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q-APOSTOL – EULER POLYNOMIALS AND q-ALTERNATING SUMS *

q-ПОЛІНОМИ АПОСТОЛА – ЕЙЛЕРА ТА q-ЗНАКОЗМІННІ СУМИ

We establish the basic properties and generating functions of the q-Apostol – Euler polynomials. We define q-alternating sums and obtain q-extensions of some formulas in [*Integral Transform. Spec. Funct.* – 2009. – **20**. – P. 377 – 391]. We also deduce an explicit relationship between the q-Apostol – Euler polynomials and the q-Hurwitz – Lerch zeta-function.

Встановлено основні властивості та твірні функції q-поліномів Апостола – Ейлера. Визначено q-знакозмінні суми та отримано q-продовження деяких формул з [*Integral Transform. Spec. Funct.* – 2009. – **20**. – P. 377 – 391]. Виведено також явне співвідношення між q-поліномами Апостола – Ейлера і q-дзета-функцією Хурвіца – Лерча.

1. Introduction and definitions. Throughout this paper, we always use the following notation: $\mathbb{N} = \{1, 2, 3, \ldots\}$ denotes the set of natural numbers, $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$ denotes the set of nonnegative integers, $\mathbb{Z}_0^- = \{0, -1, -2, -3, \ldots\}$ denotes the set of nonpositive integers, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{C}$ denotes the set of complex numbers.

The q-shifted factorial are defined by

$$(a;q)_0 = 1,\ (a;q)_k = (1-a)(1-aq)\ldots(1-aq^{k-1}), \qquad k = 1, 2, \ldots,$$

$$(a;q)_\infty = (1-a)(1-aq)\ldots(1-aq^k)\ldots = \prod_{k=0}^{\infty}(1-aq^k).$$

The q-numbers are defined by $[a]_q = \dfrac{1-q^a}{1-q}, \quad q \neq 1.$

The above q-standard notation can be found in Gasper [12, p. 7].

The classical Bernoulli polynomials and Euler polynomials are defined by means of the following generating functions (see, e.g., [1, p. 804 – 806], [11] or [25, p. 25 – 32]):

$$\frac{ze^{xz}}{e^z-1} = \sum_{n=0}^{\infty} B_n(x)\frac{z^n}{n!}, \qquad |z| < 2\pi, \tag{1.1}$$

and

$$\frac{2e^{xz}}{e^z+1} = \sum_{n=0}^{\infty} E_n(x)\frac{z^n}{n!}, \qquad |z| < \pi, \tag{1.2}$$

respectively. Obviously, $B_n := B_n(0)$ and $E_n := E_n(0)$ are the Bernoulli numbers and Euler numbers respectively.

Some interesting analogues of the classical Bernoulli polynomials were first investigated by Apostol. We begin by recalling here Apostol's definition as follows:

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Definition 1.1 [2]. *The Apostol – Bernoulli polynomials $\mathcal{B}_n(x;\lambda)$ are defined by means of the generating function*

$$\frac{ze^{xz}}{\lambda e^z - 1} = \sum_{n=0}^{\infty} \mathcal{B}_n(x;\lambda)\frac{z^n}{n!} \tag{1.3}$$

$$(|z| < 2\pi \text{ when } \lambda = 1; \quad |z| < |\log\lambda| \text{ when } \lambda \neq 1)$$

with, of course,

$$B_n(x) = \mathcal{B}_n(x;1) \qquad \text{and} \qquad \mathcal{B}_n(\lambda) := \mathcal{B}_n(0;\lambda),$$

where $\mathcal{B}_n(\lambda)$ denotes the so-called Apostol – Bernoulli numbers (*in fact, it is a function in λ*).

Recently, Luo further extended the Euler polynomials based on the Apostol's idea [2] as follows:

Definition 1.2 (cf. [18]). *The Apostol – Euler polynomials $\mathcal{E}_n(x;\lambda)$ are defined by means of the generating function*

$$\frac{2e^{xz}}{\lambda e^z + 1} = \sum_{n=0}^{\infty} \mathcal{E}_n(x;\lambda)\frac{z^n}{n!}, \qquad |z| < |\log(-\lambda)|, \tag{1.4}$$

with, of course,

$$E_n(x) = \mathcal{E}_n(x;1) \qquad \text{and} \qquad \mathcal{E}_n(\lambda) := \mathcal{E}_n(0;\lambda), \tag{1.5}$$

where $\mathcal{E}_n(\lambda)$ denote the so-called Apostol – Euler numbers.

Recently, M. Cenkci and M. Can [6] further defined the following q-extensions of the Apostol – Bernoulli polynomials, i.e., the so-called q-Apostol – Bernoulli polynomials.

Definition 1.3. *The q-Apostol – Bernoulli numbers $\mathcal{B}_{n;q}(\lambda)$ and polynomials $\mathcal{B}_n(x;\lambda)$ are defined by means of the generating functions*

$$U_{\lambda;q}(t) = -t\sum_{n=0}^{\infty}\lambda^n q^n e^{[n]_q t} = \sum_{n=0}^{\infty}\mathcal{B}_{n;q}(\lambda)\frac{t^n}{n!} \tag{1.6}$$

and

$$U_{x;\lambda;q}(t) = -t\sum_{n=0}^{\infty}\lambda^n q^{n+x} e^{[n+x]_q t} = \sum_{n=0}^{\infty}\mathcal{B}_{n;q}(x;\lambda)\frac{t^n}{n!} \tag{1.7}$$

respectively.

