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A PURSUIT PROBLEM IN AN INFINITE SYSTEM OF SECOND-ORDER DIFFERENTIAL EQUATIONS

ПРОБЛЕМА ПЕРЕСЛІДУВАННЯ В НЕСКІНЧЕННІЙ СИСТЕМІ ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ ДРУГОГО ПОРЯДКУ

We study a pursuit differential game problem for an infinite system of second-order differential equations. The control functions of players, i. e., the pursuer and the evader, are subject to integral constraints.The pursuit is completed if $z(\tau) = \dot{z}(\tau) = 0$ at some $\tau > 0$, where $z(t)$ is the state of the system. The pursuer tries to complete the pursuit and the evader tries to avoid this. A sufficient condition is obtained for completing the pursuit in the differential game when control recourse of the pursuer greater than that of the evader. To construct the strategy of the pursuer we assume that the instantaneous control employed by the evader is known to the pursuer.

Вивчається проблема переслідування в диференціальній грі для нескінченної системи диференціальних рівнянь другого порядку. Керівні функції гравців, тобто переслідувача та переслідуваного, мають деякі обмеження. Переслідування завершується, коли $z(\tau) = \dot{z}(\tau) = 0$ для деякого $\tau > 0$, де $z(t)$ — стан системи. Переслідувач намагається завершити переслідування, а переслідуваний намагається цього уникнути. Встановлено достатню умову завершення переслідування в диференціальній грі, коли зворотне управління для переслідувача більше, ніж для переслідуваного. Для побудови стратегії переслідувача вважаємо, що миттєве керування, застосоване переслідуваним, є відомим переслідувачу.

1. Introduction. The study of two person zero-sum differential games was initiated by Isaacs [1]. Since then many works with various approaches have been done in developing the theory of differential games (see, for example, [1 – 4]). Control and differential game problems in systems with distributed parameters were studied by many researchers (see, for example, [5 – 15]).

Works [12 – 15] concerned with the differential game problems described by the following infinite system of differential equations:

$$\ddot{z}_k(t) + \mu_k z_k(t) = -u_k(t) + v_k(t), \tag{1}$$

where μ_k are positive numbers, u_k, $k = 1, 2, \ldots$, are control parameters of the pursuer, and v_k, $k = 1, 2, \ldots$, are those of the Evader.

In [15], the numbers μ_k are assumed to be any positive numbers, and the control functions of the players are subject to integral constraints. In [12], a differential game described by hyperbolic equation is reduced to that described by the infinite system of differential equations (1). Here the numbers μ_k are generalized eigenvalues of the elliptic operator

$$Az = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial z}{\partial x_j}\right)$$

and satisfy the conditions

$$0 < \mu_1 \le \mu_2 \le \ldots \to \infty.$$

Authors studied differential game problems with various constraints on control functions of players.

The general purpose of the present paper is to investigate differential game problem described by the following infinite system of differential equations:

$$\begin{aligned} \ddot{x_k} &= -\alpha_k x_k - \beta_k y_k - u_{k1} + v_{k1}, \qquad x_k(0) = x_{k0}, \qquad \dot{x_k}(0) = x_{k1}, \\ \ddot{y_k} &= \beta_k x_k - \alpha_k y_k - u_{k2} + v_{k2}, \qquad y_k(0) = y_{k0}, \qquad \dot{y_k}(0) = y_{k1}, \end{aligned} \qquad k = 1, 2, \ldots, \tag{2}$$

where α_k, β_k are real numbers, $x_k, y_k, x_{k0}, y_{k0}, x_{k1}, y_{k1}, u_{k1}, u_{k2}, v_{k1}, v_{k2} \in R^1$, $u = (u_{11}, u_{12}, u_{21}, u_{22}, \ldots)$ and $v = (v_{11}, v_{12}, v_{21}, v_{22}, \ldots)$ are control parameters of the pursuer and the evader respectively. Note that the system (2) is obtained if we take

$$z_k = x_k + iy_k, \qquad \mu_k = \alpha_k - i\beta_k, \qquad u_k = u_{k1} + iu_{k2}, \qquad v_k = v_{k2} + iv_{k2}$$

in (1). In other words, we deal with so-called the complex case of the equation (1).

Pursuit is said to be completed in the game described by the infinite system of differential equations (2) if $x_k(\tau) = 0$, $y_k(\tau) = 0$, $\dot{x}_k(\tau) = 0$, $\dot{y}_k(\tau) = 0$, $k = 1, 2, \ldots$ at some $\tau > 0$. In the literature, such differential game problems are called “soft” landing or “soft” capture problems.

In the case of finite dimensional space, a number of works on this subject have been published see, e. g., [16 – 18]. In the work [16], a “soft landing” game problem is studied, where the dynamics of the players models the motion of different-type objects in a medium with friction. The goal of the pursuer is the approach of geometric coordinates and the velocities of the players (soft landing) at a certain finite instant of time. Sufficient conditions on the parameters of the conflict-control process were obtained under which the “soft landing” problem is solvable at a finite time.

