the path-components can be enumerated by the homotopy group $\pi_m(\mathbb{S}^n)$. In view of [10] (Theorem 4.1), there is a strong relation between evaluation fibrations $\omega_\alpha \colon M_\alpha(\mathbb{S}^m, \mathbb{S}^n) \to \mathbb{S}^n$ for $\alpha \in \pi_m(\mathbb{S}^n)$ and the Whitehead product $[\iota_n, \alpha]$. This was used in [10] (Theorems 5.1, 5.2) to tackle a complete homotopy classification of path-components of $M(\mathbb{S}^m, \mathbb{S}^n)$ for $m = n, n+1$ and compute the order of the homotopy group $\pi_{n-1}(M_\alpha(\mathbb{S}^n, \mathbb{S}^n))$. Homotopy properties of various $M_\alpha(\mathbb{S}^m, \mathbb{S}^n)$ have been studied in [1, 14, 20].

The purpose of this note is to extend the results of [6] for $m = n + k$ with $8 \leq k \leq 13$.

Section 1 summarizes [10, 11] and follows [16] to connect in Theorem 1.1 these classification problems for $M(\mathbb{S}^m, \mathbb{S}^n)$ with the m-th Gottlieb group $G_m(\mathbb{S}^n)$ considered in [8, 9] and then studied in [5].

Section 2 makes use of [5] to take up the systematic study of the quotient sets $\pi_{n+k}(\mathbb{S}^n)/\pm$ $\pm\, G_{n+k}(\mathbb{S}^n)$ with $0 \leq k \leq 13$. Then, our basic results stated in Propositions 2.1 – 2.6 estimate the number of homotopy types of path-components of $M(\mathbb{S}^{n+k}, \mathbb{S}^n)$ and fibre-homotopy types of evaluation fibrations $\omega_\alpha \colon M_\alpha(\mathbb{S}^{n+k}, \mathbb{S}^n) \to \mathbb{S}^n$ with $0 \leq k \leq 13$. Further, the number of strong fibre-homotopy types of $\omega_\alpha \colon M_\alpha(\mathbb{S}^{n+k}, \mathbb{S}^n) \to \mathbb{S}^n$ with $0 \leq k \leq 13$ is determined. Corollary 2.1 concludes a list of evaluation fibrations $\omega_\alpha \colon M_\alpha(\mathbb{S}^{n+k}, \mathbb{S}^n) \to \mathbb{S}^n$ which are fibre-homotopy equivalent but not strong fibre-homotopy equivalent for some $0 \leq k \leq 13$.

Those results are applied in Section 3 to estimate the number of homotopy types of path-components of $M(\mathbb{S}^{(n+1)d+k-1}, \mathbb{F}P^n)$ and (strong) fibre-homotopy types of evaluation fibrations $\omega_\alpha \colon M_\alpha(\mathbb{S}^{(n+1)d+k-1}, \mathbb{F}P^n) \to \mathbb{F}P^n$ with $0 \leq k \leq 13$ for the n-projective spaces $\mathbb{F}P^n$ for $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$. Further, we deduce that path-components of $M(\mathbb{S}^m, \mathbb{K}P^2)$ have the same homotopy type for $m \leq 21$, where $\mathbb{K}P^2$ is the Cayley projective plane.

The last Section 4 makes use of [4, 18] to present the rational homotopy type of $M(\mathbb{S}^m, \mathbb{S}^n)$ and path-components of $M(\mathcal{M}(\mathbb{A}, m), \mathbb{S}^n)$ for a Moore space $\mathcal{M}(\mathbb{A}, m)$.

1. Prerequisites. Given $x_0 \in X$ and $y_0 \in Y$, write $M(X,Y)_*$ for the space of all continuous pointed maps of X into Y. This leads to the Hurewicz fibration $M(X,Y)_* \to M(X,Y) \xrightarrow{\omega} Y$, provided X is locally compact. Recall that on the set $[X,Y]_*$ of homotopy classes of pointed maps there is an action of $\pi_1(Y, y_0)$ and $[X,Y]_*/\pi_1(Y,y_0) = [X,Y]$ [21] (Chapter I, (1.11)).

In particular, for $\pi_1(Y, y_0) = 0$ we get $[X,Y]_* = [X,Y]$, e.g., $\pi_m(\mathbb{S}^n) = [\mathbb{S}^m, \mathbb{S}^n]_* = [\mathbb{S}^m, \mathbb{S}^n]$, for $n > 1$. Further, there is the Hurewicz fibration

$$M(\mathbb{S}^m, \mathbb{S}^n)_* \to M(\mathbb{S}^m, \mathbb{S}^n) \xrightarrow{\omega} \mathbb{S}^n.$$

A *fibration* $p \colon E \to B$ with a fibre F means a Hurewicz fibration together with a fixed homotopy equivalence $i \colon F \to p^{-1}(b_0)$ over the base point $b_0 \in B$. Recall that for fibrations $p_1 \colon E_1 \to B$ and $p_2 \colon E_2 \to B$ a based map $f \colon E_1 \to E_2$ is:

1) a *fibre homotopy equivalence* (fhe) if there exists $g \colon E_2 \to E_1$ such that $g \circ f$ and $f \circ g$ are homotopic to the respective identities by based homotopies F and G satisfying $p_1 \circ F(e_1, t) = p_1(e_1)$ and $p_2 \circ G(e_2, t) = p_2(e_2)$ for $e_1 \in E_1$, $e_2 \in E_2$ and $t \in [0,1]$;

2) a *strong fibre homotopy equivalence* (sfhe) if it is a fibre homotopy equivalence and $i_2' \circ f \circ i_1$ is homotopic to the identity map id_F, where i_2' is an arbitrary homotopy inverse of i_2.

Let X be a connected and pointed space. The m-th *Gottlieb group* $G_m(X)$ [8, 9] of a space X is the subgroup of the m-th homotopy group $\pi_m(X)$ containing all elements which can be represented by a map $f\colon \mathbb{S}^m \to X$ such that $f \vee \mathrm{id}_X\colon \mathbb{S}^m \vee X \to X$ extends (up to homotopy) to a map $F\colon \mathbb{S}^m \times X \to X$. Observe that $G_m(X) = \pi_m(X)$ provided X is an H-space.