Setting $\lambda = 1$ in (1.6) and (1.7), we obtain the corresponding Carlitz's definitions for the q-Bernoulli numbers $B_{n;q}$ and q-Bernoulli polynomials $B_{n;q}(x)$ respectively.

Obviously,

$$\lim_{q\to 1} B_{n;q}(x) = B_n(x), \qquad \lim_{q\to 1} B_{n;q} = B_n$$

and

$$\lim_{q\to 1} \mathcal{B}_{n;q}(x;\lambda) = \mathcal{B}_n(x;\lambda), \qquad \lim_{q\to 1}\mathcal{B}_{n;q}(\lambda) = \mathcal{B}_n(\lambda).$$

It follows that we define the following q-extensions of the Apostol – Euler numbers and polynomials (see [3 – 6, 9]).

Definition 1.4. *The q-Apostol – Euler numbers $\mathcal{E}_{n;q}(\lambda)$ and polynomials $\mathcal{E}_n(x;\lambda)$ are defined by means of the generating functions*

$$V_{\lambda;q}(t) = 2\sum_{n=0}^{\infty}(-\lambda)^n q^n e^{[n]_q t} = \sum_{n=0}^{\infty}\mathcal{E}_{n;q}(\lambda)\frac{t^n}{n!} \tag{1.8}$$

and

$$V_{x;\lambda;q}(t) = 2\sum_{n=0}^{\infty}(-\lambda)^n q^{n+x} e^{[n+x]_q t} = \sum_{n=0}^{\infty}\mathcal{E}_{n;q}(x;\lambda)\frac{t^n}{n!} \tag{1.9}$$

respectively.

When $\lambda = 1$, then the above definitions (1.8) and (1.9) will become the corresponding definitions of the q-Euler numbers $E_{n;q}$ and q-Euler polynomials $E_{n;q}(x)$.

Clearly,

$$\lim_{q\to 1} E_{n;q}(x) = E_n(x), \qquad \lim_{q\to 1} E_{n;q} = E_n$$

and

$$\lim_{q\to 1}\mathcal{E}_{n;q}(x;\lambda) = \mathcal{E}_n(x;\lambda), \qquad \lim_{q\to 1}\mathcal{E}_{n;q}(\lambda) = \mathcal{E}_n(\lambda).$$

There are numerous recent investigations on this subject by, among many other authors, Cenki et al. [6 – 8], Choi et al. [9, 10], Kim [14 – 16], Luo and Srivastava [17 – 24], Ozden [26] and Simsek [27 – 29].

The aim of the present paper is to investigate the basic properties, generating functions, Raabe's multiplication theorem and alternating sums for the q-Apostol – Euler polynomials and to obtain some q-extensions of some formulas in [Integral Transform. Spec. Funct. – 2009. – **20**. – P. 377 – 391]. We also derive some interesting formulas and relationships between the q-Apostol – Euler polynomials, the q-Apostol – Euler polynomials and q-Hurwitz – Lerch zeta function.

2. The properties of the q-Apostol – Euler polynomials. The following *elementary* properties of the q-Apostol – Euler polynomials $\mathcal{E}_{n;q}(x;\lambda)$ are readily derived from (1.8) and (1.9). We, therefore, choose to omit the details involved.

Proposition 2.1 (the several values)**.**

$$\begin{aligned}
\mathcal{E}_{0;q}(x;\lambda) &= \frac{2}{\lambda q+1}q^x,\\
\mathcal{E}_{1;q}(x;\lambda) &= \frac{2}{\lambda q+1}q^x[x]_q - \frac{2\lambda}{(\lambda q+1)(\lambda q^2+1)}q^{2x+1},\\
\mathcal{E}_{0;q}(\lambda) &= \frac{2}{\lambda q+1},\\
\mathcal{E}_{1;q}(\lambda) &= -\frac{2\lambda q}{(\lambda q+1)(\lambda q^2+1)},\\
\mathcal{E}_{2;q}(\lambda) &= \frac{2\lambda q(\lambda q^2-1)}{(\lambda q+1)(\lambda q^2+1)(\lambda q^3+1)},\\
\mathcal{E}_{3;q}(\lambda) &= -\frac{2\lambda q(\lambda^2 q^5-4\lambda q^2+1)}{(\lambda q+1)(\lambda q^2+1)(\lambda q^3+1)(\lambda q^4+1)}.
\end{aligned} \tag{2.1}$$

Proposition 2.2. *An expansion formula of q-Apostol – Euler polynomials*

$$\mathcal{E}_{n;q}(x;\lambda) = \sum_{k=0}^{n}\binom{n}{k}\mathcal{E}_{k;q}(\lambda)q^{(k+1)x}[x]_q^{n-k}. \tag{2.2}$$

Proposition 2.3 (difference equation)**.**

$$\lambda\mathcal{E}_{n;q}(x+1;\lambda)+\mathcal{E}_{n;q}(x;\lambda) = 2q^x[x]_q^n. \tag{2.3}$$

Proposition 2.4 (differential relation)**.**

$$\frac{\partial}{\partial_x}\mathcal{E}_{n;q}(x;\lambda) = \mathcal{E}_{n;q}(x;\lambda)\log q + n\frac{\log q}{q-1}q^x\mathcal{E}_{n-1;q}(x;\lambda q). \tag{2.4}$$

