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METHOD OF LINES FOR QUASILINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

МЕТОД ЛІНІЙ ДЛЯ КВАЗІЛІНІЙНИХ ФУНКЦІОНАЛЬНО-ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ

We give a theorem on the error estimate of approximate solutions for ordinary functional differential equations. The error is estimated by a solution of an initial problem for nonlinear functional differential equation. We apply this general result to the investigation of the convergence of the numerical method of lines for evolution functional differential equations. Initial boundary-value problems for quasilinear equations are transformed by discretization in spatial variables into systems of ordinary functional differential equations. Nonlinear estimates of the Perron-type with respect to functional variables for given operators are assumed. Numerical examples are given.

Наведено теорему про оцінку похибки наближених розв'язків звичайних диференціальних рівнянь. Похибка оцінюється за допомогою розв'язку початкової задачі для нелінійного функціонально-диференціального рівняння. Цей загальний результат застосовується при дослідженні збіжності числового методу ліній для еволюції функціонально-диференціальних рівнянь. За допомогою дискретизації по просторових змінних початково-крайові задачі для квазілінійних рівнянь зводяться до систем звичайних диференціальних рівнянь. Припускається справедливість нелінійних оцінок перронівського типу відносно функціональних змінних для заданих операторів. Наведено також чисельні приклади.

1. Introduction. The numerical method of lines for partial differential or functional differential equations consists in replacing derivatives with respect to spatial variables by difference expressions. This leads to systems of ordinary differential or functional differential equations. They satisfy consistency conditions on classical solutions of original problems. The main task in these considerations is to find sufficient conditions for the stability of differential difference problems.

There is an ample literature on the method of lines. The classical papers are [7, 8, 22, 23] where parabolic equations were considered. Existence results based on the method of lines can be found in [3, 14, 19, 24, 25]. Parabolic problems and first order partial differential equations and boundary-value problems for nonlinear elliptic equations were considered. The papers [1, 2, 4, 13, 29, 30] initiated the theory of the method of lines for evolution functional differential equations. Initial problems on the Haar pyramid for Hamilton – Jacobi-type equations and parabolic equations with initial or initial boundary conditions of the Dirichlet-type were investigated. For further bibliographical informations concerning the method of lines see [9, 11, 17, 21, 28].

Results on the method of lines for evolution functional differential equations are based on the following ideas. Comparison theorems for differential difference inequalities generated by nonlinear functional differential equations are obtained. These theorems state that functions satisfying differential difference inequalities may be estimated by solutions of ordinary differential or functional differential equations. Comparison theorems are used in proofs of the existence of approximate solutions. Results on the convergence of sequences of approximate solutions are also based on comparison theorems for differential difference inequalities. Theorems on the numerical method of lines for nonlinear first partial functional differential equations [1, 2, 12, 29] and for parabolic problems [15, 30] were obtained by using the above comparison methods.

The aim of the paper is to show a new method of investigations of the numerical method of lines for evolution functional differential equations. We prove that results on the existence of approximate solutions and theorems on the convergence of the methods are consequences of simple theorems on ordinary functional differential equations.

The paper is organized as follows. Section 2 deals with ordinary functional differential equations. Numerical methods of lines for functional differential equations lead to systems of equations considered in this section. We prove a theorem on the existence of solutions to initial problems and we give a theorem on estimates of the difference between the exact an approximate solutions. The errors of approximate solutions are estimated by solutions to initial problems for nonlinear comparison equations.

We apply this general idea to investigations of the numerical method of lines for evolution functional differential equations. Initial boundary-value problems for quasilinear first order partial functional differential equations are considered in Section 3. In the next section we give results on the numerical method of lines for quasilinear parabolic functional differential problems.

Two types of assumption are needed in theorems on the convergence of the numerical methods of lines. The first type conditions concern regularity of given functions. The second type conditions concern the mesh. The authors of the papers [2, 12, 15, 29, 30] have assumed that given functions satisfy the Lipschitz condition or they satisfy nonlinear estimates of the Perron-type with respect to functional variables and these conditions are global with respect to functional variables. Our assumptions on the regularity of given functions are more general. We assume nonlinear estimates of the Perron-type and suitable estimates are local with respect to functional variables. There are differential equations with deviated variables and differential integral equations such that local estimates of the Perron-type hold and global inequalities are not satisfied. We give suitable examples.

Results presented in [1, 2, 15, 22–24, 29, 30] are not applicable to quasilinear functional problems considered in the paper. Theorems presented here are new also in the case of differential equations without the functional dependence.

Now we formulate our functional differential problems. For any metric spaces X and Y we denote by $C(X,Y)$ the class of all continuous functions from X into Y. We use vectorial inequalities with the understanding that the same inequalities hold between their corresponding components. Write

$$E_0 = [-b_0, 0] \times [-b, b], \qquad E = [0, a] \times [-b, b], \qquad \partial_0 E = [0, a] \times \big([-b, b] \setminus (-b, b)\big),$$

where $b_0 \in \mathbb{R}_+$, $\mathbb{R}_+ = [0, +\infty)$, $a > 0$ and $b = (b_1, \ldots, b_n) \in \mathbb{R}^n$, and $b_i > 0$ for $1 \le i \le n$. For each $(t,x) \in E$ we define the set $D[t,x]$ as follows:

$$D[t,x] = \big\{(\tau, y) \in \mathbb{R}^{1+n} \colon\ \tau \le 0,\ (t+\tau, x+y) \in E_0 \cup E\big\}.$$

It is clear that $D[t,x] = [-b_0 - t, 0] \times [-b - x, b - x]$. For a function $z \colon E_0 \cup E \to \mathbb{R}$ and for a point $(t,x) \in E$ we define a function $z_{(t,x)} \colon D[t,x] \to \mathbb{R}$ as follows: $z_{(t,x)}(\tau, y) = z(t+\tau, x+y)$, $(\tau, y) \in D[t,x]$. Then $z_{(t,x)}$ is the restriction of z to the set $(E_0 \cup E) \cap ([-b_0, t] \times \mathbb{R}^n)$ and this restriction is shifted to the set $D[t,x]$.

Write $r = b_0 + a$ and $B = [-r, 0] \times [-2b, 2b]$. Then $D[t,x] \subset B$ for $(t,x) \in E$. Set $\Omega = E \times C(B, \mathbb{R})$ and suppose that $F \colon \Omega \to \mathbb{R}^n$, $F = (F_1, \ldots, F_n)$, $G \colon \Omega \to \mathbb{R}$ and $\psi \colon E_0 \cup \partial_0 E \to \mathbb{R}$ are given functions. Let z be an unknown function of the variables (t,x), $x = (x_1, \ldots, x_n)$. We consider the functional differential equation

$$\partial_t z(t,x) = \sum_{i=1}^{n} F_i(t,x,z_{(t,x)})\partial_{x_i} z(t,x) + G(t,x,z_{(t,x)}) \tag{1}$$

with the initial boundary condition

$$z(t,x) = \psi(t,x) \qquad \text{on} \quad E_0 \cup \partial_0 E. \tag{2}$$

We will say that F and G satisfy the condition (V) if for each $(t,x) \in E$ and for $w,\ \tilde{w} \in C(B,\mathbb{R})$ such that $w(\tau,y) = \tilde{w}(\tau,y)$ for $(\tau,y) \in D[t,x]$ we have $F(t,x,w) = F(t,x,\tilde{w})$ and $Gt,x,w) = G(t,x,\tilde{w})$. Condition (V) means that the values of F and G at $(t,x,w) \in \Omega$ depend on (t,x) and on the restriction of w to the set $D[t,x]$ only. We assume that F and G satisfy condition (V) and we consider classical solutions to (1), (2).

Now we formulate an initial boundary-value problem for a parabolic functional differential equation. Let $M_{n\times n}$ be the class of all $n \times n$ matrices with real elements. Suppose that $\mathbf{F}\colon \Omega \to M_{n\times n}$, $F\colon \Omega \to \mathbb{R}^n$, $G\colon \Omega \to \mathbb{R}$ $\psi\colon E_0 \cup \partial_o E$ are given functions and

$$\mathbf{F} = \big[F_{ij}\big]_{i,j=1,\ldots,n}, \qquad F = (F_1,\ldots,F_n).$$

We consider the functional differential equation

$$\partial_t z(t,x) = \sum_{i,j=1}^{n} F_{ij}(t,x,z_{(t,x)})\partial_{x_i x_j} z(t,x) + \sum_{i=1}^{n} F_i(t,x,z_{(t,x)})\partial_{x_i} z(t,x) + G(t,x,z_{(t,x)}) \tag{3}$$

with the initial boundary condition (2).