Given $\alpha \in \pi_m(\mathbb{S}^n)$ we have deduced in [5] that $\alpha \in G_m(\mathbb{S}^n)$ if and only if the Whitehead product $[\iota_n, \alpha] = 0$, where ι_n denotes the homotopy class of $\mathrm{id}_{\mathbb{S}^n}$. In other words, $G_m(\mathbb{S}^n) = \ker[\iota_n, -]$ for the map $[\iota_n, -]\colon \pi_m(\mathbb{S}^n) \to \pi_{m+n-1}(\mathbb{S}^n)$ with $m \geq 1$. Write $\sharp g$ for the order of the element g in a group G. Then, by [5] (Section 2), from this interpretation of Gottlieb groups of spheres, we obtain

$$G_m(\mathbb{S}^n) = (\sharp[\iota_n, \alpha])\pi_m(\mathbb{S}^n),$$

if $\pi_m(\mathbb{S}^n)$ is a cyclic group with a generator α. It follows that $G_m(\mathbb{S}^n) = \pi_m(\mathbb{S}^n)$ (resp. $G_m(\mathbb{S}^n) = 0$) provided $\sharp[\iota_n, \alpha] = 1$ (resp. $\sharp[\iota_n, \alpha] = \infty$) for $\alpha \in \pi_m(\mathbb{S}^n)$. Furthermore, because of H-structures on the spheres $\mathbb{S}^n$ for $n = 1, 3, 7$, it holds $G_m(\mathbb{S}^n) = \pi_m(\mathbb{S}^n)$ for any $m \geq 1$.

Given a group G and its subgroup $G' < G$, write $G/\pm G'$ for the *quotient set* of G by the relation $\sim$ defined as follows: for $x, y \in G$, $x \sim y$ if and only if $xy \in G'$ or $xy^{-1} \in G'$. Observe that if $G_i' < G_i$, $i = 1, 2$, then there is a surjection

$$\phi\colon (G_1 \times G_2)/\pm(G_1' \times G_2') \to (G_1/\pm G_1') \times (G_2/\pm G_2')$$

defined by $\overline{(g_1, g_2)} \mapsto (\overline{g_1}, \overline{g_2})$, which is not injective in general, where $\bar{g}$ states for the appropriate abstract class determined by g.

***Example* 1.1.** (1) If $G_1 = G_2 = \mathbb{Z}$ and $G_1' = G_2' = 3\mathbb{Z}$ for the infinite cyclic group $\mathbb{Z}$, then $|(\mathbb{Z}\times\mathbb{Z})/\pm(3\mathbb{Z}\times 3\mathbb{Z})| = 5$ and $|\mathbb{Z}/\pm 3\mathbb{Z}|^2 = 4$. Let $\mathbb{Z}_n$ be the cyclic group with order n. If $G_1 = \mathbb{Z}_3$, $G_2 = \mathbb{Z}_6$, $G_1' = 0$ and $G_2' = \mathbb{Z}_2$ then $|(\mathbb{Z}_3 \times \mathbb{Z}_6)/\pm(0 \times \mathbb{Z}_2)| = 5$ and $|\mathbb{Z}_3/\pm 0||\mathbb{Z}_4/\pm\mathbb{Z}_2| = 4$.

(2) If $G_1' = G_1$ then the bijection holds easily.

Writing $\simeq$ for the homotopy equivalence relation, [10] (Theorems 1, 2) and [11] (Theorem 2.3) lead to:

Theorem 1.1. *Let* $m, n \geq 1$. *Then, there are surjections*:

$$\pi_m(\mathbb{S}^n)/\pm G_m(\mathbb{S}^n) \longrightarrow \{M_\alpha(\mathbb{S}^m, \mathbb{S}^n);\ \alpha \in \pi_m(\mathbb{S}^n)\}/\simeq, \tag{1.1}$$

$$\pi_m(\mathbb{S}^n)/\pm G_m(\mathbb{S}^n) \longrightarrow \{\omega_\alpha\colon M_\alpha(\mathbb{S}^m, \mathbb{S}^n) \to \mathbb{S}^n;\ \alpha \in \pi_m(\mathbb{S}^n)\}/\,\mathrm{fhe} \tag{1.2}$$

and there is a bijection

$$\pi_m(\mathbb{S}^n)/G_m(\mathbb{S}^n) \xrightarrow{\cong} \{\omega_\alpha\colon M_\alpha(\mathbb{S}^m, \mathbb{S}^n) \to \mathbb{S}^n;\ \alpha \in \pi_m(\mathbb{S}^n)\}/\,\mathrm{sfhe}. \tag{1.3}$$

We point out that a generalization of the results above has been stated in [16]. As a consequence, using the surjections (1.1) and (1.2), it is possible to obtain an upper bound for the number of homotopy types of path-components for the mapping space $M(\mathbb{S}^{n+k}, \mathbb{S}^n)$ and to the number of evaluation fibrations $\omega_\alpha\colon M_\alpha(\mathbb{S}^{n+k},\mathbb{S}^n)\to\mathbb{S}^n$, for $\alpha\in\pi_{n+k}(\mathbb{S}^n)$, up to fibre-homotopy equivalence (fhe), respectively. In addition, the bijection (1.3) gives the exactly number of evaluation fibrations, up to strong fibre-homotopy equivalence (sfhe).

***Remark* 1.1.** By [11] (Theorem 4.1) we have $M_\alpha(\mathbb{S}^m,\mathbb{S}^n)\simeq M_0(\mathbb{S}^m,\mathbb{S}^n)$ if and only if $[\iota_n,\alpha]=0$, if and only if $\alpha\in G_m(\mathbb{S}^n)$. Thus if $G_m(\mathbb{S}^n)\subsetneq\pi_m(\mathbb{S}^n)$ then *there are at least two* path-components which are not homotopy equivalent, that is, $|\pi_m(\mathbb{S}^n)/\pm G_m(\mathbb{S}^n)|\geq 2$, and there is *only one* if and only if $G_m(\mathbb{S}^n)=\pi_m(\mathbb{S}^n)$.

We close this section with the following fact (on the relation $\sim$ defined above) useful in the sequel. First, given reals x,y, write

$$\chi(x,y)=\left\lceil\frac{(x-1)(y-1)}{2}\right\rceil+\left\lceil\frac{x-1}{2}\right\rceil+\left\lceil\frac{y-1}{2}\right\rceil+1,$$

where $\lceil r\rceil=\min\{k\in\mathbb{Z};\ k\geq r\}$ for any real r.