A game problem of pursuit of a controlled object moving in a horizontal plane, by another object, moving in a three-dimensional space, is studied in [17]. Sufficient conditions on parameters of a conflict-controlled object were derived, for which the soft landing may be performed.

In the paper [18], a “soft landing” differential game of many pursuers and one evader described by the generalized Pontryagin example is studied. By definition the evader is said to be captured if its state, velocity, and acceleration coincide with those of a pursuer. Under the assumption that the roots of characteristic equation are real, sufficient capture conditions were obtained in terms of initial states.

2. Statement of problem. Let $\lambda_1, \lambda_2, \ldots$ be a sequence of positive numbers, and r be a fixed number. We introduce into the consideration the space

$$l_r^2 = \left\{ \xi = (\xi_1, \xi_2, \ldots) \colon \sum_{i=1}^{\infty} \lambda_i^r \xi_i^2 < \infty \right\}$$

with the inner product and norm

$$(\xi, \eta) = \sum_{i=1}^{\infty} \lambda_i^r \xi_i \eta_i, \quad \xi, \eta \in l_r^2, \qquad \|\xi\|_{l_r^2} = \left(\sum_{i=1}^{\infty} \lambda_i^r \xi_i^2 \right)^{1/2}.$$

From now on $\lambda_k = \sqrt{\alpha_k^2 + \beta_k^2}$, $k = 1, 2, \ldots$.

Denote

$$z(t) = \big(z_1(t), z_2(t), \ldots\big), \qquad z_k(t) = \big(x_k(t), y_k(t)\big), \qquad |z_k| = \sqrt{x_k^2 + y_k^2},$$

$$\|z\|^2_{l^2_{r+1}} = \sum_{k=1}^{\infty} \lambda_k^{r+1}\big(x_k^2 + y_k^2\big),$$

$$z_0 = (z_{10}, z_{20}, \ldots) = (x_{10}, y_{10}, x_{20}, y_{20}, \ldots), \qquad z_{k0} = (x_{k0}, y_{k0}),$$

$$z_1 = (z_{11}, z_{21}, \ldots) = (x_{11}, y_{11}, x_{21}, y_{21}, \ldots), \qquad z_{k1} = (x_{k1}, y_{k1}).$$

We assume that $z_0 \in l^2_{r+1}$, $z_1 \in l^2_r$.

Let $L_2(0, T; l^2_r)$ be the space of functions $f(t) = (f_1(t), f_2(t), \ldots)$, $f\colon [0, T] \to l^2_r$, with measurable coordinates $f_k(t) = \big(f_{k1}(t), f_{k2}(t)\big)$, $0 \le t \le T$, such that

$$\big\|f(\cdot)\big\|_{L_2(0,T;l^2_r)} = \sum_{k=1}^{\infty} \lambda_k^r \int\limits_0^T \big(f_{k1}^2(t) + f_{k2}^2(t)\big)\, dt < \infty,$$

where T is any given positive number.

Let ρ_0, ρ, and σ be given positive numbers.

Definition 1. *A function $w(\cdot) \in L_2(0, T; l^2_r)$ subjected to the the condition*

$$\sum_{k=1}^{\infty} \lambda_k^r \int\limits_0^T \big(w_{1k}^2(s) + w_{2k}^2(s)\big)\, ds \le \rho_0^2$$

is referred to as the admissible control. We denote the set of all admissible controls by $S(\rho_0)$.

Definition 2. *A function $u(\cdot) \in S(\rho)$ (respectively $v(\cdot) \in S(\sigma)$) is referred to as the admissible control of the pursuer (the evader).*

Definition 3. *A function*

$$u(t, v) = (u_1(t, v), u_2(t, v), \ldots), \qquad u\colon [0, T] \times l^2_r \to l^2_r, \qquad u_k(t, v) = \big(u_{k1}(t, v), u_{k2}(t, v)\big),$$

of the form

$$u_k(t, v) = v_k(t) + \omega_k(t), \qquad \omega(\cdot) = \big(\omega_1(\cdot), \omega_2(\cdot), \ldots\big) \in S(\rho - \sigma), \qquad \omega_k(\cdot) = \big(\omega_{k1}(\cdot), \omega_{k2}(\cdot)\big),$$

where $v(\cdot) \in S(\sigma)$, is called a strategy of the pursuer.

Definition 4. *If there exists a strategy $u(\cdot)$ of the pursuer such that $z(\tau) = 0$, $\dot z(\tau) = 0$ at some τ, $0 \le \tau \le \vartheta$, for any control of the evader, then we say that differential game (2) can be completed for the time ϑ.*

The pursuer tries to complete the game as soon as possible while the aim of the evader is opposite.