Proposition 2.5 (integral formula)**.**

$$\int_a^b q^x\mathcal{E}_{n;q}(x;\lambda q)\,\mathrm{d}x = \frac{1-q}{n+1}\int_a^b \mathcal{E}_{n+1;q}(x;\lambda)\,\mathrm{d}x + \frac{q-1}{\log q}\,\frac{\mathcal{E}_{n+1;q}(b;\lambda)-\mathcal{E}_{n+1;q}(a;\lambda)}{n+1}. \tag{2.5}$$

Proposition 2.6 (addition theorem)**.**

$$\mathcal{E}_{n;q}(x+y;\lambda) = \sum_{k=0}^{n}\binom{n}{k}\mathcal{E}_{k;q}(x;\lambda)q^{(k+1)y}[y]_q^{n-k}. \tag{2.6}$$

Proposition 2.7 (theorem of complement)**.**

$$\mathcal{E}_{n;q}(1-x;\lambda) = \frac{(-1)^n}{\lambda q^n}\mathcal{E}_{n;q^{-1}}(x;\lambda^{-1}), \tag{2.7}$$

$$\mathcal{E}_{n;q}(1+x;\lambda) = \frac{(-1)^n}{\lambda q^n}\mathcal{E}_{n;q^{-1}}(-x;\lambda^{-1}). \tag{2.8}$$

***Remark* 2.1.** If $q \to 1$, then the formulas (2.1)–(2.8) become the corresponding formulas for the Apostol–Euler polynomials (see [18, p. 918, 919], Eqs. (3)–(11) when $\alpha = 1$). So the above formulas are q-extensions of the corresponding formulas of the Apostol–Euler polynomials respectively.

***Remark* 2.2.** When $\lambda = 1$, then the formulas (2.1) – (2.8) become the corresponding formulas for the q-Euler polynomials (see [3 – 5]).

3. The generating functions of q-Apostol–Euler polynomials. By (1.8) and (1.9) yields that

$$V_{x;\lambda;q}(t) = 2\sum_{n=0}^{\infty}(-\lambda)^n q^{n+x} e^{[n+x]_q t} =$$

$$= 2e^{\frac{t}{1-q}}\sum_{n=0}^{\infty}(-\lambda)^n q^{n+x} e^{-\frac{q^{n+x}}{1-q}t} =$$

$$= 2e^{\frac{t}{1-q}}\sum_{k=0}^{\infty}\frac{(-1)^k q^{(k+1)x}}{(1-q)^k}\frac{t^k}{k!}\sum_{n=0}^{\infty}(-\lambda q^{k+1})^n =$$

$$= 2e^{\frac{t}{1-q}}\sum_{k=0}^{\infty}\frac{(-1)^k q^{(k+1)x}}{1+\lambda q^{k+1}}\left(\frac{1}{1-q}\right)^k\frac{t^k}{k!}. \tag{3.1}$$

Therefore, we obtain the generating function of $\mathcal{E}_{n;q}(x;\lambda)$ as follows:

$$V_{x;\lambda;q}(t) = 2e^{\frac{t}{1-q}}\sum_{k=0}^{\infty}\frac{(-1)^k q^{(k+1)x}}{1+\lambda q^{k+1}}\left(\frac{1}{1-q}\right)^k\frac{t^k}{k!} = \sum_{n=0}^{\infty}\mathcal{E}_{n;q}(x;\lambda)\frac{t^n}{n!}. \tag{3.2}$$

Clearly, setting $x = 0$ in (3.2) we have the generating function of $\mathcal{E}_{n;q}(\lambda)$:

$$V_{\lambda;q}(t) = 2e^{\frac{t}{1-q}}\sum_{k=0}^{\infty}\frac{(-1)^k}{1+\lambda q^{k+1}}\left(\frac{1}{1-q}\right)^k\frac{t^k}{k!} = \sum_{n=0}^{\infty}\mathcal{E}_{n;q}(\lambda)\frac{t^n}{n!}. \tag{3.3}$$

Putting $\lambda = 1$ in (3.2) and (3.3), we deduce the generating function of $E_{n;q}(x)$ and $E_{n;q}$

$$V_{x;q}(t) = 2e^{\frac{t}{1-q}}\sum_{k=0}^{\infty}\frac{(-1)^k q^{(k+1)x}}{1+q^{k+1}}\left(\frac{1}{1-q}\right)^k\frac{t^k}{k!} = \sum_{n=0}^{\infty}E_{n;q}(x)\frac{t^n}{n!} \tag{3.4}$$

and

$$V_q(t) = 2e^{\frac{t}{1-q}}\sum_{k=0}^{\infty}\frac{(-1)^k}{1+q^{k+1}}\left(\frac{1}{1-q}\right)^k\frac{t^k}{k!} = \sum_{n=0}^{\infty}E_{n;q}\frac{t^n}{n!} \tag{3.5}$$

respectively.

It is not difficult, from (3.2) and (3.3) we get the following closed formulas:

$$\mathcal{E}_{n;q}(x;\lambda) = \frac{2}{(1-q)^n}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k q^{(k+1)x}}{1+\lambda q^{k+1}} \tag{3.6}$$

and

$$\mathcal{E}_{n;q}(\lambda) = \frac{2}{(1-q)^n}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{1+\lambda q^{k+1}}. \tag{3.7}$$

***Remark* 3.1.** In the same way, we can also obtain the generating function of q-Apostol – Bernoulli polynomials as follows:

$$U_{x;\lambda;q}(t) = -te^{\frac{t}{1-q}} \sum_{k=0}^{\infty} \frac{(-1)^k q^{(k+1)x}}{1-\lambda q^{k+1}} \left(\frac{1}{1-q}\right)^k \frac{t^k}{k!} = \sum_{n=0}^{\infty} \mathcal{B}_{n;q}(x;\lambda) \frac{t^n}{n!}. \tag{3.8}$$

4. q-Raabe's multiplication theorem, q-alternating sums and their applications.