Lemma 1.1. *For positive integers m,m',n,n' with $m\mid n$, $m'\mid n'$ and $n,n'\geq 1$, let $\mathbb{Z}_m\times\mathbb{Z}_{m'}<\mathbb{Z}_n\times\mathbb{Z}_{n'}$, $m\mathbb{Z}\times\mathbb{Z}_{m'}<\mathbb{Z}\times\mathbb{Z}_{n'}$ and $m\mathbb{Z}\times m'\mathbb{Z}<\mathbb{Z}\times\mathbb{Z}$ be the obvious inclusions. Then*

$$|(\mathbb{Z}_n\times\mathbb{Z}_{n'})/\pm(\mathbb{Z}_m\times\mathbb{Z}_{m'})|=\chi\left(\frac{n}{m},\frac{n'}{m'}\right), \tag{1.4}$$

$$|(\mathbb{Z}\times\mathbb{Z}_{n'})/\pm(m\mathbb{Z}\times\mathbb{Z}_{m'})|=\chi\left(m,\frac{n'}{m'}\right), \tag{1.5}$$

$$|(\mathbb{Z}\times\mathbb{Z})/\pm(m\mathbb{Z}\times m'\mathbb{Z})|=\chi\left(m,m'\right). \tag{1.6}$$

In particular, $|(\mathbb{Z}_n\times\mathbb{Z}_{n'})/\pm(\mathbb{Z}_m\times\mathbb{Z}_{n'})|=|\mathbb{Z}_n/\pm\mathbb{Z}_m|=\chi\left(\frac{n}{m},1\right)$.

Proof. For any $(a,b)\in\mathbb{Z}_n\times\mathbb{Z}_{n'}$, $(a,b)\sim(c,d)$ where $1\leq c\leq\frac{n}{m}$, $1\leq d\leq\frac{n'}{m'}$. Furthermore, for $c\neq\frac{n}{m}$ and $d\neq\frac{n'}{m'}$, $(c,d)\sim\left(\frac{n}{m}-c,\frac{n'}{m'}-d\right)$ for $1\leq d\leq\frac{n'}{m'}-1$ and then we have $\left\lceil\frac{1}{2}\left(\frac{n}{m}-1\right)\left(\frac{n'}{m'}-1\right)\right\rceil$ nonequivalent elements. In addition, $\left(\frac{n}{m},d\right)\sim\left(\frac{n}{m},\frac{n'}{m'}-d\right)$ for $1\leq d\leq\frac{n'}{m'}-1$ and $\left(c,\frac{n'}{m'}\right)\sim\left(\frac{n}{m}-c,\frac{n'}{m'}\right)$ for $1\leq c\leq\frac{n}{m}-1$. So, we obtain more $\left\lceil\frac{1}{2}\left(\frac{n}{m}-1\right)\right\rceil+\left\lceil\frac{1}{2}\left(\frac{n'}{m'}-1\right)\right\rceil$ nonequivalent elements. Finally, since that the trivial element is $\left(\frac{n}{m},\frac{n'}{m'}\right)$, the equation (1.4) follows.

To prove (1.5) and (1.6), just replace $\frac{n}{m}$ by m and $\frac{n}{m},\frac{n'}{m'}$ by m,m' respectively.

Lemma 1.1 is proved.

2. Main results. We make use of [5] and Lemma 1.1 to estimate the cardinality

$$|\pi_{n+k}(\mathbb{S}^n)/\pm G_{n+k}(\mathbb{S}^n)| \tag{2.1}$$

for $8\leq k\leq 13$. We first recall the results from [6] for $0\leq k\leq 7$ and make some improvements of the cardinality (2.1).

Proposition 2.1. *The cardinality* $|\pi_{n+k}(\mathbb{S}^n)/\pm G_{n+k}(\mathbb{S}^n)|$ *for* $0\le k\le 7$ *is, respectively*:

one, if $n=1,3,7$; *two, if* $n\ne 1,3,7$ *is odd;* $|\mathbb{Z}|$ *if* n *is even*;

one, if $n=1,2,6$ *or* $n\equiv 3 \pmod 4$; *two, otherwise*;

one, if $n=1,5$ *or* $n\equiv 2,3 \pmod 4$; *two, otherwise*;

ten, if $n=4$; *one, if* $n\equiv 7 \pmod 8$ *or* $n=2^i-3$ *for* $i\ge 3$; *two, if* $n\equiv 1,3,5 \pmod 8$ *and* $n\ge 9$ *and* $n\ne 2^i-3$; *seven, if* $n\equiv 2 \pmod 4$ *and* $n\ge 6$ *or* $n=12$; *thirteen, if* $n\equiv 0 \pmod 4$ *and* $n\ge 8$ *and* $n\ne 12$;

one, for all $n\ge 1$;

one, if $n\ne 6$; *two, otherwise*;

one, if $n\equiv 4,5,7 \pmod 8$ *or* $n=2^i-5$ *for* $i\ge 4$; *two, otherwise*;

one, if $n=5,11$ *or* $n\equiv 15 \pmod{16}$; *two, if* n *is odd and* $n\ge 9$, *unless* $n=11$ *and* $n\equiv 15 \pmod{16}$; *eight, if* $n=4$; *thirty one, if* $n=6$; *ninety one, if* $n=8$; *one hundred twenty one, if* n *is even and* $n\ge 10$.

2.1. The case k = 8. Making use of the Gottlieb groups $G_{n+8}(\mathbb{S}^n)$ computed in [5] (Proposition 6.3) we estimate $|\pi_{n+8}(\mathbb{S}^n)/\pm G_{n+8}(\mathbb{S}^n)|$.

For $n=1,2,6,10$ or $n\equiv 3 \pmod 4$, the cardinality (2.1) is *one*.

For $n\equiv 0,1 \pmod 4$ and $n\ne 8,9$, or $n\equiv 22 \pmod{32}$ and $n\ge 54$, $G_{n+8}(\mathbb{S}^n)=0$ and then (2.1) is equal to $|\pi_{n+8}(\mathbb{S}^n)/\pm 0|$, that is: *two*, if $n=4,5$, since that $\pi_{n+8}(\mathbb{S}^n)=\{\varepsilon_n\}\cong\mathbb{Z}_2$; *four*, if $n\ge 12$, since that $\pi_{n+8}(\mathbb{S}^n)=\{\bar\nu_n,\varepsilon_n\}\cong(\mathbb{Z}_2)^2$.

For $n\equiv 2 \pmod 8$ and $n\ge 18$, $G_{n+8}(\mathbb{S}^n)=\{\varepsilon_n\}\cong\mathbb{Z}_2$ and $\pi_{n+8}(\mathbb{S}^n)=\{\bar\nu_n,\varepsilon_n\}\cong(\mathbb{Z}_2)^2$. So the cardinality (2.1) is *two*. But $[\iota_n,\bar\nu_n]\ne 0$ and then $\omega_{\bar\nu_n}$ is not fibre-homotopy equivalent to ω_0 (which is fibre-homotopy equivalent to ω_{ε_n}).