Definition 5. *Let* $w(\cdot) = (w_1(\cdot), w_2(\cdot), \ldots) \in L_2(0, T, l_r^2)$, $w_k(\cdot) = \big(w_{k1}(\cdot), w_{k2}(\cdot)\big)$. *The function* $z(t) = \big(z_1(t), z_2(t), \ldots\big)$, $0 \le t \le T$, *where each coordinate* $z_k(t)$

1) *is continuously differentiable on* $(0, T)$ *and satisfies the initial conditions* $z_k(0) = z_{k0}$, $\dot z_k(0) = z_{k1}$,

2) *has the second derivative* $\ddot z_k(t)$ *almost everywhere on* $(0, T)$ *satisfying the equation*

$$\ddot z_k(t) = D_k z_k(t) + w_k(t), \qquad D_k = \begin{bmatrix} -\alpha_k & -\beta_k \\ \beta_k & -\alpha_k \end{bmatrix},$$

almost everywhere on $[0, T]$ *is called the solution of the system*

$$\ddot z_k(t) = D_k z_k(t) + w_k(t), \qquad z_k(0) = z_{k0}, \qquad \dot z_k(0) = z_{k1}, \quad k = 1, 2, \ldots. \tag{3}$$

Let

$$A_{k1}(t) = e^{r_{1k}t} \begin{bmatrix} \cos(r_{2k}t) & -\sin(r_{2k}t) \\ \sin(r_{2k}t) & \cos(r_{2k}t) \end{bmatrix}, \qquad A_{k2}(t) = A_{k1}(-t), \qquad R_k = \begin{bmatrix} r_{1k} & -r_{2k} \\ r_{2k} & r_{1k} \end{bmatrix},$$

$$A_k(t) = \frac{1}{2}\big(A_{k1}(t) + A_{k2}(t)\big), \qquad B_k(t) = \frac{1}{2} R_k^{-1}\big(A_{k1}(t) - A_{k2}(t)\big),$$

$$r_{1k} = \sqrt{\frac{-\alpha_k + \sqrt{\alpha_k^2 + \beta_k^2}}{2}}, \qquad r_{2k} = \sqrt{\frac{\alpha_k + \sqrt{\alpha_k^2 + \beta_k^2}}{2}}, \quad k = 1, 2, \ldots.$$

Clearly, $r_k = \sqrt{r_{1k}^2 + r_{2k}^2} = \sqrt[4]{\alpha_k^2 + \beta_k^2} = \sqrt{\lambda_k}$.

It can be shown that the matrices $A_{k1}(t)$, $A_{k2}(t)$ have the following properties:

$$A_{k1}(t+h) = A_{k1}(t)A_{k1}(h) = A_{k1}(h)A_{k1}(t), \qquad \big|A_{k1}(t)z_k\big| = \big|A_{k1}^*(t)z_k\big| = e^{r_{1k}t}|z_k|,$$

$$A_{k2}(t+h) = A_{k2}(t)A_{k2}(h) = A_{k2}(h)A_{k2}(t), \qquad \big|A_{k2}(t)z_k\big| = \big|A_{k2}^*(t)z_k\big| = e^{-r_{1k}t}|z_k|,$$

where A^* denotes the transpose of the matrix A, and E_2 does the identity (2×2)-matrix.

It is easy to verify that

$$\dot A_{k1}(t) = R_k A_{k1}(t), \qquad \dot A_{k2}(t) = -R_k A_{k2}(t), \qquad \dot A_k(t) = R_k^2 B_k(t), \qquad \dot B_k(t) = A_k(t).$$

By using the properties

$$A_{k1}(t+h) = A_{k1}(t)A_{k1}(h), \qquad A_{k2}(t+h) = A_{k2}(t)A_{k2}(h),$$

$$A_{k1}(t)A_{k2}(h) = A_{k1}(t-h), \qquad A_{k1}(t) = A_{k2}(-t)$$

it can be easily proved that

$$A_k^2(t) - R_k^2 B_k^2(t) = E_2, \tag{4}$$

$$A_k(t)B_k(t-s) - B_k(t)A_k(t-s) = -B_k(s), \tag{5}$$

$$A_k(t)A_k(t-s) - R_k^2 B_k(t)B_k(t-s) = A_k(s). \tag{6}$$

3. Control problem. Consider the infinite system of differential equations (3). Let $C(0,T;l_r^2)$ be the space of continuous functions $z(t)$, $0 \le t \le T$, with the values in the space l_r^2. It can be shown similarly to [19] that the following assertion is true.