Theorem 4.1 (q-Apostol-Raabe's multiplication theorem)**.** *For $m, n \in \mathbb{N}$, $\lambda \in \mathbb{C}$, then we have*

$$\mathcal{E}_{n;q}(mx;\lambda) = \begin{cases} [m]_q^n \sum\limits_{j=0}^{m-1} (-\lambda)^j \mathcal{E}_{n;q^m}\left(x + \dfrac{j}{m}; \lambda^m\right), & m \text{ is odd}, \\ -\dfrac{2}{n+1} [m]_q^n \sum\limits_{j=0}^{m-1} (-\lambda)^j \mathcal{B}_{n+1;q^m}\left(x + \dfrac{j}{m}; \lambda^m\right), & m \text{ is even}. \end{cases} \tag{4.1}$$

Proof. If m is odd, we compute the following sum by (3.2):

$$\sum_{n=0}^{\infty} \left[[m]_q^n \sum_{j=0}^{m-1} (-\lambda)^j \mathcal{E}_{n;q^m}\left(x + \frac{j}{m}; \lambda^m\right)\right] \frac{t^n}{n!} =$$

$$= 2e^{\frac{t}{1-q}} \sum_{k=0}^{\infty} \frac{(-1)^k q^{(k+1)mx}}{1+(\lambda q^{k+1})^m} \left(\frac{1}{1-q}\right)^k \frac{t^k}{k!} \sum_{j=0}^{m-1} \left(-\lambda q^{k+1}\right)^j =$$

$$= 2e^{\frac{t}{1-q}} \sum_{k=0}^{\infty} \frac{(-1)^k q^{(k+1)mx}}{1+\lambda q^{k+1}} \left(\frac{1}{1-q}\right)^k \frac{t^k}{k!} =$$

$$= \sum_{n=0}^{\infty} \mathcal{E}_{n;q}(mx;\lambda) \frac{t^n}{n!}. \tag{4.2}$$

Comparing the coefficients of $\dfrac{t^n}{n!}$ on the both sides of (4.2), we obtain the first formula of Theorem 4.1.

If m is *even*, we calculate the following sum by (3.8) and (3.2):

$$\sum_{n=0}^{\infty} \left[-\frac{2}{n+1} [m]_q^n \sum_{j=0}^{m-1} (-\lambda)^j \mathcal{B}_{n+1;q^m}\left(x + \frac{j}{m}; \lambda^m\right)\right] \frac{t^n}{n!} =$$

$$= 2e^{\frac{t}{1-q}} \sum_{k=0}^{\infty} \frac{(-1)^k q^{(k+1)mx}}{1-(\lambda q^{k+1})^m} \left(\frac{1}{1-q}\right)^k \frac{t^k}{k!} \sum_{j=0}^{m-1} \left(-\lambda q^{k+1}\right)^j =$$

$$= 2e^{\frac{t}{1-q}} \sum_{k=0}^{\infty} \frac{(-1)^k q^{(k+1)mx}}{1+\lambda q^{k+1}} \left(\frac{1}{1-q}\right)^k \frac{t^k}{k!} =$$

$$= \sum_{n=0}^{\infty} \mathcal{E}_{n;q}(mx;\lambda) \frac{t^n}{n!}. \tag{4.3}$$

Comparing the coefficients of $\frac{t^n}{n!}$ on the both sides of (4.3), we obtain the second formula of Theorem 4.1.

Theorem 4.1 is proved.

Clearly, the above formulas (4.1) of Theorem 4.1 are a q-extensions of the multiplication formulas in [20, p. 386] (Eq. (43)).

Taking $\lambda = 1$ in (4.1), we obtain the following corollary.

***Corollary* 4.1** (q-Raabe's multiplication theorem). *For $m, n \in \mathbb{N}$, then we have*

$$E_{n;q}(mx) = \begin{cases} [m]_q^n \sum\limits_{j=0}^{m-1} (-1)^j E_{n;q^m}\left(x + \dfrac{j}{m}\right), & m \text{ is odd}, \\ \\ -\dfrac{2}{n+1}[m]_q^n \sum\limits_{j=0}^{m-1} (-1)^j B_{n+1;q^m}\left(x + \dfrac{j}{m}\right), & m \text{ is even}. \end{cases} \tag{4.4}$$

Obviously, the above formulas (4.4) are a q-extensions of the classical Raabe's multiplication theorem of Euler polynomials in [20, p. 386] (Eq. (45)).

We now define the following alternating sums:

$$Z_{k;q}(m;n;\lambda) = \sum_{j=1}^{m} (-1)^{j+1} \lambda^j q^{j(n-k)} [j]_q^k =$$

$$= \lambda q^{n-k}[1]_q^k - \lambda^2 q^{2(n-k)}[2]_q^k + \ldots + (-1)^{m+1} \lambda^m q^{m(n-k)} [m]_q^k, \tag{4.5}$$

$$Z_{k;q}(m;n) = \sum_{j=1}^{m} (-1)^{j+1} q^{j(n-k)} [j]_q^k = q^{n-k}[1]_q^k - q^{2(n-k)}[2]_q^k + \ldots + (-1)^{m+1} q^{m(n-k)} [m]_q^k, \tag{4.6}$$

$$Z_k(m;\lambda) = \sum_{j=1}^{m} (-1)^{j+1} \lambda^j j^k = \lambda 1^k - \lambda^2 2^k + \ldots + (-1)^{m+1} \lambda^m m^k, \tag{4.7}$$

$$Z_k(m) = \sum_{j=1}^{m} (-1)^{j+1} j^k = 1^k - 2^k + \ldots + (-1)^{m+1} m^k, \tag{4.8}$$

$$m, n, k \in \mathbb{N}; \ n \geq k; \ \lambda \in \mathbb{C},$$

which are called the *q-λ-alternating sums, q-alternating sums, λ-alternating sums* and *alternating sums* respectively.