We will say that $\mathbf{F}$ satisfies the condition (V) if for each $(t,x) \in E$ and for $w,\tilde{w} \in C(B,\mathbb{R})$ such that $w(\tau,y) = \tilde{w}(\tau,y)$ for $(\tau,y) \in D[t,x]$ we have $\mathbf{F}(t,x,w) = \mathbf{F}(t,x,\tilde{w})$. We assume that $\mathbf{F}$, F and G satisfy condition (V) and we consider classical solutions to (3), (2).

Sufficient conditions for the existence and uniqueness of classical or generalized solutions of evolution functional differential equations can be found in [3, 5, 6, 10, 16, 18, 20, 26].

Differential equations with deviated variables and differential integral equations can be derived from (1) and (3) by specializing given operators. Information on applications of functional differential equations can be found in [11, 27].

2. Approximate solutions of ordinary functional differential systems. For any spaces X and Y we denote by $F(X,Y)$ the class of all functions defined on X and taking values in Y. Let $\mathbb{N}$ and $\mathbb{Z}$ be the sets of natural numbers and integers respectively. We define a mesh with respect to spatial variables in the following way. Let $(h_1,\ldots,h_n) = h$, $h_i > 0$ for $1 \le i \le n$, stand for steps of the mesh. For $m = (m_1,\ldots,m_n) \in \mathbb{Z}^n$ we put

$$x^{(m)} = \big(x_1^{(m_1)},\ldots,x_n^{(m_n)}\big) = (m_1 h_1,\ldots,m_n h_n)$$

and

$$\mathbb{R}^{1+n}_{t.h} = \big\{(t,x^{(m)})\colon\ t \in \mathbb{R},\ m \in \mathbb{Z}^n\big\}.$$

Write

$$E_h = E \cap \mathbb{R}^{1+n}_{t.h}, \qquad E_{0.h} = E_0 \cap \mathbb{R}^{1+n}_{t.h}, \qquad \partial_0 E_h = \partial_0 E \cap \mathbb{R}^{1+n}_{t.h}$$

and

$$B_h = B \cap \mathbb{R}^{1+n}_{t.h}, \qquad D_h[t,m] = D\big[t, x^{(m)}\big] \cap \mathbb{R}^{1+n}_{t.h}.$$

For $z\colon E_{0.h} \cup E_h \to \mathbb{R}$ and $w\colon B_h \to \mathbb{R}$ we write $z^{(m)}(t) = z(t, x^{(m)})$ on $E_{0.h} \cup E_h$ and $w^{(m)}(t) = w(t, x^{(m)})$ on B_h. Let us denote by H the set of all $h = (h_1, \ldots, h_n)$ satisfying the condition: there is $K = (K_1, \ldots, K_n) \in \mathbb{N}^n$ such that $(K_1 h_1, \ldots, K_n h_n) = b$. Set

$$\mathbb{K} = \{m \in \mathbb{Z}^n\colon -K \le m \le K\}, \qquad \operatorname{Int}\mathbb{K} = \{m \in \mathbb{Z}^n\colon -K < m < K\}.$$

For $x \in \mathbb{R}^n$, $A \in M_{n\times n}$ where $x = (x_1, \ldots, x_n)$, $A = \big[a_{ij}\big]_{i,j=1,\ldots,n}$, we put

$$\|x\| = \sum_{i=1}^n |x_i| \qquad \text{and} \qquad \|A\|_{n\times n} = \max\left\{\sum_{j=1}^n |a_{ij}|\colon\ 1 \le i \le n\right\}.$$

Let $F_c(E_{0.h} \cup E_h, \mathbb{R})$ be the class of all $z\colon E_{0.h} \cup E_h \to \mathbb{R}$ such that $z(\,\cdot\,, x^{(m)}) \in C([-b_0, a], \mathbb{R})$ for $m \in \mathbb{K}$. In a similar way we define the spaces $F_c(B_h, \mathbb{R})$ and $F_c(E_{0.h} \cup \partial_0 E_h, \mathbb{R})$.

For $z\colon E_{0.h} \cup E_h \to \mathbb{R}$ and $(t, x^{(m)}) \in E_h$ we define a function $z_{[t,m]}\colon D_h[t,m] \to \mathbb{R}$ in the following way: $z_{[t,m]}(\tau, y) = z(t+\tau, x^{(m)} + y)$, $(\tau, y) \in D_h[t,m]$. Write

$$\Lambda = \big\{\lambda = (\lambda_1, \ldots, \lambda_n)\colon\ \lambda_i \in \{-1, 0, 1\} \ \text{ for } \ 1 \le i \le n \ \text{ and } \ \|\lambda\| \le 2\big\},$$

$$\Lambda' = \Lambda \setminus \{\,\theta\,\}, \qquad \theta = (0, \ldots, 0) \in \mathbb{R}^n,$$

and $\kappa = 1 + 2n^2$. Note that κ is the number of elements of Λ. Let $\pi\colon \Lambda \to \{1, \ldots, \kappa\}$ be a function such that $\pi(\lambda) \ne \pi(\tilde\lambda)$ for $\lambda \ne \tilde\lambda$. We assume that $\prec$ is an order in Λ defined in the following way: $\lambda \prec \tilde\lambda$ if $\pi(\lambda) < \pi(\tilde\lambda)$. Elements of the space $\mathbb{R}^\kappa$ will be denoted by $\xi = \{\xi_\lambda\}_{\lambda\in\Lambda}$. Write

$$A_h = \big\{x^{(m)}\colon\ m = (m_1, \ldots, m_n) \in \Lambda\big\}.$$

For $\zeta\colon A_h \to \mathbb{R}$ we put $\zeta^{(m)} = \zeta(x^{(m)})$.

For $z\colon E_{0.h} \cup E_h \to \mathbb{R}$ and $(t, x^{(m)}) \in E_h$, $m \in \operatorname{Int}\mathbb{K}$, we define a function $z_{\langle t,m\rangle}\colon A_h \to \mathbb{R}$ in the following way: $z_{\langle t,m\rangle}(y) = z(t, x^{(m)} + y)$, $y \in A_h$. Write $\Omega_h = E_h \times F_c(B_h, \mathbb{R})$ and suppose that

$$G_h\colon \Omega_h \to \mathbb{R}, \qquad F_h\colon \Omega_h \to \mathbb{R}^\kappa, \qquad F_h = \{F_{h.\lambda}\}_{\lambda\in\Lambda},$$

are given functions. For $(t, x, w) \in \Omega_h$, $\zeta \in F(A_h, \mathbb{R})$ we put

$$F_h(t, x, w) \circ \zeta = \sum_{\lambda\in\Lambda} F_{h.\lambda}(t, x, w)\, \zeta^{(\lambda)}.$$

Set

$$\mathbf{F}_h(t, x, w, \zeta) = G_h(t, x, w) + F_h(t, x, w) \circ \zeta.$$

Given $\psi_h\colon E_{0.h} \cup \partial_0 E_h \to \mathbb{R}$, we consider the functional differential equations

$$\frac{d}{dt} z^{(m)}(t) = \mathbf{F}_h\big(t, x^{(m)}, z_{[t,m]}, z_{\langle t,m\rangle}\big), \quad m \in \operatorname{Int}\mathbb{K}, \tag{4}$$

with the initial boundary condition

$$z^{(m)}(t) = \psi_h^{(m)}(t) \qquad \text{on} \quad E_{0.h} \cup \partial_0 E_h. \tag{5}$$

We will say that F_h and G_h satisfy the condition (V_h) if for each $(t, x^{(m)}) \in E_h$ and for $w, \tilde{w} \in F_c(B_h, \mathbb{R})$ such that $w(\tau, y) = \tilde{w}(\tau, y)$ for $(\tau, y) \in D[t, m]$ we have $F_h(t, x^{(m)}, w) = F_h(t, x^{(m)}, \tilde{w})$ and $G_h(t, x^{(m)}, w) = G_h(t, x^{(m)}, \tilde{w})$. We assume that F_h and G_h satisfy condition (V_h) and we consider classical solutions to (4), (5).

Let $W_h\colon F_c(B_h, \mathbb{R}) \to C([-r, 0], \mathbb{R}_+)$ be an operator defined by

$$W_h[w](t) = \max\left\{|w(t, x^{(m)})|\colon x^{(m)} \in [-2b, 2b]\right\}, \quad t \in [-r, 0].$$

The maximum norms in $C(B, \mathbb{R})$ and $F_c(B_h, \mathbb{R})$ are denoted by $\|\cdot\|_B$ and $\|\cdot\|_{B_h}$ respectively. For $\omega\colon [-r, a] \to \mathbb{R}$ and $t \in [0, a]$ we define $\omega_t\colon [-r, 0]$ as follows: $\omega_t(\tau) = \omega(t + \tau)$, $\tau \in [-r, 0]$. If $\eta, \tilde{\eta} \in C([-r, 0], \mathbb{R})$ then the inequality $\eta \le \tilde{\eta}$ states that $\eta(\tau) \le \tilde{\eta}(\tau)$ for $\tau \in [-r, 0]$.