Assertion. If $\{r_{1k}\}_{k\in N}$ is a bounded above sequence, then the infinite system of differential equations (3) has a unique solution $z(\cdot) \in C(0,T;l_{r+1}^2)$ defined by

$$z_k(t) \doteq A_k(t)z_{k0} + B_k(t)z_{k1} + \int\limits_0^t B_k(t-s)w_k(s)ds, \quad k = 1,2,\ldots. \tag{7}$$

It can be verified that

$$\dot z_k(t) = R_k^2 B_k(t)z_{k0} + A_k(t)z_{k1} + \int\limits_0^t A_k(t-s)w_k(s)ds.$$

We transform the system

$$\begin{aligned} z_k(t) &= A_k(t)z_{k0} + B_k(t)z_{k1} + \int\limits_0^t B_k(t-s)w_k(s)ds, \\ &\qquad\qquad\qquad\qquad k = 1,2,\ldots, \\ \dot z_k(t) &= R_k^2 B_k(t)z_{k0} + A_k(t)z_{k1} + \int\limits_0^t A_k(t-s)w_k(s)ds, \end{aligned} \tag{8}$$

by setting

$$\begin{bmatrix} \eta_k(t) \\ \xi_k(t) \end{bmatrix} = \begin{bmatrix} A_k(t) & -R_kB_k(t) \\ -R_kB_k(t) & A_k(t) \end{bmatrix} \begin{bmatrix} R_kz_k(t) \\ \dot z_k(t) \end{bmatrix}, \qquad \begin{bmatrix} \eta_{k0} \\ \xi_{k0} \end{bmatrix} = \begin{bmatrix} R_kz_{k0} \\ z_{k1} \end{bmatrix}. \tag{9}$$

Then using (4), (5) and (6), we obtain

$$\eta_k(t) = R_kA_k(t)z_k(t) - R_kB_k(t)\dot z_k(t) =$$

$$= R_k\big(A_k^2(t) - R_k^2B_k^2(t)\big)z_{k0} + R_k\big(A_k(t)B_k(t) - B_k(t)A_k(t)\big)z_{k1} +$$

$$+ \int\limits_0^t R_k\left(A_k(t)B_k(t-s) - B_k(t)A_k(t-s)\right)w_k(s)ds =$$

$$= R_kz_{k0} - \int\limits_0^t R_kB_k(s)w_k(s)ds = \eta_{k0} - \int\limits_0^t R_kB_k(s)w_k(s)ds,$$

$$\xi_k(t) = -R_k^2B_k(t)z_k(t) + A_k(t)\dot z_k(t) =$$

$$= \left(-R_k^2 B_k(t) A_k(t) + A_k(t) R_k^2 B_k(t)\right) z_{k0} + \left(-R_k^2 B_k^2(t) + A_k^2(t)\right) z_{k1} +$$

$$+ \int_0^t \left(-R_k^2 B_k(t) B_k(t-s) + A_k(t) A_k(t-s)\right) w_k(s) ds =$$

$$= z_{k1} + \int_0^t A_k(s) w_k(s) ds = \xi_{k0} + \int_0^t A_k(s) w_k(s) ds.$$

Our goal is to realize $\eta_k(t) = 0$, $\xi_k(t) = 0$, for all $k = 1, 2, \ldots$, at some time t. They are equivalent to

$$\begin{aligned} \eta_{k0} &= \int_0^t R_k B_k(s) w_k(s) ds, \\ -\xi_{k0} &= \int_0^t A_k(s) w_k(s) ds, \end{aligned} \qquad k = 1, 2, \ldots. \tag{10}$$

We shall find a condition on η_{k0}, ξ_{k0}, $k = 1, 2, \ldots$, to be found a control $\big(w_1(\cdot), w_2(\cdot), \ldots\big) \in$ $\in S(\sigma)$ guaranteeing (10). To this end we study some properties of the set

$$X_k(\vartheta, \sigma_k) = \left\{ (\eta, \xi) \ \mid \ \eta = \int_0^\vartheta R_k B_k(s) w_k(s) ds, \qquad \xi = \int_0^\vartheta A_k(s) w_k(s) ds, \ w_k(.) \in S_\vartheta(\sigma_k) \right\},$$

where

$$\sum_{k=1}^\infty \sigma_k^2 = \sigma^2, \quad \sigma_k \ge 0 \qquad \text{and} \qquad S_\vartheta(\sigma_k) = \left\{ w_k(\cdot) \,\middle|\, \int_0^\vartheta \left|w_k(s)\right|^2 ds \le \sigma_k^2 \right\}.$$