For $n=22$, or $n\equiv 14 \pmod{16}$, or $n\equiv 6 \pmod{32}$ and $n\ge 14$, $G_{n+8}(\mathbb{S}^n)=\{\eta_n\sigma_{n+1}\}\cong$ $\cong\mathbb{Z}_2$ and $\pi_{n+8}(\mathbb{S}^n)=\{\bar\nu_n,\varepsilon_n\}\cong(\mathbb{Z}_2)^2$. Thus, the cardinality (2.1) is *two*. In view of [17] (Lemma 6.4), it holds $\eta_n\sigma_{n+1}=\bar\nu_n+\varepsilon_n\in G_{n+8}(\mathbb{S}^n)$ for $n\ge 9$ and the bilinearity of the Whitehead product yields $[\iota_n,\bar\nu_n]=-[\iota_n,\varepsilon_n]$. By [10] (Theorem 2.3), $\omega_{\bar\nu_n}$ and ω_{ε_n} are fibre-homotopy equivalent as well as ω_0 and $\omega_{\bar\nu_n+\varepsilon_n}$.

For $n=8$, the Gottlieb group is $G_{16}(\mathbb{S}^8)=\{(E\sigma')\eta_{15},\sigma_8\eta_{15}+\bar\nu_8+\varepsilon_8\}\cong(\mathbb{Z}_2)^2$ and the homotopy group is $\pi_{16}(\mathbb{S}^8)=\{(E\sigma')\eta_{15},\sigma_8\eta_{15},\bar\nu_8,\varepsilon_8\}\cong(\mathbb{Z}_2)^4$. We replace the generator $\sigma_8\eta_{15}\in$ $\in\pi_{16}(\mathbb{S}^8)$ by the sum $\sigma_8\eta_{15}+\bar\nu_8+\varepsilon_8$ and then (2.1) is *four*.

For $n=9$, $G_{17}(\mathbb{S}^9)=\{[\iota_9,\iota_9]\}\cong\mathbb{Z}_2$ and $\pi_{17}(\mathbb{S}^9)=\{\sigma_9\eta_{16},\bar\nu_9,\varepsilon_9\}\cong(\mathbb{Z}_2)^3$. Although the generators for $n=9$ are different from that ones for $n=8$, but (2.1) is *four* as well.

We can summarize the results above and estimate the number of homotopy types of path-components of the mapping space $M(\mathbb{S}^{n+8},\mathbb{S}^n)$ and fibre-homotopy equivalence types of evaluation fibrations $\omega_\alpha\colon M_\alpha(\mathbb{S}^{n+8},\mathbb{S}^n)\to\mathbb{S}^n$ for $\alpha\in\pi_{n+8}(\mathbb{S}^n)$.

Proposition 2.2. *The cardinality* $|\pi_{n+8}(\mathbb{S}^n)/\pm G_{n+8}(\mathbb{S}^n)|$ *is*:

one, if $n=1,2,6,10$ *or* $n\equiv 3 \pmod 4$;

two, if $n=4,5,22$, *or* $n\equiv 2 \pmod 8$ *and* $n\ge 18$, *or* $n\equiv 14 \pmod{16}$, *or* $n\equiv 6 \pmod{32}$ *and* $n\ge 14$;

four, if $n\equiv 0,1 \pmod 4$ *and* $n\ge 8$, *or* $n\equiv 22 \pmod{32}$ *and* $n\ge 54$.

2.2. The case $k = 9$. In view of [5] (Proposition 6.4), we estimate the cardinality

$$|\pi_{n+9}(\mathbb{S}^n)/\pm G_{n+9}(\mathbb{S}^n)|.$$

For $n = 1, 2, 6$ or $n \equiv 3 \pmod 4$, $|\pi_{n+9}(\mathbb{S}^n)/\pm G_{n+9}(\mathbb{S}^n)| = 1$.

For $n \equiv 0 \pmod 8$ and $n \geq 16$, $G_{n+9}(\mathbb{S}^n) = 0$ and then (2.1) is

$$|\pi_{n+9}(\mathbb{S}^n)/\pm 0| = |(\mathbb{Z}_2)^3/\pm 0| = 8.$$

For $n \equiv 2 \pmod 4$ and $n \geq 14$, or $n = 2^i - 7$ with $i \geq 5$, or $n \equiv 5 \pmod 8$ and $n \not\equiv 53 \pmod{64}$, $G_{n+9}(\mathbb{S}^n) \cong (\mathbb{Z}_2)^2$ and $\pi_{n+9}(\mathbb{S}^n) \cong (\mathbb{Z}_2)^3$ and then (2.1) is *two*.

For $n \equiv 4 \pmod 8$, or $n \equiv 53 \pmod{64}$ and $n \geq 117$, or $n \equiv 1 \pmod 8$ and $n \geq 17$ and $n \neq 2^i - 7$, $G_{n+9}(\mathbb{S}^n) \cong \mathbb{Z}_2$ and $\pi_{n+9}(\mathbb{S}^n) \cong (\mathbb{Z}_2)^3$. So (2.1) is *four*.

For $n = 8$, $G_{17}(\mathbb{S}^8) = \{(E\sigma')\eta_{15}^2, \sigma_8\eta_{15}^2 + \nu_8^3 + \eta_8\varepsilon_9\} \cong (\mathbb{Z}_2)^2$ and $\pi_{17}(\mathbb{S}^8) = \{(E\sigma')\eta_{15}^2, \sigma_8\eta_{15}^2, \nu_8^3, \mu_8, \eta_8\varepsilon_9\} \cong (\mathbb{Z}_2)^5$. Replacing the generator $\sigma_8\eta_{15}^2 \in \pi_{17}(\mathbb{S}^8)$ by the sum $\sigma_8\eta_{15}^2 + \nu_8^3 + \eta_8\varepsilon_9$, (2.1) is $|\{\nu_8^3, \mu_8, \eta_8\varepsilon_9\}/\pm 0| = |(\mathbb{Z}_2)^3/\pm 0| = 8$.

For $n = 9$, the Gottlieb group is $G_{18}(\mathbb{S}^9) = \{\sigma_9\eta_{16}^2, \nu_9^3, \eta_9\varepsilon_{10}\} \cong (\mathbb{Z}_2)^3$ and the homotopy group is $\pi_{18}(\mathbb{S}^9) = \{\sigma_9\eta_{16}^2, \nu_9^3, \mu_9, \eta_9\varepsilon_{10}\} \cong (\mathbb{Z}_2)^4$. In a similar way we conclude that (2.1) is *two*.

Finally, for $n = 10$, $G_{19}(\mathbb{S}^{10}) = \{3[\iota_{10}, \iota_{10}], \nu_{10}^3, \eta_{10}\varepsilon_{11}\} \cong 3\mathbb{Z} \oplus (\mathbb{Z}_2)^2$ and $\pi_{19}(\mathbb{S}^{10}) = \{\Delta(\iota_{21}), \nu_{10}^3, \mu_{10}, \eta_{10}\varepsilon_{11}\} \cong \mathbb{Z} \oplus (\mathbb{Z}_2)^3$. So (2.1) is $|(\mathbb{Z} \oplus (\mathbb{Z}_2)^3)/\pm(3\mathbb{Z} \oplus (\mathbb{Z}_2)^2)| = 4$, by Lemma 1.1.