It is easy to obtain the following generating functions of $Z_k(m;\lambda)$ and $Z_k(m)$ respectively:

$$\sum_{k=0}^{\infty} Z_k(m;\lambda) \frac{x^k}{k!} = \sum_{j=1}^{m} (-1)^{j+1} \lambda^j e^{jx} = \frac{(-\lambda)^{m+1} e^{(m+1)x} + \lambda e^x}{\lambda e^x + 1}, \tag{4.9}$$

$$\sum_{k=0}^{\infty} Z_k(m) \frac{x^k}{k!} = \sum_{j=1}^{m} (-1)^{j+1} e^{jx} = \frac{(-1)^{m+1} e^{(m+1)x} + e^x}{e^x + 1}. \tag{4.10}$$

Theorem 4.2. *Let m be odd. For $m, n \in \mathbb{N}; \lambda \in \mathbb{C}$, the recursive formula of q-Apostol – Euler numbers*

$$[m]_q^n \mathcal{E}_{n;q^m}(\lambda^m) - \mathcal{E}_{n;q}(\lambda) = \sum_{k=0}^{n} \binom{n}{k} [m]_q^k \mathcal{E}_{k;q^m}(\lambda^m) Z_{n-k;q}(m-1; n+1; \lambda) \tag{4.11}$$

holds true in terms of the q-λ-alternating sums defined by (4.5).

Proof. If m is odd, taking $x = 0$ in (4.1) we obtain

$$\mathcal{E}_{n;q}(\lambda) = [m]_q^n \sum_{j=0}^{m-1} (-\lambda)^j \mathcal{E}_{n;q^m}\left(\frac{j}{m}; \lambda^m\right) =$$

$$= \sum_{k=0}^{n} \binom{n}{k} [m]_q^k \mathcal{E}_{k;q^m}(\lambda^m) \sum_{j=0}^{m-1} (-\lambda)^j q^{(k+1)j} [j]_q^{n-k} =$$

$$= -\sum_{k=0}^{n} \binom{n}{k} [m]_q^k \mathcal{E}_{k;q^m}(\lambda^m) Z_{n-k;q}(m-1; n+1; \lambda) + [m]_q^n \mathcal{E}_{n;q^m}(\lambda^m). \tag{4.12}$$

The formula (4.11) follows.

Theorem 4.2 is proved.

Clearly, the above formula (4.11) is an q-extension of the formula in [20, p. 389] (Eq. (60)).

Putting $\lambda = 1$ in (4.11), we obtain the following corollary.

***Corollary* 4.2.** *Let m be odd. For $m, n \in \mathbb{N}$, the recursive formula of Apostol – Euler numbers holds*

$$[m]_q^n E_{n;q^m} - E_{n;q} = \sum_{k=0}^{n} \binom{n}{k} [m]_q^k E_{k;q^m} Z_{n-k;q}(m-1; n+1) \qquad (m \textit{ is odd}). \tag{4.13}$$

Clearly, the above formula (4.13) is a q-extension of the formula in [20, p. 389] (Eq. (61)).

Theorem 4.3. *Let m be even. For $m, n \in \mathbb{N}$, $\lambda \in \mathbb{C}$, the formula of q-Apostol – Euler numbers*

$$[m]_q \mathcal{E}_{n;q}(\lambda) + \frac{2}{n+1} [m]_q^{n+1} \mathcal{B}_{n+1;q^m}(\lambda^m) =$$

$$= \frac{2}{n+1} \sum_{k=0}^{n+1} \binom{n+1}{k} [m]_q^k \mathcal{B}_{k;q^m}(\lambda^m) Z_{n+1-k;q}(m-1; n+1; \lambda), \tag{4.14}$$

holds true in terms of the q-λ-alternating sums defined by (4.5).

Proof. If m is *even*, setting $x = 0$ in (4.1) we have

$$\mathcal{E}_{n;q}(\lambda) = -\frac{2}{n+1} [m]_q^n \sum_{j=0}^{m-1} (-\lambda)^j \mathcal{B}_{n+1;q^m}\left(\frac{j}{m}; \lambda^m\right) =$$

$$= \frac{2}{n+1} \sum_{k=0}^{n+1} \binom{n+1}{k} [m]_q^{k-1} \mathcal{B}_{k;q^m}(\lambda^m) \sum_{j=0}^{m-1} (-1)^{j+1} \lambda^j q^{kj} [j]_q^{n+1-k} =$$

$$= \frac{2}{n+1}\sum_{k=0}^{n+1}\binom{n+1}{k}[m]_q^{k-1}\mathcal{B}_{k;q^m}(\lambda^m)Z_{n-k+1;q}(m-1;n+1;\lambda) - \frac{2}{n+1}[m]_q^n\mathcal{B}_{n+1;q^m}(\lambda^m). \tag{4.15}$$

The formula (4.3) follows.

Theorem 4.3 is proved.