Let $\psi_k \in X_k(\vartheta, \sigma_k)$ be any point and $e \in R^4$ be a unit vector. We find a control $w_k(\cdot)$ for that $\psi_k = (\eta_k, \xi_k)$ belongs to the boundary of $X_k(\vartheta, \sigma_k)$. By using the Cauchy – Schwartz inequality we get

$$\langle \psi_k, \ e \rangle = \int_0^\vartheta \left\langle \begin{pmatrix} R_k B_k(s) \\ A_k(s) \end{pmatrix} w_k(s), \ e \right\rangle ds =$$

$$= \int_0^\vartheta \langle C_k(s) w_k(s), \ e \rangle \, ds = \int_0^\vartheta \langle w_k(s), \ C_k^*(s) e \rangle \, ds \le$$

$$\le \sigma_k \left(\int_0^\vartheta |C_k^*(s) e|^2 ds \right)^{1/2} = \sigma_k F_k^{1/2}(\vartheta, e),$$

where

$$C_k(s) = \begin{pmatrix} R_k B_k(s) \\ A_k(s) \end{pmatrix}, \qquad F_k(\vartheta, e) = \int_0^\vartheta \left| C_k^*(s)e \right|^2 ds.$$

Note that the equality occurs if

$$w_k(s) = \frac{\sigma_k}{\sqrt{F_k(\vartheta, e)}}\, C_k^*(s)e, \tag{11}$$

almost everywhere on $[0, \vartheta]$. It can be shown that the point ψ_k is on the boundary $\partial X_k(\vartheta, \sigma_k)$ of the set $X_k(\vartheta, \sigma_k)$ whenever the control $w_k(\cdot)$ has the form (11).

We obtain

$$F_k(\vartheta, e) = \int_0^\vartheta \left| C_k^*(s)e \right|^2 ds = \left\langle e, \int_0^\vartheta C_k(s) C_k^*(s) e\, ds \right\rangle = \langle e, P_k(\vartheta)e \rangle, \tag{12}$$

where

$$P_k(\vartheta) = \int_0^\vartheta C_k(s) C_k^*(s) ds = \begin{bmatrix} c_{1k} & 0 & c_{2k} & c_{3k} \\ 0 & c_{1k} & c_{3k} & c_{2k} \\ c_{2k} & c_{3k} & c_{4k} & 0 \\ -c_{3k} & c_{2k} & 0 & c_{4k} \end{bmatrix},$$

$$c_{1k} = \frac{1}{4r_{1k}} \sinh(2r_{1k}\vartheta) - \frac{1}{4r_{2k}} \sin(2r_{2k}\vartheta), \qquad c_{2k} = \frac{1}{2r_{1k}} \sinh^2(r_{1k}\vartheta),$$

$$c_{3k} = \frac{1}{2r_{2k}} \sin^2(r_{2k}\vartheta), \qquad c_{4k} = \frac{1}{4r_{1k}} \sinh(2r_{1k}\vartheta) + \frac{1}{4r_{2k}} \sin(2r_{2k}\vartheta).$$

We study now some properties of eigenvalues and eigenvectors of the matrix $P_k(\vartheta)$. It is not difficult to verify that the eigenvalues of the matrix $P_k(\vartheta)$ are

$$m_1(\vartheta) = m_2(\vartheta) = \frac{1}{4r_{1k}} \sinh(2r_{1k}\vartheta) - \sqrt{\frac{1}{4r_{1k}^2} \sinh^4(r_{1k}\vartheta) + \frac{1}{4r_{2k}^2} \sin^4(r_{2k}\vartheta)},$$

$$m_3(\vartheta) = m_4(\vartheta) = \frac{1}{4r_{1k}} \sinh(2r_{1k}\vartheta) + \sqrt{\frac{1}{4r_{1k}^2} \sinh^4(r_{1k}\vartheta) + \frac{1}{4r_{2k}^2} \sin^4(r_{2k}\vartheta)}.$$

Eigenvectors associated with these eigenvalues are

$$e_1(\vartheta) = \begin{bmatrix} \dfrac{c_{2k} m_1(\vartheta) - c_{2k} c_{4k}}{c_{2k}^2 + c_{3k}^2} \\ \dfrac{c_{3k} m_1(\vartheta) - c_{3k} c_{4k}}{c_{2k}^2 + c_{3k}^2} \\ 1 \\ 0 \end{bmatrix}, \qquad e_2(\vartheta) = \begin{bmatrix} \dfrac{-c_{3k} m_1(\vartheta) + c_{3k} c_{4k}}{c_{2k}^2 + c_{3k}^2} \\ \dfrac{c_{2k} m_1(\vartheta) - c_{2k} c_{4k}}{c_{2k}^2 + c_{3k}^2} \\ 0 \\ 1 \end{bmatrix},$$

$$e_3(\vartheta) = \begin{bmatrix} \dfrac{c_{2k}m_3(\vartheta) - c_{2k}c_{4k}}{c_{2k}^2 + c_{3k}^2} \\ \dfrac{c_{3k}m_3(\vartheta) - c_{3k}c_{4k}}{c_{2k}^2 + c_{3k}^2} \\ 1 \\ 0 \end{bmatrix}, \qquad e_4(\vartheta) = \begin{bmatrix} \dfrac{-c_{3k}m_3(\vartheta) + c_{3k}c_{4k}}{c_{2k}^2 + c_{3k}^2} \\ \dfrac{c_{2k}m_3(\vartheta) - c_{2k}c_{4k}}{c_{2k}^2 + c_{3k}^2} \\ 0 \\ 1 \end{bmatrix}.$$

Hence, $P_k(\vartheta)e_i(\vartheta) = m_i(\vartheta)e_i(\vartheta),\ i = 1, 2, 3, 4$. Note that $e_i(\vartheta),\ i = 1, 2, 3, 4$, is an orthonormal system in R^4.