Then, we summarize the results above as follows:

Proposition 2.3. *The cardinality* $|\pi_{n+9}(\mathbb{S}^n)/\pm G_{n+9}(\mathbb{S}^n)|$ *is*:

one, if $n = 1, 2, 6$, *or* $n \equiv 3 \pmod 4$;

two, if $n = 9$, *or* $n \equiv 2 \pmod 4$ *and* $n \geq 14$, *or* $n = 2^i - 7$ *with* $i \geq 5$, *or* $n \equiv 5 \pmod 8$ *and* $n \not\equiv 53 \pmod{64}$;

four, if $n = 10$, *or* $n \equiv 4 \pmod 8$, *or* $n \equiv 53 \pmod{64}$ *and* $n \geq 117$, *or* $n \equiv 1 \pmod 8$ *and* $n \geq 17$ *and* $n \neq 2^i - 7$;

eight, if $n \equiv 0 \pmod 8$.

2.3. The cases $k = 10, 11$. Following the same ideas as above and making use of Lemma 1.1, we can also compute the appropriate quotient set to estimate its cardinality to state the next results:

Proposition 2.4. *The cardinality* $|\pi_{n+10}(\mathbb{S}^n)/\pm G_{n+10}(\mathbb{S}^n)|$ *is*:

one, if $n = 1, 2, 5$, *or* $n \equiv 3 \pmod 4$;

two, if $n \equiv 2 \pmod 4$, *or* $n \equiv 1 \pmod 4$ *and* $n \geq 9$;

four, if $n \equiv 0 \pmod 4$.

Proposition 2.5. *The cardinality* $|\pi_{n+11}(\mathbb{S}^n)/\pm G_{n+11}(\mathbb{S}^n)|$ *is*:

one, if $n \equiv 1 \pmod 2$ *and* $n \not\equiv 115 \pmod{128}$;

two, if $n \equiv 115 \pmod{128}$ *and* $n \geq 243$;

twenty two, two hundred fifty-four, seven hundred fifty seven, if $n = 4, 8, 12$ *respectively*;

two hundred fifty-three, if $n \equiv 0 \pmod 4$ *and* $n \geq 16$;

one hundred twenty-seven, if $n \equiv 2 \pmod 4$ *and* $n \geq 6$.

2.4. The cases $k = 12, 13$. Following [5] (Section 6), we have $G_{n+12}(\mathbb{S}^n) = \pi_{n+12}(\mathbb{S}^n)$ for $n \neq 10$ and $G_{n+13}(\mathbb{S}^n) = \pi_{n+13}(\mathbb{S}^n)$ for $n = 2$ or n odd. So the cardinality (2.1) is *one*. For $k = 12,\ n = 10$ or $k = 13,\ n$ even and $n \neq 2, 4, 14$, the cardinality (2.1) is *two*. For $k = 13,\ n = 4$, the cardinality (2.1) is *four* and for $k = 13,\ n = 14$ it is *five*.

In resume:

Proposition 2.6. *The cardinality $|\pi_{n+k}(\mathbb{S}^n)/\pm G_{n+k}(\mathbb{S}^n)|$ is:*

one, for $k = 12$ and $n \neq 10$, or $k = 13$ and $n = 2$ or n odd;

two, for $k = 12$ and $n = 10$, or $k = 13$ and n even, $n \neq 2, 4, 14$;

four, for $k = 13$ and $n = 4$;

five, for $k = 13$ and $n = 14$.

***Remark* 2.1.** We observe that the cases $k = 9,\ n = 53$ and $k = 11,\ n = 115$ are missing because the Gottlieb groups $G_{62}(\mathbb{S}^{53})$ and $G_{126}(\mathbb{S}^{115})$ are unknown. On the other hand, the 2-primary component of the homotopy group $\pi_{126}(\mathbb{S}^{115})$ is $\pi^{115}_{126} = \{\zeta_{115}\}$ [19] (Theorem 7.4) and in view of [15] (Theorem 3.1) the Kervaire invariant θ_6 exists in the stable homotopy group π^s_{126} if and only if $[\zeta_{115}, \iota_{115}] = 0$.

We recall that in [10] (Example 1), two fhe evaluation fibrations $\omega_\alpha\colon M_\alpha(\mathbb{S}^2 \vee \mathbb{S}^2, \mathbb{S}^2) \to \mathbb{S}^2$ and $\omega_\beta\colon M_\beta(\mathbb{S}^2 \vee \mathbb{S}^2, \mathbb{S}^2) \to \mathbb{S}^2$ for $\alpha, \beta \in [\mathbb{S}^2 \vee \mathbb{S}^2, \mathbb{S}^2]$ not being sfhe are constructed. From the results above, we get:

***Corollary* 2.1.** *There are evaluation fibrations $\omega_\alpha\colon M_\alpha(\mathbb{S}^{n+k}, \mathbb{S}^n) \to \mathbb{S}^n$ for some $\alpha \in \pi_{n+k}(\mathbb{S}^n)$ and $0 \leq k \leq 13$ being* fhe *and not* sfhe.

At the end of this section, we notice that:

***Remark* 2.2.** The procedure above leads to an estimation of the number of homotopy types of path-components of $M(\mathbb{S}^{n+k}, \mathbb{S}^n)_*$ and fibre-homotopy types of evaluation fibrations $\omega_\alpha\colon M_\alpha(\mathbb{S}^{n+k}, \mathbb{S}^n)_* \to \mathbb{S}^n$ with $0 \leq k \leq 13$.

3. Applications to projective spaces.. Let $\mathbb{R}$ and $\mathbb{C}$ be the fields of real and complex numbers, respectively and $\mathbb{H}$ the skew $\mathbb{R}$-algebra of quaternions. In this section we apply the results above to study the path-components of $M(\mathbb{S}^m, \mathbb{F}P^n)$ for $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ and $M(\mathbb{S}^m, \mathbb{K}P^2)$, where $\mathbb{K}$ denotes the Cayley algebra.