Assumption $H[\Upsilon]$. The function $\Upsilon\colon [0, a] \times C([-r, 0], \mathbb{R}_+) \to \mathbb{R}_+$ is continuous and it is nondecreasing with respect to the second variable and

1) for $t \in [0, a]$ and for $\upsilon, \tilde{\upsilon} \in C([-r, 0], \mathbb{R}_+)$ such that $\upsilon(\tau) = \tilde{\upsilon}(\tau)$ for $\tau \in [-b_0 - t, 0]$ we have $\Upsilon(t, \upsilon) = \Upsilon(t, \tilde{\upsilon})$,

2) for each $\mu \in C([-b_0, 0], \mathbb{R}_+)$ the *maximal solution* of the Cauchy problem

$$\omega'(t) = \Upsilon(t, \omega_t), \qquad \omega(t) = \mu(t) \qquad \text{for} \quad t \in [-b_0, 0], \tag{6}$$

is defined on $[-b_0, a]$. By a maximal solution of a Cauchy problem we mean a solution which majorizes any other solution of the same problem on the intersection of respective domains.

***Remark* 2.1.** Condition 1 of Assumption $H[\Upsilon]$ states that the value of Υ at $(t, \upsilon) \in [0, a] \times C([-r, 0], \mathbb{R}_+$ depends on t and on the restriction of υ to $[-b_0 - t, 0]$ only.

Assumption $H[F_h, G_h, \psi_h]$. The functions $F_h\colon \Omega_h \to \mathbb{R}^\kappa$, $G_h\colon \Omega_h \to \mathbb{R}$, $\psi_h\colon E_{0.h} \cup \partial_0 E_h \to \mathbb{R}$ satisfy the conditions:

1) F_h and G_h are continuous and they satisfy condition (V_h),

2) there is $\Upsilon\colon [0, a] \times C([-r, 0], \mathbb{R}_+) \to \mathbb{R}_+$ such that Assumption $H[\Upsilon]$ is satisfied and

$$\left| G_h(t, x, w) \right| \le \Upsilon(t, W_h[w]) \qquad \text{on} \quad \Omega_h,$$

3) for $(t, x, w) \in \Omega_h$ we have

$$F_{h.\lambda}(t, x, w) \ge 0 \qquad \text{for} \quad \lambda \in \Lambda' \qquad \text{and} \qquad \sum_{\lambda \in \Lambda} F_{h.\lambda}(t, x, w) \le 0,$$

4) $\psi_h \in F_c(E_{0.h} \cup \partial_0 E_h, \mathbb{R})$ and $\eta_h \in C([-r, 0], \mathbb{R}_+)$ satisfies the conditions: $\left|\psi_h^{(m)}(t)\right| \le \eta_h(t)$ on $E_{0.h}$ and

$$\left|\psi_h^{(m)}(t)\right| \le \omega(t, \eta_h) \qquad \text{on} \quad \partial_0 E_h,$$

where $\omega(\,\cdot\,, \eta_h)$ is the maximal solution to (6) with $\mu = \eta_h$.

Lemma 2.1. *If $h \in H$ and Assumption $H[F_h, G_h, \psi_h]$ is satisfied then there is a solution $z_h\colon E_{0.h} \cup E_h \to \mathbb{R}$ to* (4), (5) *and*

$$\left|z_h^{(m)}(t)\right| \le \omega(t, \eta_h) \qquad \textit{on} \quad E_h. \tag{7}$$

Proof. From classical theorems on functional differential equations it follows that there is $\tilde{\varepsilon} > 0$ such that the solution z_h to (4), (5) is defined on $(E_{0.h} \cup E_h) \cap ([-b_0, \tilde{\varepsilon}) \times \mathbb{R}^n)$. Suppose that z_h is defined on $(E_{0.h} \cup E_h) \cap ([-b_0, \tilde{a}) \times \mathbb{R}^n)$ and it is non continuable. We prove that

$$\left|z_h^{(m)}(t)\right| \le \omega(t, \eta_h) \qquad \text{on} \quad E_h \cap ([0, \tilde{a}) \times \mathbb{R}^n). \tag{8}$$

For $\varepsilon > 0$ we denote by $\omega(\,\cdot\,, \eta_h, \varepsilon)$ the maximal solution of the Cauchy problem

$$\omega'(t) = \Upsilon(t, \omega_t) + \varepsilon, \qquad \omega(t) = \eta_h(t) + \varepsilon \qquad \text{for} \quad t \in [-b_0, 0].$$

There is $\tilde{\varepsilon} > 0$ such that for $0 < \varepsilon < \tilde{\varepsilon}$ the solution $\omega(\,\cdot\,, \eta_h, \varepsilon)$ is defined on $[-b_0, \tilde{a}]$ and

$$\lim_{\varepsilon \to 0} \omega(t, \eta_h, \varepsilon) = \omega(t, \eta_h) \qquad \text{uniformly on} \quad [-b_0, \tilde{a}].$$

We prove that

$$|z_h^{(m)}(t)|_\infty < \omega(t, \eta_h, \varepsilon) \qquad \text{on} \quad E_h \cap ([0, \tilde{a}) \times \mathbb{R}^n), \tag{9}$$

where $0 < \varepsilon < \tilde{\varepsilon}$. Set

$$\omega_h(t) = \max\left\{|z_h^{(m)}(t)| \colon m \in \mathbb{K}\right\}, \quad t \in [-b_0, \tilde{a}).$$

It is clear that $\omega_h(t) < \omega(t, \eta_h, \varepsilon)$ for $t \in [-b_0, 0]$. Suppose by contradiction that (9) fails to be true. Then there is $t \in (0, \tilde{a})$ such that

$$\omega_h(\tau) < \omega(\tau, \eta_h, \varepsilon) \qquad \text{for} \quad \tau \in [-b_0, t) \qquad \text{and} \qquad \omega_h(t) = \omega(t, \eta_h, \varepsilon).$$

This gives

$$D_-\omega_h(t) \geq \omega'(t, \eta_h, \varepsilon), \tag{10}$$

where D_- is the left-hand lower Dini derivative. There is $m \in \mathbb{K}$ such that $\omega_h(t) = z_h^{(m)}(t)$ or $\omega_h(t) = -z_h^{(m)}(t)$. Let us consider the first case. We deduce from condition 4 of Assumption $H[F_h, G_h, \psi_h]$ that $m \in \operatorname{Int}\mathbb{K}$. Then we have

$$D_-\omega_h(t) \leq \frac{d}{dt} z_h^{(m)}(t) = G_h(t, x^{(m)}, (z_h)_{[t,m]}) + F_h(t, x^{(m)}, (z_h)_{[t,m]}) \circ (z_h)_{\langle t,m\rangle} \leq$$

$$\leq \Upsilon(t, \omega_t(\,\cdot\,, \eta_h, \varepsilon)) + \omega_h(t) \sum_{\lambda \in \Lambda} F_{h.\lambda}(t, x^{(m)}, (z_h)_{[t,m]}) < \Upsilon(t, \omega_t(\,\cdot\,, \eta_h, \varepsilon)) + \varepsilon = \omega'(t, \eta_h, \varepsilon),$$

which contradicts (10). The case $\omega_h(t) = -z_h^{(m)}(t)$ can be treated in a similar way. Hence, the proof of (9) is completed. From (9) we obtain in the limit, letting ε tend to 0, inequality (8).