***Property* 1.** *The eigenvalues of the matrix $P_k(\vartheta)$ are positive for all $\vartheta > 0$.*

Proof. It is sufficient to show that $m_1(\vartheta) > 0$ for all $\vartheta > 0$.

We have

$$m_1(\vartheta)m_3(\vartheta) = g(\vartheta)h(\vartheta),$$

where

$$g(\vartheta) = \frac{1}{2r_1}\sinh(r_1\vartheta) - \frac{1}{2r_2}\sin^2(r_2\vartheta), \qquad h(\vartheta) = \frac{1}{2r_1}\sinh(r_1\vartheta) + \frac{1}{2r_2}\sin^2(r_2\vartheta).$$

It is obvious that

$$g'(\vartheta) = \frac{1}{2}\left(\cosh(r_1\vartheta) - \sin(2r_2\vartheta)\right) > 0, \quad \vartheta > 0,$$

since $\cosh(t) > 1 \geq \sin(t)$ for all $t > 0$. As $g(0) = 0,\ g'(\vartheta) > 0,\ \vartheta > 0$, then $g(\vartheta) > 0,\ \vartheta > 0$. Since $g(\vartheta) > 0,\ m_3(\vartheta) > 0$, and $h(\vartheta) > 0$ for all $\vartheta > 0$, we obtain $m_1(\vartheta) > 0$.

***Property* 2.** *The set $\partial X_k(\vartheta, \sigma_k)$, $k \in \{1, 2, \ldots\}$, is an ellipsoid in R^4.*

Proof. Let $\psi_k \in \partial X_k(\vartheta, \sigma_k)$ and $e(\vartheta) = \sum_{i=1}^{4} d_i e_i(\vartheta)$, where the numbers d_i satisfy the condition $\sum_{i=1}^{4} d_i^2 = 1$. By (11) we obtain

$$\psi_k = \int_0^\vartheta C_k(s)w_k(s)ds = \frac{\sigma_k}{\sqrt{F_k(\vartheta, e(\vartheta))}}\left(\int_0^\vartheta C_k(s)C_k^*(s)ds\right)e(\vartheta) = \frac{\sigma_k}{\sqrt{F_k(\vartheta, e(\vartheta))}}P_k(\vartheta)e(\vartheta).$$

It follows from (12) that

$$F_k(\vartheta, e(\vartheta)) = \langle\, e(\vartheta), P_k(\vartheta)e(\vartheta)\rangle = \left\langle \sum_{i=1}^{4} d_i e_i(\vartheta),\ \sum_{i=1}^{4} m_i(\vartheta)d_i e_i(\vartheta) \right\rangle = \sum_{i=1}^{4} m_i(\vartheta)d_i^2. \qquad (13)$$

Hence

$$\psi_k = \frac{\sigma_k}{\sqrt{F_k(\vartheta, e(\vartheta))}}\left(\sum_{i=1}^{4} m_i(\vartheta)d_i e_i(\vartheta)\right).$$

Let $\psi_{ik}(\vartheta) = \langle \psi_k, e_i(\vartheta)\rangle$, $i = 1, 2, 3, 4$. Then

$$\psi_{ik}(\vartheta) = \frac{\sigma_k m_i(\vartheta) d_i}{\sqrt{F_k(\vartheta, e(\vartheta))}}.$$

Combining this formula with (13) we conclude that

$$\sum_{i=1}^{4} \left(\frac{\psi_{ik}(\vartheta)}{\sigma_k \sqrt{m_i(\vartheta)}} \right)^2 = \frac{\sum_{i=1}^{4} m_i(\vartheta) d_i^2}{F_k(\vartheta, e(\vartheta))} = 1, \tag{14}$$

and hence

$$\partial X_k(\vartheta, \sigma_k) = \left\{ \psi_k \;\middle|\; \sum_{i=1}^{4} \frac{\psi_{ik}^2(\vartheta)}{\sigma_k^2 m_i(\vartheta)} = 1, \; \psi_k = \sum_{i=1}^{4} \psi_{ik}(\vartheta) e_i(\vartheta) \right\}$$

so it is an ellipsoid.