Denote by $\mathbb{F}P^n$ the n-projective space over $\mathbb{F}$. Put $d = \dim_{\mathbb{R}} \mathbb{F}$, write $i_{m,n}\colon \mathbb{F}P^m \hookrightarrow \mathbb{F}P^n$, $m \leq n$, for the inclusion map, $\gamma_n = \gamma_{n,\mathbb{F}}\colon \mathbb{S}^{(n+1)d-1} \to \mathbb{F}P^n$ for the quotient map and set $i_{\mathbb{F}} = i_{1,n}\colon \mathbb{F}P^1 = \mathbb{S}^d \hookrightarrow \mathbb{F}P^n$. Let EX be the suspension of a space X and denote by $E\colon \pi_m(X) \to \pi_{m+1}(EX)$ the suspension homomorphism. Next, write $\Delta = \Delta_{\mathbb{F}P}\colon \pi_m(\mathbb{F}P^n) \to \pi_{m-1}(\mathbb{S}^{d-1})$ for the connecting map. By [3] (Theorem (2.1)) it holds:

$$\Delta(i_{\mathbb{F}*}E) = \mathrm{id}_{\pi_{m-1}(\mathbb{S}^{d-1})}$$

and

$$\pi_m(\mathbb{F}P^n) = \gamma_{n*}\pi_m(\mathbb{S}^{d(n+1)-1}) \oplus i_{\mathbb{F}*}E\pi_{m-1}(\mathbb{S}^{d-1}).$$

Hence, $\pi_m(\mathbb{R}P^1) \cong \pi_m(\mathbb{S}^1)$ and $\pi_m(\mathbb{C}P^1) \cong \pi_m(\mathbb{S}^2)$ for $m \geq 0$. Further, for $n > 1$, we derive

$$\pi_m(\mathbb{R}P^n) = \begin{cases} 0, & \text{if } m = 0, \\ \mathbb{Z}_2, & \text{if } m = 1, \\ \gamma_{n*}\pi_m(\mathbb{S}^n), & \text{if } m > 1, \end{cases}$$

and

$$\pi_m(\mathbb{C}P^n) = \begin{cases} 0, & \text{if } m = 0, 1, \\ \mathbb{Z}, & \text{if } m = 2, \\ \gamma_{n*}\pi_m(\mathbb{S}^{2n+1}), & \text{if } m > 2. \end{cases}$$

The path-connected components of $M(\mathbb{S}^m, \mathbb{F}P^n)$ are in one-to-one correspondence with the set $[\mathbb{S}^m, \mathbb{F}P^n]$ of (free) homotopy classes. Because $\mathbb{C}P^n$ and $\mathbb{H}P^n$ are 1-connected, $[\mathbb{S}^m, \mathbb{R}P^n] \cong \cong \pi_m(\mathbb{R}P^n)/\pi_1(\mathbb{R}P^n)$ and $[\mathbb{S}^m, \mathbb{C}P^n] \cong \pi_m(\mathbb{C}P^n)$, $[\mathbb{S}^m, \mathbb{H}P^n] \cong \pi_m(\mathbb{H}P^n)$.

By [2] (Corollary (7.4)) and [3] ((4.1)–(4.3)), we obtain a formula:

Lemma 3.1. *Let $h_0\alpha \in \pi_m(\mathbb{S}^{2n-1})$ be the 0-th Hopf–Hilton invariant for $\alpha \in \pi_m(\mathbb{S}^n)$. Then*

$$[\gamma_n\alpha, i_{\mathbb{R}}] = \begin{cases} 0 & \textit{for odd } n; \\ (-1)^m\gamma_n(-2\alpha + [\iota_n, \iota_n] \circ h_0\alpha) & \textit{for even } n. \end{cases}$$

Let $\tau_\eta(\xi) \in \pi_m(X)$ be the operation of $\eta \in \pi_1(X)$ on $\xi \in \pi_m(X)$. Then, in view of [21] (Chapter X, (7.6)), it holds

$$[\xi, \eta] = (-1)^m(\tau_\eta(\xi) - \xi).$$

Hence, by Lemma 3.1, the action of $\pi_1(\mathbb{R}P^n)$ on $\pi_m(\mathbb{R}P^n)$ is trivial for odd n and we get $[\mathbb{S}^m, \mathbb{R}P^n] \cong \pi_m(\mathbb{R}P^n) = \gamma_{n*}\pi_m(\mathbb{S}^n)$. Further, the map $\gamma_n\colon \mathbb{S}^{(n+1)d-1} \to \mathbb{F}P^n$ leads to commutative diagrams of surjective maps

$$\begin{array}{ccc} \pi_m(\mathbb{S}^n)/\pm G_m(\mathbb{S}^n) & \longrightarrow & \{M_\alpha(\mathbb{S}^m, \mathbb{S}^n);\ \alpha \in \pi_m(\mathbb{S}^n)\}/\simeq \\ \downarrow & & \downarrow \\ \pi_m(\mathbb{R}P^n)/\pm\gamma_{n*}G_m(\mathbb{S}^n) & \longrightarrow & \{M_\alpha(\mathbb{S}^m, \mathbb{R}P^n);\ \alpha \in \pi_m(\mathbb{R}P^n)\}/\pi_1(\mathbb{R}P^n)/\simeq \end{array}$$

and

$$\begin{array}{ccc} \pi_m(\mathbb{S}^{2n+1})/\pm G_m(\mathbb{S}^{2n+1}) & \longrightarrow & \{M_\alpha(\mathbb{S}^m, \mathbb{S}^{2n+1});\ \alpha \in \pi_m(\mathbb{S}^{2n+1})\}/\simeq \\ \downarrow & & \downarrow \\ \pi_m(\mathbb{C}P^n)/\pm\gamma_{n*}G_m(\mathbb{S}^{2n+1}) & \longrightarrow & \{M_\alpha(\mathbb{S}^m, \mathbb{C}P^n);\ \alpha \in \pi_m(\mathbb{C}P^n)\}/\simeq . \end{array}$$

Further, $\pi_m(\mathbb{H}P^n) = \gamma_{n*}\pi_m(\mathbb{S}^{4n+3}) \oplus i_{\mathbb{H}*}E\pi_{m-1}(\mathbb{S}^3)$. Because $G_m(\mathbb{S}^3) = \pi_m(\mathbb{S}^3)$, the path-components $M_\alpha(\mathbb{S}^m, \mathbb{H}P^n)$ for $\alpha \in i_{\mathbb{H}*}E\pi_{m-1}(\mathbb{S}^3)$ have the same homotopy type. This yields the next commutative diagram of surjective maps

$$\begin{array}{ccc} \pi_m(\mathbb{S}^{4n+3})/\pm G_m(\mathbb{S}^{4n+3}) & \longrightarrow & \{M_\alpha(\mathbb{S}^m, \mathbb{S}^{4n+3});\ \alpha \in \pi_m(\mathbb{S}^{4n+3})\}/\simeq \\ \downarrow & & \downarrow \\ \pi_m(\mathbb{H}P^n)/\pm\gamma_{n*}G_m(\mathbb{S}^{4n+3}) & \longrightarrow & \{M_\alpha(\mathbb{S}^m, \mathbb{H}P^n);\ \alpha \in \pi_m(\mathbb{H}P^n)\}/\simeq . \end{array}$$

Consequently, the main result presented in Section 2 leads to estimations of $|\{M_\alpha(\mathbb{S}^{(n+1)d-1+k}, \mathbb{F}P^n)\}/\simeq|$ for $k \leq 13$ and $\mathbb{F} = \mathbb{R},\mathbb{C},\mathbb{H}$. Then, the results [9] (Theorems 1, 2) and [10] (Theorem 2.3) lead also to:

***Remark* 3.1.** There are estimations of fibre-homotopy types of evaluation fibrations $\omega_\alpha\colon M_\alpha(\mathbb{S}^{(n+1)d-1+k},\mathbb{F}P^n) \to \mathbb{F}P^n$ and their strong fibre-homotpy types for $k \leq 13$ and $\mathbb{F} = \mathbb{R},\mathbb{C},\mathbb{H}$ as well.