We prove that there are the limits $\lim_{t \to \tilde{a}} z_h^{(m)}(t)$ for $m \in \operatorname{Int}\mathbb{K}$. Write

$$\hat{\omega}(t, \tilde{t}) = \max\left\{\left|z_h^{(m)}(t) - z_h^{(m)}(\tilde{t}\,)\right| \colon m \in \operatorname{Int}\mathbb{K}\right\},$$

where $t, \tilde{t} \in [0, \tilde{a})$. We prove that

$$\hat{\omega}(t, \tilde{t}) \leq \left|\omega(t, \eta_h) - \omega(\tilde{t}, \eta_h)\right| \qquad \text{for} \quad t,\, \tilde{t} \in [0, \tilde{a}). \tag{11}$$

Suppose that $t \geq \tilde{t}$. There is $m \in \operatorname{Int}\mathbb{K}$ such that $\hat{\omega}(t, \tilde{t}) = z_h^{(m)}(t) - z_h^{(m)}(\tilde{t})$ or $\hat{\omega}(t, \tilde{t}\,) = -\big[z_h^{(m)}(t) - z_h^{(m)}(\tilde{t})\big]$. Let us consider the first case. Then we have

$$\hat{\omega}(t, \tilde{t}) = \int_{\tilde{t}}^{t} \Big[G_h(\tau, x^{(m)}, (z_h)_{[\tau,m]}) + F_h(\tau, x^{(m)}, (z_h)_{[\tau,m]}) \circ (z_h)_{\langle \tau,m\rangle}\Big]\, d\tau \leq$$

$$\leq \int_{\tilde t}^{t} \Upsilon(\tau, \omega_\tau(\,\cdot\,, \eta_h))\, d\tau + \omega(t, \eta_h) \int_{\tilde t}^{t} \sum_{\lambda\in\Lambda} F_{h.\lambda}(\tau, x^{(m)}, (z_h)_{[\tau,m]})\, d\tau \leq \omega(t, \eta_h) - \omega(\tilde t, \eta_h).$$

The case $\hat\omega(t, \tilde t) = -\big[z_h^{(m)}(t) - z_h^{(m)}(\tilde t)\big]$ can be treated in a similar way. Then (11) is proved.

It follows from (11) that there are the limits

$$\lim_{t\to\tilde a} z_h^{(m)}(t) = z_h^{(m)}(\tilde a), \quad m \in \operatorname{Int} \mathbb{K}.$$

Then the solution z_h is defined on $(E_{0.h} \cup E_h) \cap \big([-b_0, \tilde a] \times \mathbb{R}^n\big)$. If $\tilde a < a$ then there is $\bar a > \tilde a$ such that z_h is defined on $(E_{0.h} \cup E_h) \cap \big([-b_0, \bar a] \times \mathbb{R}^n\big)$. This contradicts our assumption that z_h given on $(E_{0.h} \cup E_h) \cap \big([-b_0, \tilde a) \times \mathbb{R}^n\big)$ is non continuable. It follows that z_h is defined on $E_{0.h} \cup E_h$ and estimate (7) is satisfied.

Lemma 2.1 is proved.

We will consider approximate solutions to (4), (5). Let $X_h \subset F_c(B_h, \mathbb{R})$ and $Y_h \subset F(A_h, \mathbb{R})$ be fixed subspaces. Suppose that $\tilde z_h \in F_c(E_{0.h} \cup E_h, \mathbb{R})$ and there exists $\dfrac{d}{dt}\tilde z_h^{(m)}(t)$ on E_h and there are $\vartheta, \gamma\colon H \to \mathbb{R}_+$ such that

$$\left|\frac{d}{dt}\tilde z_h^{(m)}(t) - \mathbf{F}_h\big(t, x^{(m)}, (\tilde z_h)_{[t,m]}, (\tilde z_h)_{\langle t,m\rangle}\big)\right| \leq \gamma(h), \quad m \in \operatorname{Int}\mathbb{K}, \tag{12}$$

$$\left|\tilde z_h^{(m)}(t) - \psi_h^{(m)}(t)\right| \leq \vartheta(h) \qquad \text{for} \quad (t, x^{(m)}) \in E_{0.h} \cup \partial_0 E_h, \tag{13}$$

$$\lim_{h\to 0} \gamma(h) = 0, \qquad \lim_{h\to 0} \vartheta(h) = 0, \tag{14}$$

and

$$\big((\tilde z_h)_{[t,m]}, (\tilde z_h)_{\langle t,m\rangle}\big) \in X_h \times Y_h \qquad \text{for} \quad (t, x^{(m)}) \in E_h. \tag{15}$$

The function $\tilde z_h$ satisfying the above relations is treated as an approximate solution to (4), (5). It is important in our considerations that we look for approximate solutions to (4), (5) such that condition (15) is satisfied with a fixed subspaces $X_h \times Y_h \subset F_c(B_h, \mathbb{R}) \times F(A_h, \mathbb{R})$. Remarks 2.2 and 3.3 contain additional comments on (15).

We give a theorem on the estimate of the difference between the exact and approximate solutions to (4), (5).

Assumption $\boldsymbol{H[\sigma]}$. The function $\sigma\colon [0, a] \times C([-r, 0], \mathbb{R}_+) \to \mathbb{R}_+$ satisfies the conditions:

1) σ is continuous and for each $t \in [0, a]$ the function $\sigma(t, \cdot)\colon C([-r, 0], \mathbb{R}_+) \to \mathbb{R}_+$ is nondecreasing;

2) for $t \in [0, a]$ and for $\upsilon, \tilde\upsilon \in C([-r, 0], \mathbb{R}_+)$ such that $\upsilon(\tau) = \tilde\upsilon(\tau)$ for $\tau \in [-b_0 - t, 0]$ we have $\sigma(t, \upsilon) = \sigma(t, \tilde\upsilon)$,

3) for each $c \geq 1$ the maximal solution of the Cauchy problem

$$\omega'(t) = c\,\sigma(t, \omega_t), \qquad \omega(t) = 0 \qquad \text{for} \quad t \in [-b_0, 0],$$

is $\tilde\omega(t) = 0$ for $t \in [-b_0, a]$.

Theorem 2.1. *Suppose that*

1) *$h \in H$ and Assumption $H[F_h, G_h, \psi_h]$ is satisfied and $z_h\colon E_{0.h} \cup E_h \to \mathbb{R}$ is the solution to* (4), (5) *and*

$$(z_h)_{[t,m]} \in X_h \qquad \textit{for} \quad (t, x^{(m)}) \in E_h,$$

2) *$\tilde{z}_h \in F_c(E_{0.h} \cup E_h, \mathbb{R})$, the derivatives $\dfrac{d}{dt}\tilde{z}_h^{(m)}(t)$ exist on E_h and there are $\vartheta, \gamma\colon H \to \mathbb{R}_+$ such that condition* (12)–(15) *are satisfied,*

3) *there exists $\sigma\colon [0,a] \times C([-r,0], \mathbb{R}_+) \to \mathbb{R}_+$ such that Assumption $H[\sigma]$ is satisfied and for $w, \tilde{w} \in X_h$, $\zeta \in Y_h$ we have*

$$\big|\mathbf{F}_h(t,x,w,\zeta) - \mathbf{F}_h(t,x,\tilde{w},\zeta)\big| \le \sigma(t, W_h[w - \tilde{w}]),$$

where $(t,x) \in E_h$,

4) *the maximal solution $\omega(\,\cdot\,, \gamma, \upsilon)$ of the Cauchy problem*

$$\omega'(t) = \sigma(t, \omega_t) + \gamma(h), \qquad \omega(t) = \vartheta(h) \qquad \textit{for} \quad t \in [-b_0, 0], \tag{16}$$

is defined on $[-b_0, a]$.

Under these assumptions we have

$$\big|\tilde{z}^{(m)}(t) - z_h^{(m)}(t)\big| \le \omega(t, \gamma, \vartheta) \qquad \textit{on} \quad E_h. \tag{17}$$

Proof. For $\varepsilon > 0$ we denote by $\omega(\,\cdot\,, \gamma, \vartheta, \varepsilon)$ the maximal solution of the Cauchy problem

$$\omega'(t) = \sigma(t, \omega_t) + \gamma(h) + \varepsilon,\ \omega(t) = \vartheta(h) + \varepsilon \qquad \text{for} \quad t \in [-r, 0].$$

There exists $\tilde{\varepsilon} > 0$ such that for every $0 < \varepsilon < \tilde{\varepsilon}$ the solution $\omega(\,\cdot\,, \gamma, \vartheta, \varepsilon)$ is defined on $[-b_0, a]$ and

$$\lim_{\varepsilon \to 0} \omega(t, \gamma, \vartheta, \varepsilon) = \omega(t, \gamma, \vartheta) \qquad \text{uniformly on} \quad [-b_0, a].$$

Set

$$\hat{\omega}_h(t) = \max\big\{\big|z_h^{(m)}(t) - \tilde{z}_h^{(m)}(t)\big|\colon\ m \in \mathbb{K}\big\}, \quad t \in [-b_0, a].$$