***Property* 3.** *The eigenvalue $m_1(\vartheta)$ is bounded above.*

Proof. It follows from the boundedness of the limit

$$\lim_{\vartheta \to \infty} m_1(\vartheta) = \lim_{\vartheta \to \infty} \left(\frac{1}{4r_1} \sinh(2r_1\vartheta) - \frac{1}{2} \sqrt{\frac{1}{r_1^2} \sinh^4(r_1\vartheta) + \frac{1}{r_2^2} \sin^4(r_2\vartheta)} \right) = \frac{1}{4r_1}$$

and the fact that $m_1(\vartheta)$ is a continuous function of ϑ.

Proposition 1. *If $0 \le \vartheta_1 < \vartheta_2$, then*

$$X(\vartheta_1, \sigma) \subset X(\vartheta_2, \sigma), \qquad X(\vartheta, \sigma) = \bigcup_{(\sigma_1, \sigma_2, \ldots)} \prod_{k=1}^{\infty} X_k(\vartheta, \sigma_k),$$

where union is taken over all the sequences

$$(\sigma_1, \sigma_2, \ldots), \quad \sigma_i \ge 0, \quad i = 1, 2, \ldots, \qquad \sum_{k=1}^{\infty} \sigma_k^2 = \sigma^2.$$

Proof. Assume that $(\eta, \xi) = (\eta_1, \xi_1, \eta_2, \xi_2, \ldots) \in X(\vartheta_1, \sigma)$. Then there exists an admissible control $w(\cdot) = (w_1(\cdot), w_2(\cdot), \ldots)$, $w_k(\cdot) \in S_{\vartheta_1}(\sigma_k)$ such that

$$(\eta_k, \xi_k) = \left(\int_0^{\vartheta_1} R_k B_k(s) w_k(s) ds, \int_0^{\vartheta_1} A_k(s) w_k(s) ds \right) \in X_k(\vartheta_1, \sigma_k), \quad k = 1, 2, \ldots.$$

Now define a new control $\widetilde{w}(\cdot) = \big(\widetilde{w}_1(\cdot), \widetilde{w}_2(\cdot), \ldots\big)$ as follows:

$$\widetilde{w}_k(t) = \begin{cases} w_k(t), & 0 \le t \le \vartheta_1, \\ 0, & \vartheta_1 < t \le \vartheta_2. \end{cases}$$

It is obvious that $\widetilde{w}_k(\cdot) \in S_{\vartheta_2}(\sigma_k)$ and

$$(\eta_k, \xi_k) = \left(\int_0^{\vartheta_1} R_k B_k(s) w_k(s) ds, \int_0^{\vartheta_1} A_k(s) w_k(s) ds \right) =$$

$$= \left(\int_0^{\vartheta_2} R_k B_k(s) \widetilde{w}_k(s) ds, \int_0^{\vartheta_2} A_k(s) \widetilde{w}_k(s) ds \right) \in X_k(\vartheta_2, \sigma_k).$$

Hence $X(\vartheta_1, \sigma) \subset X(\vartheta_2, \sigma)$.

Proposition 2. *Let* $\zeta = (0, 0, \zeta_3, \zeta_4) \in R^4$ *satisfy* $\zeta_3^2 + \zeta_4^2 > M$. *Then* $\zeta \notin X_k(\vartheta, \sigma_k)$, $k = 1, 2, \ldots,$ *for all* $\vartheta \geq 0$, *where* $M = \sup_{\vartheta \geq 0} \sigma^2 m_1(\vartheta)$.

Proof. Since

$$\zeta_1(\vartheta) = \langle \zeta,\ e_1(\vartheta) \rangle = \zeta_3, \qquad \zeta_2(\vartheta) = \langle \zeta,\ e_2(\vartheta) \rangle = \zeta_4,$$

then

$$\zeta_1^2(\vartheta) + \zeta_2^2(\vartheta) = \zeta_3^2 + \zeta_4^2 > M.$$

Hence for all $\vartheta \geq 0$ $\big($recall that $m_1(\vartheta) = m_2(\vartheta)$ and $m_3(\vartheta) = m_4(\vartheta)\big)$

$$\sum_{i=1}^{4} \frac{\zeta_i^2(\vartheta)}{\sigma_k^2 m_i} = \frac{\zeta_1^2(\vartheta) + \zeta_2^2(\vartheta)}{\sigma_k^2 m_1} + \frac{\zeta_3^2(\vartheta) + \zeta_4^2(\vartheta)}{\sigma_k^2 m_3} \geq \frac{\zeta_3^2 + \zeta_4^2}{M} + \frac{\zeta_3^2(\vartheta) + \zeta_4^2(\vartheta)}{\sigma_k^2 m_3} > 1.$$

Thus $\zeta \notin X_k(\vartheta, \sigma_k)$. In other words, the point ζ can not be steered into the origin.