Clearly, the above formula (4.3) is a q-extension of the formula (see [20, p. 390], Eq. (63) for $\ell = 1$):

$$m\mathcal{E}_n(\lambda)+\frac{2}{n+1}m^{n+1}\mathcal{B}_{n+1}(\lambda^m) = \frac{2}{n+1}\sum_{k=0}^{n+1}\binom{n+1}{k}m^k\mathcal{B}_k(\lambda^m)Z_{n+1-k}(m-1;\lambda) \quad (m \text{ is even}), \tag{4.16}$$

where λ-alternating sums defined by (4.7).

Putting $\lambda = 1$ in (4.3), we have the following corollary.

***Corollary* 4.3.** *For m be even, $m, n \in \mathbb{N}$; $\lambda \in \mathbb{C}$, the formula of q-Euler numbers*

$$[m]_qE_{n;q}+\frac{2}{n+1}[m]_q^{n+1}B_{n+1;q^m} = \frac{2}{n+1}\sum_{k=0}^{n+1}\binom{n+1}{k}[m]_q^kB_{k;q^m}Z_{n+1-k;q}(m-1;n+1), \tag{4.17}$$

holds true in terms of the q-alternating sums defined by (4.6).

Clearly, the above formula (4.17) is a q-extension of the formula in [20, p. 390] (Eq. (64) for $\ell = 1$).

***Remark* 4.1.** Setting $\lambda = 1$ in (4.16) and noting that $Z_0(m-1) = 1$ for m even, we derive the following interesting sum formula:

$$\sum_{k=0}^{n}\binom{n+1}{k}m^kB_kZ_{n+1-k}(m-1) = \frac{m(n+1)}{2}E_n \qquad (n \in \mathbb{N};\quad m \text{ is even}). \tag{4.18}$$

Applying the relation $E_n = \frac{2}{n+1}(1-2^{n+1})B_{n+1}$ to (4.18), we find that

$$\sum_{k=0}^{n}\binom{n+1}{k}m^kB_kZ_{n+1-k}(m-1) = m(1-2^{n+1})B_{n+1} \qquad (n \in \mathbb{N};\quad m \text{ is even}), \tag{4.19}$$

which is just the formula of Howard (see [13, p. 167], Eq. (33)).

***Remark* 4.2.** Separatting the odd and even terms in (4.19), and noting that $B_0 = 1$, $B_1 = \frac{1}{2}$, $B_{2n+1} = 0$, $n \in \mathbb{N}$, we have

$$\sum_{k=1}^{n-1}\binom{2n}{2k}m^{2k}B_{2k}Z_{2n-2k}(m-1) = m(1-2^{2n})B_{2n} + mnZ_{2n-1}(m-1) - Z_{2n}(m-1) \tag{4.20}$$

$$(n \in \mathbb{N};\quad m \text{ is even})$$

and

$$\sum_{k=1}^{n}\binom{2n+1}{2k}m^{2k}B_{2k}Z_{2n-2k+1}(m-1) = \frac{m(2n+1)}{2}Z_{2n}(m-1) - Z_{2n+1}(m-1) \tag{4.21}$$

$$(n \in \mathbb{N};\ m \text{ is even}),$$

where the alternating sums $Z_k(m)$ defined by (4.8).

Below we give the evaluations for the alternating sums (4.5), (4.6) given by (4.22) and (4.24) respectively.

Theorem 4.4. *For $m, n \in \mathbb{N}$, $\lambda \in \mathbb{C}$, the following formula of q-λ-alternating sums*:

$$Z_{n;q}(m; n+1; \lambda) = \sum_{j=0}^{m} (-1)^{j+1} \lambda^j q^j [j]_q^n = \frac{(-\lambda)^{m+1} \mathcal{E}_{n;q}(m+1; \lambda) - \mathcal{E}_{n;q}(\lambda)}{2}, \tag{4.22}$$

holds true in terms of the q-Apostol – Euler polynomials.

Proof. It is easy to observe that

$$(-\lambda)^{m+1} \sum_{j=0}^{\infty} (-\lambda)^j q^{m+j+1} e^{[m+j+1]_q t} + \sum_{j=0}^{\infty} (-\lambda)^j q^j e^{[j]_q t} = \sum_{j=0}^{m} (-1)^{j+1} \lambda^j q^j e^{[j]_q t}. \tag{4.23}$$

By (1.8), (1.9) and (4.23), via simple computation, we arrive at the desire (4.22) immediately.

Theorem 4.4 is proved.

Clearly, the above formula (4.22) is a q-extension of the formula in [20, p. 388] (Eq. (55)).

Setting $\lambda = 1$ in (4.22), then we have

$$Z_{n;q}(m; n+1) = \sum_{j=0}^{m} (-1)^{j+1} q^j [j]_q^n = -\frac{(-1)^m E_{n;q}(m+1) + E_{n;q}}{2}, \tag{4.24}$$

which is a q-extension of the well-known formula in [20, p. 388] (Eq. (56)) and [1, p. 804], (23.1.4).