We prove that

$$\hat{\omega}_h(t) < \omega(t, \gamma, \vartheta, \varepsilon) \qquad \text{for} \quad t \in [-b_0, a], \tag{18}$$

where $0 < \varepsilon < \tilde{\varepsilon}$. It is clear that $\hat{\omega}_h(t) < \omega(t, \gamma, \vartheta, \varepsilon)$ for $t \in [-r, 0]$. Suppose by contradiction that assertion (18) fails to be true. Then the set

$$I_+ = \big\{t \in (0, a]\colon\ \hat{\omega}_h(t) \ge \omega(t, \gamma, \vartheta, \varepsilon)\big\}$$

is not empty. If we put $t = \min I_+$, it is clear that $t > 0$ and

$$D_-\hat{\omega}_h(t) \ge \omega'(t, \gamma, \vartheta, \varepsilon). \tag{19}$$

There is $m \in \mathbb{K}$ such that $\hat{\omega}_h(t) = z_h^{(m)}(t) - \tilde{z}_h^{(m)}(t)$ or $\hat{\omega}_h(t) = -\big[z_h^{(m)}(t) - \tilde{z}_h^{(m)}(t)\big]$. Let us consider the first case. We conclude from (13) that $m \in \operatorname{Int}\mathbb{K}$. It follows from Assumption $H[F_h, G_h, \psi_h]$ that

$$D_-\hat{\omega}_h(t) \le \frac{d}{dt}\big[z_h^{(m)}(t) - \tilde{z}_h^{(m)}(t)\big] \le$$

$$\leq \sigma(t, \omega_t(\,\cdot\,, \gamma, \vartheta), \varepsilon) + \gamma(h) + \hat{\omega}_h(t) \sum_{\lambda \in \Lambda} F_{h.\lambda}\big(t, x^{(m)}, (z_h)_{[t,m]}\big).$$

This gives

$$D_{-}\hat{\omega}(t) < \sigma(t, \omega(t, \gamma, \vartheta, \varepsilon)) + \varepsilon = \omega'(t, \gamma, \vartheta, \varepsilon)$$

which contradicts (19). The case $\hat{\omega}_h(t) = -\big[z_h^{(m)}(t) - \tilde{z}_h^{(m)}(t)\big]$ can be treated in a similar way. Then $I_{+} = \varnothing$ and (18) is proved. From (18) we obtain in the limit, letting ε tend to 0, inequality (17).

Theorem 2.1 is proved.

***Remark* 2.2.** Let us consider the following condition:

3A) there exists $\sigma\colon [0, a] \times C([-r, 0], \mathbb{R}_{+}) \to \mathbb{R}_{+}$ such that Assumption $H[\sigma]$ is satisfied and for w, $\tilde{w} \in F_c(B_h, \mathbb{R})$ and for $\zeta \in F(A_h, \mathbb{R})$ we have

$$\big|\mathbf{F}_h(t, x, w, \zeta) - \mathbf{F}_h(t, x, \tilde{w}, \zeta)\big| \leq \sigma(t, W_h[w - \tilde{w}]),$$

where $(t, x) \in E_h$, It is clear that Theorem 2.1 remains true if condition 3 is replaced by 3A. We show in Sections 3 and 4 that assumption 3, is important in our considerations. The operators $\mathbf{F}_h$ generated by (1) or (3) satisfy condition 3 and they do not satisfy 3A.

***Remark* 2.3.** Suppose that the assumptions 1, 2 of Theorem 2.1 are satisfied and there exists $L \in \mathbb{R}_{+}$ such that and for w, $\tilde{w} \in X_h$, $\zeta \in Y_h$ we have

$$\big|\mathbf{F}_h(t, x, w, \zeta) - \mathbf{F}_h(t, x, \tilde{w}, \zeta)\big| \leq\; L\|w - \tilde{w}\|,$$

where $(t, x) \in E_h$. Then

$$\big|\tilde{z}^{(m)}(t) - z_h^{(m)}(t)\big| \leq \tilde{\alpha}(h) \qquad \text{on} \quad E_h$$

where

$$\tilde{\alpha}(h) = \vartheta(h) e^{La} + \gamma(h) \frac{e^{La} - 1}{L} \qquad \text{if} \quad L > 0, \tag{20}$$

and

$$\tilde{\alpha}(h) = \vartheta(h) + a\gamma(h) \qquad \text{if} \quad L = 0. \tag{21}$$

The above estimates are obtained by solving problem (16).

3. First order partial functional differential equations. We construct a numerical method of lines for (1), (2). The following assumptions of given functions are needed in our considerations.

Assumption $H_0[F, G]$. The functions $F\colon \Omega \to \mathbb{R}^n$, $G\colon \Omega \to \mathbb{R}$ are continuous and they satisfy condition (V) and there is $\tilde{x} \in (-b, b)$, $\tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_n)$, such that

$$(x_i - \tilde{x}_i)\, F_i(t, x, w) \geq 0 \qquad \text{on} \quad \Omega \qquad \text{for} \quad 1 \leq i \leq n. \tag{22}$$

***Remark* 3.1.** Two types of assumptions are needed in theorems on the existence and uniqueness of classical or generalized solutions to (1), (2). The first type conditions deal with regularity of given functions. It is assumed in theorems on uniqueness of solutions that F and G are continuous and they satisfy nonlinear estimates of the Perron-type with respect to functional variables. More restrictive conditions are needed in theorems on the existence of solutions.

The assumptions of the second type concern the bicharacteristics and they have the following form. Suppose that $z \in C(E_0 \cup E, \mathbb{R})$. Let us denote by $g[z](\,\cdot\,, t, x) = (g_1(\,\cdot\,, t, x), \ldots, g_n(\,\cdot\,, t, x))$ the solution of the Cauchy problem

$$y'(\tau) = F(\tau, y(\tau), z_{(\tau, y(\tau))}), \quad y(t) = x,$$

where $(t,x) \in E$. Condition (22) asserts that the function $g_i(\,\cdot\,, t, x)$ is non increasing if $\tilde{x}_i \le x_i \le b_i$ and it is non decreasing if $-b_i \le x_i < \tilde{x}_i$. This property of bicharacteristics and assumptions on regularity of given functions ensure the existence and uniqueness of classical or generalized solutions to (1), (2) (see [6] and [11] (Chapter 5)).

Let us denote by $\hat{H}$ the set of all $h \in H$ satisfying the condition $\|h\| < \min\{b_\star, b^\star\}$, where $b_\star = \min\{b_i - \tilde{x}_i \colon 1 \le i \le n\}$ and $b^\star = \min\{b_i + \tilde{x}_i \colon 1 \le i \le n\}$.

Solutions of differential difference equations corresponding to (1), (2) are defined on $E_{0.h} \cup E_h$. Equation (1) contains the functional variable $z_{(t,x)}$ which is an element of the space $C(D[t,x], \mathbb{R})$. Then we need an interpolating operator $T_h \colon F_c(B_h, \mathbb{R}) \to C(B, \mathbb{R})$. We assume that T_h satisfies the following condition (V): if $w, \tilde{w} \in F_c(B_h, \mathbb{R})$, $(t, x^{(m)}) \in E_h$ and $w(\tau, y) = \tilde{w}(\tau, y)$ for $(\tau, y) \in$ $\in D_h[t, m]$ then $(T_h w)(\tau, y) = (T_h \tilde{w})(\tau, y)$ for $(\tau, y) \in D[t, x^{(m)}]$. In the next part of the paper we adopt additional assumptions on T_h.

Write $e_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{R}^n$ with 1 standing on the i th place. For $z \colon E_{0.h} \cup E_h \to \mathbb{R}$ and $(t, x^{(m)}) \in E_h$, $m \in \operatorname{Int} \mathbb{K}$, we write

$$\delta_i^{(m)}(t) = \frac{1}{h_i}\big[z^{(m+e_i)}(t) - z^{(m)}(t)\big] \qquad \text{if} \quad x_i^{(m_i} \ge \tilde{x}_i,$$

$$\delta_i^{(m)}(t) = \frac{1}{h_i}\big[z^{(m)}(t) - z^{(m-e_i)}(t)\big] \qquad \text{if} \quad x_i^{(m_i} < \tilde{x}_i,$$

and we put $i = 1, \ldots, n$ in the above definitions. Set

$$\mathbb{F}_h[z]^{(m)}(t) = \sum_{i=1}^n F_i(t, x^{(m)}, T_h z_{[t,m]}) \delta_i z^{(m)}(t) + G(t, x^{(m)}, T_h z_{[t,m]})$$

and suppose that $\psi_h \colon E_{0.h} \cup \partial_0 E_h \to \mathbb{R}$ is a given function. We approximate classical solutions to (1), (2) with solutions of differential difference equations

$$\frac{d}{dt} z^{(m)}(t) = \mathbb{F}_h[z]^{(m)}(t), \quad m \in \operatorname{Int} \mathbb{K}, \tag{23}$$

with the initial boundary conditions

$$z^{(m)}(t) = \psi_h^{(m)}(t) \qquad \text{on} \quad E_{0.h} \cup \partial_0 E_h. \tag{24}$$