Theorem 1. *Let* $\psi_0 = (\psi_{10}, \psi_{20}, \ldots)$, $\psi_{k0} = (\eta_{k0}, -\xi_{k0})$, $k = 1, 2, \ldots$. *If*

$$\psi_0 \in X(\vartheta, \sigma) = \bigcup_{(\sigma_1, \sigma_2, \ldots)} \prod_{k=1}^{\infty} X_k(\vartheta, \sigma_k),$$

where union is taken over all the sequences $(\sigma_1, \sigma_2, \ldots)$, $\sigma_i \geq 0$, $i = 1, 2, \ldots,$ $\sum_{k=1}^{\infty} \sigma_k^2 = \sigma^2$. *Then there exists a control* $w(t) = \big(w_1(t), w_2(t), \ldots\big)$, $0 \leq t \leq \vartheta$, *such that* $z(\vartheta) = \dot{z}(\vartheta) = 0$, *for the state* $z(t)$ *of the system* (3).

Proof. Since $\psi_0 = (\psi_{10}, \psi_{20}, \ldots) \in X(\vartheta, \sigma)$, then there exists a sequence $(\sigma_1, \sigma_2, \ldots)$, $\sigma_i \geq 0$, $i = 1, 2, \ldots,$ $\sum_{k=1}^{\infty} \sigma_k^2 = \sigma^2$ such that $\psi_{k0} \in X_k(\vartheta, \sigma_k)$, $k = 1, 2, \ldots$. Hence

$$\psi_{k0} = (\eta_{k0}, -\xi_{k0}) = \left(\int_0^{\vartheta} R_k B_k(s) w_k(s) ds, \int_0^{\vartheta} A_k(s) w_k(s) ds \right)$$

for some $w_k(\cdot) \in S_\vartheta(\sigma_k)$. This means

$$\eta_k(\vartheta) = \eta_{k0} - \int_0^{\vartheta} R_k B_k(s) w_k(s) ds = 0,$$

$$\xi_k(\vartheta) = \xi_{k0} + \int_0^{\vartheta} A_k(s) w_k(s) ds = 0.$$

Then from (8) and (9) we get $z_k(\vartheta) = \dot{z}_k(\vartheta) = 0$.

Theorem 1 is proved.

4. Pursuit differential game. In this section, we study the differential game (2). The following theorem is true.

Theorem 2. *Let*

$$\psi_0 \in X(\vartheta, \rho - \sigma) = \bigcup_{(\sigma_1,\sigma_2,\ldots)} \prod_{k=1}^{\infty} X_k(\vartheta, \sigma_k),$$

where union is taken over all the sequences $(\sigma_1, \sigma_2, \ldots)$, $\sigma_i \geq 0$, $i = 1, 2, \ldots$, $\sum_{k=1}^{\infty} \sigma_k^2 = (\rho - \sigma)^2$. *Then pursuit can be completed from the position* ψ_0 *for the time* ϑ.

Proof. As $\psi_0 \in X(\vartheta, \rho - \sigma)$, then there exists a sequence $(\sigma_1, \sigma_2, \ldots)$, $\sigma_i \geq 0$, $i = 1, 2, \ldots$, $\sum_{k=1}^{\infty} \sigma_k^2 = (\rho - \sigma)^2$ such that $\psi_{k0} \in X_k(\vartheta, \sigma_k)$, $k = 1, 2, \ldots$, for some θ. It follows from Theorem 1 that there exists a control

$$w^0(t) = \left(w_1^0(t), w_2^0(t), \ldots\right), \quad 0 \leq t \leq \vartheta, \qquad \sum_{k=1}^{\infty} \lambda_k^r \int_0^T \left|w_k^0(s)\right|^2 ds \leq (\rho - \sigma)^2,$$

such that $z(\vartheta) = \dot{z}(\vartheta) = 0$ in (3). We show that pursuit can be completed for the time ϑ. To this end we offer to the pursuer the following strategy:

$$u_k(t, v) = v_k(t) - w_k^0(t), \quad k = 1, 2, \ldots, \tag{15}$$

where $v(\cdot)$ is any admissible control of the evader. Then it is clear that $z(\vartheta) = \dot{z}(\vartheta) = 0$ for the system (2) (see the proof of Theorem 1).

What is left is to show the admissibility of the strategy (15). It can be shown by using the Minkowski inequality as follows:

$$\left(\sum_{k=1}^{\infty} \lambda_k^r \int_0^{\vartheta} |u_k(t, v)|^2 dt\right)^{1/2} = \left(\sum_{k=1}^{\infty} \lambda_k^r \int_0^{\vartheta} |v_k(t) - w_k^0(t)|^2 dt\right)^{1/2} \leq$$

$$\leq \left(\sum_{k=1}^{\infty} \lambda_k^r \int_0^{\vartheta} |v_k(t)|^2 dt\right)^{1/2} + \left(\sum_{k=1}^{\infty} \lambda_k^r \int_0^{\vartheta} |w_k^0(t)|^2 dt\right)^{1/2} \leq \sigma + \rho - \sigma = \rho.$$

Theorem 2 is proved.

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