5. Some relationships between the q-Apostol – Euler polynomials and q-Hurwitz – Lerch zeta-function. The Hurwitz – Lerch zeta-function $\Phi(z, s, a)$ defined by (cf., e.g., [29, p. 121]).

$$\Phi(z, s, a) := \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s}$$

$$\left(a \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ s \in \mathbb{C} \text{ when } |z| < 1;\ \Re(s) > 1 \text{ when } |z| = 1\right)$$

contains, as its *special* cases, not only the Riemann and Hurwitz (or generalized) zeta-functions

$$\zeta(s) := \Phi(1, s, 1) = \zeta(s, 1) = \frac{1}{2^s - 1}\, \zeta\left(s, \frac{1}{2}\right),$$

$$\zeta(s, a) := \Phi(1, s, a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}, \qquad \Re(s) > 1, \quad a \notin \mathbb{Z}_0^-,$$

and the Lerch zeta-function:

$$l_s(\xi) := \sum_{n=1}^{\infty} \frac{e^{2n\pi i \xi}}{n^s} = e^{2\pi i \xi}\, \Phi\left(e^{2\pi i \xi}, s, 1\right), \qquad \xi \in \mathbb{R}, \quad \Re(s) > 1,$$

but also such other functions as the polylogarithmic function:

$$\mathrm{Li}_s(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^s} = z\, \Phi(z, s, 1)$$

$$\big(s \in \mathbb{C} \quad \text{when} \quad |z| < 1;\ \Re(s) > 1 \quad \text{when} \quad |z| = 1\big)$$

and the Lipschitz – Lerch zeta-function (cf. [29, p. 122], Eq. 2.5 (11)):

$$\phi(\xi, a, s) := \sum_{n=0}^{\infty} \frac{e^{2n\pi i\xi}}{(n+a)^s} = \Phi\left(e^{2\pi i\xi}, s, a\right) =: L\left(\xi, s, a\right)$$

$$\big(a \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ \Re(s) > 0 \quad \text{when} \quad \xi \in \mathbb{R} \setminus \mathbb{Z};\ \Re(s) > 1 \quad \text{when} \quad \xi \in \mathbb{Z}\big).$$

We define the q-Hurwitz – Lerch zeta-functions as follows:

Definition 5.1. *For $\Re(a) > 0$, q-Hurwitz – Lerch zeta-function is defined by*

$$\Phi_q(z, s, a) := \sum_{n=0}^{\infty} \frac{z^n q^{n+a}}{[n+a]_q^s}, \qquad \Re\left(a\right) > 0, \quad a \notin \mathbb{Z}_0^-.$$

Theorem 5.1. *The following relationship*:

$$\mathcal{E}_{n;q}\left(a;\lambda\right) = 2\Phi_q(-\lambda, -n, a), \qquad n \in \mathbb{N}, \quad |\lambda| \le 1, \quad a \in \mathbb{C} \setminus \mathbb{Z}_0^-, \tag{5.1}$$

holds true between the q-Apostol – Euler polynomials and the q-Hurwitz – Lerch zeta-function.

Proof. We differentiate the both sides of (1.9) with respect to the variable t yields that

$$\mathcal{E}_{n;q}(a;\lambda) = \frac{d^n}{dt^n} V_{a;\lambda;q}(t)\bigg|_{t=0} = 2\sum_{k=0}^{\infty} (-\lambda)^k q^{k+a} \frac{d^n}{dt^n}\left\{e^{[k+a]_q t}\right\}\bigg|_{t=0} =$$

$$= 2\sum_{k=0}^{\infty} (-\lambda)^k q^{k+a} \big([k+a]_q\big)^n = 2\sum_{k=0}^{\infty} \frac{(-\lambda)^k q^{k+a}}{[k+a]_q^{-n}}.$$

Theorem 5.1 is proved.

Letting $q \to 1$ in (5.1), we have the following corollary.

***Corollary* 5.1.** *The following relationship*:

$$\mathcal{E}_n\left(a;\lambda\right) = 2\Phi(-\lambda, -n, a), \qquad n \in \mathbb{N}, \quad |\lambda| \le 1, \quad a \in \mathbb{C} \setminus \mathbb{Z}_0^-,$$

holds true between the Apostol – Euler polynomials and Hurwitz – Lerch zeta-function.

On the other hand, we define an analogue of the Hurwitz zeta-function as follows:

Definition 5.2. *For $\Re(s) > 1$, $a \notin \mathbb{Z}_0^-$, L-function is defined by*

$$L(s, a) := \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+a)^s}.$$

Clearly, $L(s, a) = \phi\left(\frac{1}{2}, a, s\right) = \Phi(e^{\pi i}, s, a) = L\left(\frac{1}{2}, s, a\right)$.

Next we define an q-extension of the L-function.

Definition 5.3. *For $\Re(s) > 1$, $a \notin \mathbb{Z}_0^-$, the q-L-function is defined by*

$$L_q(s, a) := \sum_{n=0}^{\infty} \frac{(-1)^n q^{n+a}}{[n+a]_q^s}.$$

In the same way, we can obtain the following relationships.

Theorem 5.2. *The following relationship*:

$$E_{n,q}\left(a\right)=2L_{q}(-n,a),\qquad n\in\mathbb{N},\quad a\in\mathbb{C}\setminus\mathbb{Z}_{0}^{-},$$

holds true between the q-Euler polynomials and q-L-function.

***Corollary* 5.2.** *The following relationship*:

$$E_{n}\left(a\right)=2L(-n,a),\qquad n\in\mathbb{N},\quad a\in\mathbb{C}\setminus\mathbb{Z}_{0}^{-},$$

holds true between the Euler polynomials and the L-function.

We define an analogue of the Riemann zeta-function:

Definition 5.4. *The l-function is defined by*

$$l(s):=\sum_{n=1}^{\infty}\frac{(-1)^{n}}{n^{s}},\qquad \Re\left(s\right)>1.$$

Obviously, $l(s)=l_{s}\left(\dfrac{1}{2}\right)=\mathrm{Li}_{s}(-1)$.