We claim that we have obtained a functional differential problem which is a particular case of (4), (5). For $m \in \operatorname{Int} \mathbb{K}$ we put

$$I_+[m] = \big\{i \in \{1, \ldots, n\} \colon \; \tilde{x}_i \le x_i^{(m_i)} < b_i\big\},$$

$$I_-[m] = \big\{i \in \{1, \ldots, n\} \colon \; -b_i < x_i^{(m_i)} < \tilde{x}_i\big\}$$

and

$$\mathbf{F}_h(t, x^{(m)}, w, \zeta) = G(t, x^{(m)}, T_h w) + \sum_{i=1}^n F_i(t, x^{(m)}, T_h w) \delta_i \zeta^{(\theta)}, \tag{25}$$

where $(t, x^{(m)}, w, \zeta) \in \Omega_h \times F(A_h, \mathbb{R})$. The expressions $\delta_i \zeta^{(\theta)}$, $1 \le i \le n$, are defined in the following way:

$$\delta_i \zeta^{(\theta)} = \frac{1}{h_i}\big[\zeta^{(e_i)} - \zeta^{(\theta)}\big] \qquad \text{for} \quad i \in I_+[m],$$

$$\delta_i \zeta^{(\theta)} = \frac{1}{h_i}\big[\zeta^{(\theta)} - \zeta^{(-e_i)}\big] \qquad \text{for} \quad i \in I_-[m],$$

and we put $i = 1, \ldots, n$ in the above formulas. It is clear that problem (23), (24) is equivalent to (4), (5) with the above defined $\mathbf{F}_h$.

We formulate assumptions on F, G, ψ and T_h. Let us denote by $W\colon C(B, \mathbb{R}) \to C([-r, 0], \mathbb{R}_+)$ the operator given by

$$W[w](t) = \max\big\{|w(t, x)|\colon x \in [-2b, 2b]\big\}, \quad t \in [-r, 0].$$

Assumption $H_\star[F, G, \psi]$. The functions $F\colon \Omega \to \mathbb{R}^n$, $G\colon \Omega \to \mathbb{R}$ satisfy Assumption $H_0[F, G]$ and

1) there is $\Upsilon\colon [0, a] \times C([-r, 0], \mathbb{R}_+) \to \mathbb{R}_+$ such that Assumption $H[\Upsilon]$ is satisfied and

$$\big|G(t, x, w)\big| \le \Upsilon(t, W[w]) \qquad \text{on} \quad \Omega,$$

2) $\psi \in C(E_0 \cup \partial_0 E, \mathbb{R})$, $\psi_h \in F_c(E_{0.h} \cup \partial_0 E, \mathbb{R})$ and there is $\alpha_0\colon \hat{H} \to \mathbb{R}_+$ such that

$$\big|\psi^{(m)}(t) - \psi_h^{(m)}(t)\big| \le \alpha_0(h) \qquad \text{on} \quad E_{0.h} \cup \partial_0 E_h \qquad \text{and} \qquad \lim_{h \to 0} \alpha_0(h) = 0,$$

3) $\eta \in C([-b_0, 0], \mathbb{R}_+)$ and

$$\big|\psi^{(m)}(t)\big| \le \eta(t), \qquad \big|\psi_h^{(m)}(t)\big| \le \eta(t) \qquad \text{on} \quad E_{0.h}$$

and the maximal solution $\omega(\,\cdot\,, \eta)$ to (6) with $\mu = \eta$ satisfies the conditions

$$\big|\psi^{(m)}(t)\big| \le \omega(t, \eta), \qquad \big|\psi_h^{(m)}(t)\big| \le \omega(t, \eta) \qquad \text{on} \quad \partial_0 E_h.$$

Assumption $H[T_h]$. The operator $T_h\colon F_c(B_h, \mathbb{R}) \to C(B, \mathbb{R})$ satisfies the conditions:

1) if $w, \tilde{w} \in F_c(B_h, \mathbb{R})$, $(t, x^{(m)}) \in E_h$ and $w(\tau, y) = \tilde{w}(\tau, y)$ for $(\tau, y) \in D_h[t, m]$, then $(T_h w)(\tau, y) = (T_h \tilde{w})(\tau, y)$ for $(\tau, y) \in D[t, x^{(m)}]$,

2) for $w, \tilde{w} \in F_c(B_h, \mathbb{R})$ we have

$$\|T_h w - T_h \tilde{w}\|_B \le \|w - \tilde{w}\|_{B_h},$$

3) if $\theta_h\colon B_h \to \mathbb{R}$ is given by $\theta_h(\tau, y) = 0$ for $(\tau, y) \in B_h$, then $T_h[\theta_h](\tau, y) = 0$ for $(\tau, y) \in B$,

4) for each $w\colon B \to \mathbb{R}$ which is of class C^1 there is $\tilde{c} \in \mathbb{R}_+$ such that

$$\|w - T_h w_h\|_B \le \tilde{c}\|h\|,$$

where w_h is the restriction of w to B_h.

***Remark* 3.2.** An example of the operator T_h satisfying Assumption $H[T_h]$ can be found in [11], Chapter 5.

Lemma 3.1. *If Assumptions $H_\star[F,G,\psi]$, $H[T_h]$ are satisfied then for each $h\in\hat H$ there exists a solution $\hat z_h\colon E_{0.h}\cup E_h\to\mathbb{R}$ to* (23), (24) *and*

$$\left|\hat z_h^{(m)}(t)\right|\le\omega(t,\eta)\qquad on\quad E.$$

Proof. We apply Lemma 2.1. Let us define $G_h\colon\Omega_h\to\mathbb{R}$, $F_h\colon\Omega_h\to\mathbb{R}^\kappa$, $F_h=\{\,F_{h.\lambda}\,\}_{\lambda\in\Lambda}$, in the following way:

$$G_h(t,x^{(m)},w)=G(t,x^{(m)},T_hw),$$

$$F_{h,\theta}(t,x^{(m)},w)=\sum_{i\in I_-[m]}\frac{1}{h_i}F_i(t,x^{(m)},T_hw)-\sum_{i\in I_+[m]}\frac{1}{h_i}F_i(t,x^{(m)},T_hw)$$

and

$$F_{h.e_i}(t,x^{(m)},w)=\frac{1}{h_i}F_i(t,x^{(m)},T_hw)\qquad\text{for}\quad i\in I_+[m],$$

$$F_{h.-e_i}(t,x^{(m)},w)=-\frac{1}{h_i}F_i(t,x^{(m)},T_hw)\qquad\text{for}\quad i\in I_i[m]$$

and we put $i=1,\ldots,n$ in the above definitions. Set

$$F_{h.\lambda}(t,x^{(m)},w)=0\qquad\text{for}\quad\lambda\in\Lambda\setminus\big\{I_+[m]\cup I_-[m]\cup\{\,\theta\,\}\big\},$$

and $\mathbf{F}_h(t,x^{(m)},w,\zeta)=G_h(t,x^{(m)},w)+F_h(t,x^{(m)},w)\circ\zeta$. Then Assumption $H[F_h,G_h,\psi_h]$ is satisfied. Our lemma follows from Lemma 2.1.

Now we construct estimates of solutions to (1), (2).