Next, write $\mathbb{K}P^2 = \mathbb{S}^8 \cup_{\sigma_8} e^{16}$ for the Cayley projective plane and $i_{\mathbb{K}}\colon \mathbb{S}^8 \hookrightarrow \mathbb{K}P^2$ for the inclusion map, where $\sigma_8\colon \mathbb{S}^{15} \to \mathbb{S}^8$ is the Hopf map. Then, in view of [17], it holds $\pi_m(\mathbb{K}P^2) = i_{\mathbb{K}*}E\pi_{n-1}(\mathbb{S}^7) \cong \pi_{m-1}(\mathbb{S}^7)$ for $m \leq 21$. Because $G_m(\mathbb{S}^7) = \pi_m(\mathbb{S}^7)$, all path-components of $M(\mathbb{S}^m,\mathbb{K}P^2)$ have the same homotopy type for $m \leq 21$.

4. Miscellanea on mapping spaces. Homotopy properties of various path-components $M_\alpha(\mathbb{S}^m,\mathbb{S}^n)$ have been studied in [1, 14, 20] and then some homotopy groups $\pi_k(M_\alpha(\mathbb{S}^m,\mathbb{S}^n))$ computed. However, the rational type of $M(\mathbb{S}^m,\mathbb{S}^n)$ and $M(\mathbb{S}^m,\mathbb{S}^n)_*$ has been fully described in [4, 18] as follows:

Theorem 4.1. (i) *For n odd and any m*:

$$M(\mathbb{S}^m,\mathbb{S}^n) \cong_{\mathbb{Q}} \begin{cases} \mathbb{S}^n \times \mathcal{K}(\mathbb{Z}, n-m), & \textit{if } n > m,\\ \coprod_{k=1}^{\infty} \mathbb{S}^n, & \textit{if } n = m,\\ \mathbb{S}^n, & \textit{if } n < m, \end{cases}$$

$$M(\mathbb{S}^m,\mathbb{S}^n)_* \cong_{\mathbb{Q}} \begin{cases} \mathcal{K}(\mathbb{Z}, n-m), & \textit{if } n > m,\\ \coprod_{k=1}^{\infty} *, & \textit{if } n = m,\\ *, & \textit{if } n < m. \end{cases}$$

(ii) *For n even and any m*:

$$M(\mathbb{S}^m,\mathbb{S}^n) \cong_{\mathbb{Q}} \begin{cases} Y, & \textit{if } n > m,\\ \mathbb{S}^n \times \mathcal{K}(\mathbb{Z}, 2n-m-1)\coprod_{k=1}^{\infty} \mathbb{S}^{2n-1}, & \textit{if } n = m,\\ \mathbb{S}^n \times \mathcal{K}(\mathbb{Z}, 2n-m-1), & \textit{if } n < m < 2n-1,\\ \coprod_{k=1}^{\infty} \mathbb{S}^n, & \textit{if } m = 2n-1,\\ \mathbb{S}^n, & \textit{if } m > 2n-1, \end{cases}$$

where Y is given by the fibration $\mathbb{S}^n \times \mathcal{K}(\mathbb{Z}, n-m) \to Y \to \mathcal{K}(\mathbb{Z}, 2n-m-1)$;

$$M(\mathbb{S}^m, \mathbb{S}^n)_* \cong_{\mathbb{Q}} \begin{cases} \mathcal{K}(\mathbb{Z}, n-m) \times \mathcal{K}(\mathbb{Z}, 2n-m-1), & \textit{if } n > m, \\ \coprod_{k=1}^{\infty} \mathcal{K}(\mathbb{Z}, 2n-m-1), & \textit{if } n = m, \\ \mathcal{K}(\mathbb{Z}, 2n-m-1), & \textit{if } n < m < 2n-1, \\ \coprod_{k=1}^{\infty} *, & \textit{if } m = 2n-1, \\ *, & \textit{if } m > 2n-1. \end{cases}$$

Now, let $\mathbb{A}$ be an abelian group and $n \geq 1$. A space $\mathcal{M}(\mathbb{A}, n)$ such that

$$\tilde{H}_i(\mathcal{M}(\mathbb{A}, n)) = \begin{cases} \mathbb{A}, & \text{if } i = n, \\ 0, & \text{otherwise} \end{cases}$$

is called a *Moore space* of type $(\mathbb{A}, n)$. If $\mathbb{A} = \mathbb{Z}_k$ is a cyclic group of order k then such space can be constructed from the n-sphere $\mathbb{S}^n$ by attaching an $(n+1)$-cell e^{n+1} via a map $f\colon \mathbb{S}^n \to \mathbb{S}^n$ of degree k.

Proposition 4.1 ([12], Proposition 4H.2)**.** *For any $n > 1$, and any abelian group $\mathbb{A}$ and a pointed space X there are natural short exact sequences*

$$0 \to \operatorname{Ext}(\mathbb{A}, \pi_{n+1}(X)) \to [\mathcal{M}(\mathbb{A}, n), X]_* \to \operatorname{Hom}(\mathbb{A}, \pi_n(X)) \to 0. \tag{4.1}$$

Notice that for $\mathbb{A} = \mathbb{Z}_k$, we get

$$\operatorname{Ext}(\mathbb{Z}_k, \pi_{n+1}(X)) \cong \mathbb{Z}_k \otimes \pi_{n+1}(X) \cong \pi_{n+1}(X)/k\pi_{n+1}(X)$$

and

$$\operatorname{Hom}(\mathbb{Z}_k, \pi_n(X)) = {}_k\pi_n(X) = \{\alpha \in \pi_n(X); k\alpha = 0\}.$$

Hence, the sequence (4.1) leads to

$$0 \to \pi_{n+1}(X)/k\pi_{n+1}(X) \to [\mathcal{M}(\mathbb{Z}_k, n), X]_* \to {}_k\pi_n(X) \to 0,$$

which we use to compute $[\mathcal{M}(\mathbb{Z}_k, n), \mathbb{S}^m]_*$ (in fact $[\mathcal{M}(\mathbb{Z}_k, n), \mathbb{S}^m]$) for some m, n.