It follows that we define an q-extension of the l-function:

Definition 5.5. *The q-l-function is defined by*

$$l_{q}(s):=\sum_{n=1}^{\infty}\frac{(-1)^{n}q^{n}}{[n]_{q}^{s}},\qquad \Re\left(s\right)>1.$$

Similarly, we can obtain the following explicit relationship:

Theorem 5.3. *The following relationship*:

$$E_{n,q}=2l_{q}(-n),\qquad n\in\mathbb{N},$$

holds true between the q-Euler numbers and q-l-function.

***Corollary* 5.3.** *The following relationship*:

$$E_{n}=2\;l(-n),\qquad n\in\mathbb{N},$$

holds true between the Euler numbers and l-function.

6. Some explicit relationships between the q-Apostol – Bernoulli and q-Apostol – Euler polynomials. In this section, we will investigate some relationships between the q-Apostol – Bernoulli and q-Apostol – Euler polynomials. We also obtain an q-extension of Howard's result.

It is easy to observe that

$$2\sum_{n=0}^{\infty}\lambda^{2n}q^{2n+x}e^{[2n+x]_{q}t}-\sum_{n=0}^{\infty}\lambda^{n}q^{n+x}e^{[n+x]_{q}t}=\sum_{n=0}^{\infty}(-\lambda)^{n}q^{n+x}e^{[n+x]_{q}t}.$$

By (1.7) and (1.9), via the simple computation, we obtain

$$\mathcal{E}_{n;q}(x;\lambda)=\frac{2}{n+1}\left[\mathcal{B}_{n+1;q}(x;\lambda)-2[2]_{q}^{n}\mathcal{B}_{n+1;q^{2}}\left(\frac{x}{2};\lambda^{2}\right)\right],\tag{6.1}$$

which is just an q-extension of the formula of Luo and Srivastava (see [19, p. 636], Eq. (38))

$$\mathcal{E}_n(x;\lambda) = \frac{2}{n+1}\left[\mathcal{B}_{n+1}(x;\lambda) - 2^{n+1}\mathcal{B}_{n+1}\left(\frac{x}{2};\lambda^2\right)\right].$$

Setting $x = 0$ in (6.1), we get

$$\mathcal{E}_{n;q}(\lambda) = \frac{2}{n+1}\left[\mathcal{B}_{n+1;q}(\lambda) - 2[2]_q^n\mathcal{B}_{n+1;q^2}\left(\lambda^2\right)\right]. \tag{6.2}$$

Putting $\lambda = 1$ in (6.1), we have

$$E_{n;q}(x) = \frac{2}{n+1}\left[B_{n+1;q}(x) - 2[2]_q^n B_{n+1;q^2}\left(\frac{x}{2}\right)\right], \tag{6.3}$$

which is an q-extension of the well-known formula (see [1])

$$E_n(x) = \frac{2}{n+1}\left[B_{n+1}(x) - 2^{n+1}B_{n+1}\left(\frac{x}{2}\right)\right].$$

***Remark* 6.1.** If taking $x = 0$ in (6.3), we obtain

$$E_{n;q} = \frac{2}{n+1}\left(B_{n+1;q} - 2[2]_q^n B_{n+1;q^2}\right)$$

is an q-extension of the formula (see [1, p. 805] (Entry (23.1.20)) and [25, p. 29])

$$E_n = \frac{2}{n+1}(1 - 2^{n+1})B_{n+1}.$$

***Remark* 6.2.** By (4.3) and (6.2), we easily obtain the following explicit recursive formula for the q-Apostol–Bernoulli numbers:

$$[m]_q\left(\mathcal{B}_{n;q}(\lambda) - 2[2]_q^{n-1}\mathcal{B}_{n;q^2}(\lambda^2)\right) = \sum_{k=0}^{n}\binom{n}{k}[m]_q^k\mathcal{B}_{k;q^m}(\lambda^m)Z_{n-k;q}(m-1;n;\lambda) - [m]_q^n\mathcal{B}_{n;q^m}(\lambda^m). \tag{6.4}$$

***Remark* 6.3.** Setting $\lambda = 1$ in (6.4), we have

$$[m]_q\left(B_{n;q} - 2[2]_q^{n-1}B_{n;q^2}\right) = \sum_{k=0}^{n}\binom{n}{k}[m]_q^k B_{k;q^m}Z_{n-k;q}(m-1;n) - [m]_q^n B_{n;q^m} \qquad (m \text{ is even})$$

is an q-extension of Howard's formula [13, p. 167] (Eq. (33))

$$m(1-2^n)B_n = \sum_{k=0}^{n-1}\binom{n}{k}B_k m^k Z_{n-k}(m-1) \qquad (m \text{ is even}).$$

***Remark* 6.4.** Letting $q \to 1$ in (6.4), we obtain a new formula for the Apostol–Bernoulli numbers as follows:

$$m\left(\mathcal{B}_n(\lambda) - 2^n\mathcal{B}_n(\lambda^2)\right) = \sum_{k=0}^{n}\binom{n}{k}m^k\mathcal{B}_k(\lambda^m)Z_{n-k}(m-1;\lambda) - m^n\mathcal{B}_n(\lambda^m) \qquad (m \text{ is even}). \tag{6.5}$$

Obviously, by setting $\lambda = 1$ in (6.5) we get a new recurrence formula for the Bernoulli numbers:

$$B_n = \sum_{k=0}^{n}\binom{n}{k}\frac{m^{k-1}Z_{n-k}(m-1)}{1-2^n+m^{n-1}}B_k \qquad (m \text{ is even}).$$

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