Lemma 3.2. *Suppose that Assumption $H_\star[F,G,\psi]$ is satisfied and $v\colon E_0\cup E\to\mathbb{R}$ is a solution to* (1), (2) *and v is of class C^1 on E. Then*

$$|v(t,x)|\le\omega(t,\eta)\quad on\quad E. \tag{26}$$

Proof. For $\varepsilon>0$ we denote by $\omega(\,\cdot\,,\eta,\varepsilon)$ the maximal solution of the Cauchy problem

$$\omega'(t)=\Upsilon(t,\omega_t)+\varepsilon,\qquad\omega(t)=\eta(t)+\varepsilon\qquad\text{for}\quad t\in[-b_0,0]. \tag{27}$$

There is $\varepsilon_0>0$ such that for $0<\varepsilon<\varepsilon_0$ the solution $\omega(\,\cdot\,,\eta,\varepsilon)$ is defined on $[-b_0,a]$ and

$$\lim_{\varepsilon\to0}\omega(t,\eta,\varepsilon)=\omega(t,\eta)\qquad\text{uniformly on}\quad[-b_0,a].$$

Write

$$\tilde\omega(t)=\max\big\{|v(t,x)|\colon x\in[-b,b]\big\},\quad t\in[-b_0,a].$$

We prove that

$$\tilde\omega(t)<\omega(t,\eta,\varepsilon)\qquad\text{for}\quad t\in[-b_0,a]. \tag{28}$$

It is clear that $\tilde\omega(\tau)<\omega(\tau,\eta,\varepsilon)$ for $\tau\in[-b_0,0]$. Suppose by contradiction that (28) fails to be true. Then there is $t\in(0,a]$ such that

$$\tilde\omega(\tau)<\omega(\tau,\eta,\varepsilon)\qquad\text{for}\quad\tau\in[0,t)\qquad\text{and}\qquad\tilde\omega(t)=\omega(t,\eta,\varepsilon).$$

Then we have

$$D_-\tilde{\omega}(t) \geq \omega'(t,\eta,\varepsilon). \tag{29}$$

There is $x \in [-b,b]$ such that $\tilde{\omega}(t) = |v(t,x)|$. It follows from condition 3 of Assumption $H_\star[F,G,\psi]$ that $(t,x) \notin \partial_0 E$ and consequently $\partial_x v(t,x) = \theta$. Let us consider the case when $\tilde{\omega}(t) = v(t,x)$. Then we have

$$D_-\tilde{\omega}(t) \leq \partial_t v(t,x) \leq \Upsilon(t,\tilde{\omega}_t) < v(t,\omega_t(\,\cdot\,,\eta,\varepsilon) + \varepsilon = \omega'(t,\eta,\varepsilon),$$

which contradicts (29). The case $\tilde{\omega}(t) = -v(t,x)$ can be treated in a similar way. This completes the proof of (28). From (28) we obtain in the limit, letting ε tend to 0, estimate (26).

Lemma 3.2 is proved.

Suppose that Assumptions $H_\star[F,G,\psi]$ and $H[T_h]$ are satisfied. Write $\hat{c} = \omega(a,\eta)$ and

$$X[\hat{c}] = \big\{w \in C(B,\mathbb{R})\colon \|w\|_B \leq \hat{c}\big\}.$$

Assumption $H[F,G,\psi]$. The functions $F\colon \Omega \to \mathbb{R}^n$, $G\colon \Omega \to \mathbb{R}$, $\psi\colon E_0 \cup \partial_0 E \to \mathbb{R}$ satisfy Assumption $H_\star[F,G,\psi]$ and there is $\sigma\colon [0,a] \times C([-r,0],\mathbb{R}_+) \to \mathbb{R}_+$ such that Assumption $G[\sigma]$ is satisfied and

$$\big\|F(t,x,w) - F(t,x,\tilde{w})\big\| \leq \sigma(t, W[w-\tilde{w}]),$$

$$\big|G(t,x,w) - G(t,x,\tilde{w})\big| \leq \sigma(t, W[w-\tilde{w}]),$$

for $(t,x) \in E$ and $w,\tilde{w} \in X[\hat{c}]$.

***Remark* 3.3.** It is important that we have assumed nonlinear estimates of the Perron-type for $\|w\|_B$, $\|\tilde{w}\|_B \leq \hat{c}$. There are differential equations with deviated variables and differential integral equations such that Assumption $H[F,G,\psi$ holds and global estimates are not satisfied. We give suitable examples.

Suppose that $\tilde{F}\colon E\times\mathbb{R} \to \mathbb{R}^n$, $\tilde{F} = (\tilde{F}_1,\ldots,\tilde{F}_n)$, and $\tilde{G}\colon E\times\mathbb{R} \to \mathbb{R}$ are given functions of the variables (t,x,p). Suppose that $\phi_0 \in C(E,\mathbb{R})$, $\phi \in C(E,\mathbb{R}^n)$ and $0 \leq \phi_0(t,x) \leq t$, $\phi(t,x) \in [-b,b]$ for $(t,x) \in E$. Write $\varphi(t,x) = (\phi_0(t,x),\phi(t,x))$ on E. Let $F\colon \Omega \to \mathbb{R}^n$, $G\colon \Omega \to \mathbb{R}$ be defined by

$$F(t,x,w) = \tilde{F}(t,x,w(\varphi(t,x)-(t,x)), \qquad G(t,x,w) = \tilde{G}(t,x,w(\varphi(t,x)-(t,x)). \tag{30}$$

Then (1) reduces to the differential equation with deviated variables

$$\partial_t z(t,x) = \sum_{i=1}^n \tilde{F}_i(t,x,z(\varphi(t,x)))\partial_{x_i} z(t,x) + \tilde{G}(t,x,z(\varphi(t,x))).$$

For the above $\tilde{F}$ and $\tilde{G}$ we put

$$F(t,x,w) = \tilde{F}\left(t,x,\int\limits_{D[t,x]} w(\tau,y)dy\,d\tau\right), \qquad G(t,x,w) = \tilde{G}\left(t,x,\int\limits_{D[t,x]} w(\tau,y)dy\,d\tau\right). \tag{31}$$

Then (1) is equivalent to the differential integral equation

$$\partial_t z(t,x) = \sum_{i=1}^n \tilde{F}_i\left(t,x,\int\limits_{-b_0}^{t}\int\limits_{-x}^{x} z(\tau,y)dy\,d\tau\right)\partial_{x_i} z(t,x) + \tilde{G}\left(t,x,\int\limits_{-b_0}^{t}\int\limits_{-x}^{x} z(\tau,y)dy\,d\tau\right).$$

Suppose that

1) $\tilde{F} \in C(E \times \mathbb{R}, \mathbb{R}^n)$, $\tilde{G} \in C(E \times \mathbb{R}, \mathbb{R})$ and there are $\alpha, \beta \in \mathbb{R}_+$ such that

$$\left|\tilde{G}(t, x, p)\right| \le \alpha + \beta |p| \qquad \text{on} \quad E \times \mathbb{R},$$

2) the derivatives $\partial_p \tilde{F}$, $\partial_p \tilde{G}$ exist on $E \times \mathbb{R}$ and $\partial_p \tilde{F} \in C(E \times \mathbb{R}, \mathbb{R}^n)$, $\partial_p \tilde{G} \in C(E \times \mathbb{R}, \mathbb{R})$,

3) the functions $\partial_p \tilde{F}$ and $\partial_p \tilde{G}$ are unbounded on $E \times \mathbb{R}$.

Then the functions F and G given by (30) and by (31) satisfy Assumption $H[F, G, \psi]$ and they do not satisfy global estimates with respect to functional variables.

Theorem 3.1. *Suppose that Assumption $H[F, G, \psi]$ and $H[T_h]$ are satisfied and*

1) $v\colon E_0 \cup E \to \mathbb{R}$ *is a solution to* (1), (2) *and v is of class C^1,*

2) $h \in \hat{H}$ *and* $z_h\colon E_{0.h} \cup E_h \to \mathbb{R}$ *is a solution to* (23), (24).

Then there is $\alpha\colon \hat{H} \to \mathbb{R}_+$ such that

$$\left|v_h^{(m)}(t) - z_h^{(m)}(t)\right| \le \alpha(h) \qquad \text{on} \quad E_h \qquad \text{and} \qquad \lim_{h \to 0} \alpha(h) = 0, \tag{32}$$

where v_h is the restriction of v to $E_{0.h} \cup E_h$.

Proof. We apply Theorem 2.1 to prove (32). Let Γ_h be defined by the relations

$$\frac{d}{dt} v_h^{(m)}(t) = \mathbb{F}_h[v_h]^{(m)}(t) + \Gamma_h^{(m)}(t), \qquad t \in [0, a], \quad m \in \operatorname{Int} \mathbb{K}.$$

It follows from Assumption $H[T_h]$ and from the definition of $\delta v_h = (\delta_1 v_h, \ldots, \delta_n v_h)$ that there is $\gamma\colon \hat{H} \to \mathbb{R}_+$ such that

$$\left|\Gamma_h^{(m)}(t)\right| \le \gamma(h) \qquad \text{for} \quad t \in [0, a], \quad m \in \operatorname{Int} \mathbb{K} \qquad \text{and} \qquad \lim_{h \to 0} \gamma(h) = 0.$$

There is $\bar{c} \in \mathbb{R}_+$ such that $\|\partial_x v(t, x)\| \le \bar{c}$ on E. Let us denote by Y_h that class of all $\zeta \in F(A_h, \mathbb{R})$ such that

$$\left|\frac{1}{h_i}\left[\zeta^{(e_i)} - \zeta^{(\theta)}\right]\right| \le \bar{c}, \qquad \left|\frac{1}{h_i}\left[\zeta^{(\theta)} - \zeta^{(-e_i)}\right]\right| \le \bar{c}, \quad 1 \le i \le n.$$

Then $(v_h)_{\langle t, m\rangle} \in X_h$ for $(t, x^{(m)} \in E_h$. It follows from Lemmas 3.1 and 3.2 that

$$(v_h)_{[t,m]}, (z_h)_{[t,m]} \in X_h \qquad \text{and} \qquad (v_h)_{\langle t,m\rangle} \in Y_h \qquad \text{for} \quad (t, x^{(m)}) \in E_h.$$

The operator $\mathbf{F}_h$ given by (25) satisfies the condition:

$$\left|\mathbf{F}_h(t, x, w, \zeta) - \mathbf{F}_h(t, x, \tilde{w}, \zeta)\right| \le (1 + \bar{c}) \sigma(t, W_h[w - \tilde{w}]) \tag{33}$$

for $(t, x) \in E_h$, $w, \tilde{w} \in X_h$, $\zeta \in Y_h$. Then all the assumptions of Theorem 2.1 are satisfied and assertion (32) follows.