The case $m = 1$ is simple: if $n = 1$ then ${}_k\pi_n(\mathbb{S}^1) = 0$ and $\pi_{n+1}(\mathbb{S}^1) = \pi_n(\mathbb{S}^1) = 0$ for $n > 1$. Thus, we have $[\mathcal{M}(\mathbb{Z}_k, n), \mathbb{S}^1]_* = [\mathcal{M}(\mathbb{Z}_k, n), \mathbb{S}^1] = 0$.

From now on, we assume that $m > 1$. So, $\pi_1(\mathbb{S}^m) = 0$ and $[\mathcal{M}(\mathbb{Z}_k, n), \mathbb{S}^m]_* = [\mathcal{M}(\mathbb{Z}_k, n), \mathbb{S}^m]$.

***Case* 1.** If $n + 1 < m$, then $\pi_n(\mathbb{S}^m) = \pi_{n+1}(\mathbb{S}^m) = 0$. So, $[\mathcal{M}(\mathbb{Z}_k, n), \mathbb{S}^m] = 0$.

***Case* 2.** If $n + 1 = m$, then $\pi_{n+1}(\mathbb{S}^m) \cong \mathbb{Z}$ and $\pi_n(\mathbb{S}^m) = 0$ which imply that $[\mathcal{M}(\mathbb{Z}_k, n), \mathbb{S}^m] \cong \cong \mathbb{Z}_k$.

***Case* 3.** If $n + 1 > m$, then $n = m + l - 1$, for some $l > 0$. Now we study the short exact sequences below for $l > 0$

$$0 \to \pi_{m+l}(\mathbb{S}^m)/k\pi_{m+l}(\mathbb{S}^m) \to [\mathcal{M}(\mathbb{Z}_k, m+l-1), \mathbb{S}^m] \to {}_k\pi_{m+l-1}(\mathbb{S}^m) \to 0. \tag{4.2}$$

First, if $l = 1$, then ${}_k\pi_{m+l-1}(\mathbb{S}^m) = 0$ and we have to consider the cases $m = 2$ and $m > 2$ separately, since $\pi_3(\mathbb{S}^2) \cong \mathbb{Z}$ and $\pi_{m+1}(\mathbb{S}^m) \cong \mathbb{Z}_2$, respectively. More precisely,

$$[\mathcal{M}(\mathbb{Z}_k, m), \mathbb{S}^m] \cong \pi_{m+1}(\mathbb{S}^m)/k\pi_{m+1}(\mathbb{S}^m) \cong \begin{cases} \mathbb{Z}_k, & \text{if } m = 2, \\ \mathbb{Z}_2, & \text{if } m > 2 \text{ and } k \text{ is even}, \\ 0, & \text{if } m > 2 \text{ and } k \text{ is odd}. \end{cases}$$

Next, if $l = 2$, then $\pi_{m+l}(\mathbb{S}^m) \cong \mathbb{Z}_2$ and $\pi_{m+l-1}(\mathbb{S}^m) \cong \mathbb{Z}$ for $m = 2$ and $\pi_{m+l-1}(\mathbb{S}^m) \cong \mathbb{Z}_2$ for $m > 2$. If $m = 2$, then the sequence (4.2) yields

$$[\mathcal{M}(\mathbb{Z}_k, 3), \mathbb{S}^2] \cong \mathbb{Z}_2/k\mathbb{Z}_2 \cong \begin{cases} \mathbb{Z}_2, & \text{if } k \text{ is even}, \\ 0, & \text{if } k \text{ is odd}. \end{cases}$$

If $m > 2$, then (4.2) becomes $0 \to \mathbb{Z}_2/k\mathbb{Z}_2 \to [\mathcal{M}(\mathbb{Z}_k, m+1), \mathbb{S}^m] \to {}_k\mathbb{Z}_2 \to 0$ and if k is odd, then $[\mathcal{M}(\mathbb{Z}_k, m+1), \mathbb{S}^m] = 0$, while if k is even, then $0 \to \mathbb{Z}_2 \to [\mathcal{M}(\mathbb{Z}_k, m+1), \mathbb{S}^m] \to \mathbb{Z}_2 \to 0$. So, we get $|[\mathcal{M}(\mathbb{Z}_k, m+1), \mathbb{S}^m]| = 4$.

Further, if $l = 3$, then

$$\pi_{m+3}(\mathbb{S}^m) \cong \begin{cases} \mathbb{Z}_2, & \text{if } m = 2, \\ \mathbb{Z}_{12}, & \text{if } m = 3, \\ \mathbb{Z} \oplus \mathbb{Z}_{12}, & \text{if } m = 4, \\ \mathbb{Z}_{24}, & \text{if } m \geq 5, \end{cases}$$

and $\pi_{m+2}(\mathbb{S}^m) \cong \mathbb{Z}_2$. Since ${}_k\mathbb{Z}_2 = 0$ for any odd k, we obtain

$$[\mathcal{M}(\mathbb{Z}_k, m+2), \mathbb{S}^m] \cong \begin{cases} 0, & \text{if } m = 2, \\ \mathbb{Z}_4, & \text{if } m = 3 \text{ and } 3 \mid k, \\ 0, & \text{if } m = 3 \text{ and } 3 \nmid k, \\ (\mathbb{Z} \oplus \mathbb{Z}_{12})/k(\mathbb{Z} \oplus \mathbb{Z}_{12}), & \text{if } m = 4, \\ \mathbb{Z}_{24}/k\mathbb{Z}_{24}, & \text{if } m \geq 5. \end{cases}$$

If k is even, then ${}_k\mathbb{Z}_2 = \mathbb{Z}_2$ and in view of (4.2) we get

$$0 \to \pi_{m+3}(\mathbb{S}^m)/k\pi_{m+3}(\mathbb{S}^m) \to [\mathcal{M}(\mathbb{Z}_k, m+2), \mathbb{S}^m] \to \mathbb{Z}_2 \to 0$$

which leads to the value of $|[\mathcal{M}(\mathbb{Z}_k, m+2), \mathbb{S}^m]|$. Following the procedure above and using the homotopy groups $\pi_{m+l}(\mathbb{S}^m)$ (see, e.g., [19]), it is possible to determine $|[\mathcal{M}(\mathbb{Z}_k, m+l), \mathbb{S}^m]|$ for other values of $l > 3$ as well.

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