***Remark* 3.4.** Note that estimate (33) is not satisfied for all $\zeta \in F(A_h, \mathbb{R})$.

Let us suppose that all the assumptions of Theorem 3.1 are satisfied and there is $\tilde{L} \in \mathbb{R}_+$ such that

$$\|F(t, x, w) - F(t, x, \tilde{w})\|, \ |G(t, x, w) - G(t, x, \tilde{w})| \le \tilde{L} \|w - \tilde{w}\|_B,$$

for $(t, x) \in E$, $w, \tilde{w} \in X[\hat{c}]$. Then there is $L \in \mathbb{R}_+$ such that

$$\left|v_h^{(m)}(t) - z_h^{(m)}(t)\right| \le \tilde{\alpha}(h) \qquad \text{on} \quad E_h,$$

where $\tilde{\alpha}\colon \hat{H} \to \mathbb{R}_+$ is given by (20), (21).

We apply the results on the numerical method of lines to differential equations with deviated variables and to differential integral equations. We have transformed initial boundary-value problems into systems of ordinary functional differential equations. The system such obtained is solved by using the explicit Euler method. Let us denote by ε_h the maximal error of the difference method. In the tables we give experimental values for ε_h.

***Example* 3.1.** Put $n = 2$ and $E = [0, 0.25] \times [-0.5, 0.5] \times [-0.5, 0.5]$. Consider the differential equation

$$\partial_t z(t,x,y) = x\left[1 + \cos\pi y \int\limits_{-0.5}^{0.5} z(t,x,s)ds - \frac{2}{\pi} z(t,x,y)\right]^2 \partial_x z(t,x,y) +$$

$$+ y\left[1 - \cos\pi x \int\limits_{-0.5}^{0.5} z(t,s,y)ds + \frac{2}{\pi} z(t,x,y)\right]^2 \partial_y z(t,x,y) +$$

$$+ x\pi^2 \int\limits_{0}^{x} z(t,s,y)ds + y\pi^2 \int\limits_{0}^{y} z(t,x,s)ds + z(t,x,y) + \cos\pi x\, \cos\pi y \tag{34}$$

with the initial boundary conditions

$$z(0,x,y) = 0, \qquad (x,y) \in [-0.5, 0.5] \times [-0.5, 0.5], \tag{35}$$

$$z(t,-0,5,y) = z(t,0.5,y) = 0, \quad t \in [0, 0.25], \quad y \in [-0.5, 0.5], \tag{36}$$

$$z(t,x,-0.5) = z(t,x,0.5) = 0, \quad t \in [0,a], \quad x \in [-0.5, 0.5]. \tag{37}$$

The solution to (34)–(37) is known. It is $v(t,x,y) = (e^t - 1)\cos\pi x\, \cos\pi y$. Table 3.1 gives the maximal errors for several step sizes $h = (h_0, h_1, h_2)$.

Table 3.1

(h_0, h_1, h_2)	ε_h	Time $[s]$
$(2^{-8}, 2^{-6}, 2^{-6})$	$1.47690152 \cdot 10^{-3}$	0.037
$(2^{-9}, 2^{-7}, 2^{-7})$	$7.56321595 \cdot 10^{-4}$	0.196
$(2^{-10}, 2^{-8}, 2^{-8})$	$3.83651338 \cdot 10^{-4}$	4.731
$(2^{-11}, 2^{-9}, 2^{-9})$	$1.93483845 \cdot 10^{-4}$	41.351
$(2^{-12}, 2^{-10}, 2^{-10})$	$9.72385359 \cdot 10^{-5}$	497.0004

***Example* 3.2.** Put $n = 2$, $E = [0, 0.25] \times [-1, 1] \times [-1, 1]$. Consider the equation

$$\partial_t z(t,x,y) = \frac{x}{4}\big[1 + z(t, \sin(x+y), \cos(x+y))\big]^2 \partial_x z(t,x,y) +$$

$$+\frac{y}{4}\big[1 + z(t, \sin(x-y), \cos(x-y))\big]^2 \partial_y z(t,x,y) +$$

$$+z(t, 0.5(x+y), 0.5(x-y)) \sin z(t,x,y) + f(t,x,y) z(t,x,y),$$

$$f(t,x,y) = (x^2+y^2)(1-2t) - 1 - \exp\left\{-t\left(\frac{x^2}{2} + \frac{x^2}{2}\right)\right\} \sin \exp\big\{t(x^2+y^2-1)\big\} \tag{38}$$

with the initial boundary conditions

$$z(0,x,y) = 1, \qquad (x,y) \in [-1,1], \tag{39}$$

$$z(t,-1,y) = z(t,1,y) = \exp\{ty^2\}, \qquad (t,y) \in [0,a] \times [-1,1], \tag{40}$$

$$z(t,x,-1) = z(t,x,1) = \exp\{tx^2\}, \qquad (t,x) \in [0,a] \times [-1,\ 1]]. \tag{41}$$

The solution to (38)–(41) is known. It is $\tilde{z}(t,x,y) = \exp\{t(x^2+y^2-1)\}$. Table 3.2 gives the maximal errors for several step sizes $h = (h_0, h_1, h_2)$.

Table 3.2

(h_0, h_1, h_2)	ε_h	Time $[s]$
$(2^{-6}, 2^{-5}, 2^{-5})$	$1.47049139 \cdot 10^{-3}$	0.048
$(2^{-7}, 2^{-6}, 2^{-6})$	$7.53238436 \cdot 10^{-4}$	0.319
$(2^{-8}, 2^{-7}, 2^{-7})$	$3.81321493 \cdot 10^{-4}$	2.924
$(2^{-9}, 2^{-8}, 2^{-8})$	$1.91826460 \cdot 10^{-4}$	26.837
$(2^{-10}, 2^{-9}, 2^{-9})$	$9.62069745 \cdot 10^{-5}$	255.386

Note that the right-hand sides of equations (34) and (38) satisfy the assumptions of Theorem 3.1. The local Lipschitz condition with respect to unknown function holds and the global Lipschitz condition is not satisfied.

4. Parabolic functional differential equations. We formulate a differential difference problem corresponding to (3), (2). Write

$$J = \big\{(i,j) \colon i,j = 1,\ldots,n,\ i \neq j\big\}$$

and suppose that we have defined the sets $J_+, J_- \subset J$ such that $J_+ \cup J_- = J$, $J_+ \cap J_- = \varnothing$. We assume that $(i,j) \in J_+$ if $(j,i) \in J_+$. In particular, it may happen that $J_+ = \varnothing$ or $J_- = \varnothing$.

Relations between the sets J_+, J_- and equation (3) are given in Remark 4.1. Let us denote by $\tilde{H}$ the set of all $h \in H$ satisfying the condition: there is $\tilde{d} > 0$ such that $h_i \le \tilde{d} h_j$ for $(i, j) \in J$.

For $z\colon E_{0.h} \cup E_h \to \mathbb{R}$, $(t, x^{(m)}) \in E_h$, $m \in \operatorname{Int} \mathbb{K}$ we write

$$\delta_i^+ z^{(m)}(t) = \frac{1}{h_i}\big[z^{(m+e_i)} - z^{(m)}(t)\big], \qquad \delta_i^- z^{(m)}(t) = \frac{1}{h_i}\big[z^{(m))} - z^{(m-e_i)}(t)\big],$$

where $i = 1, \ldots, n$. The difference operators $\delta = (\delta_1, \ldots, \delta_n)$ and $\delta^{(2)} = \big[\delta_{ij}\big]_{i,j=1,\ldots,n}$ are defined in the following way. Set

$$\delta_i z^{(m)}(t) = \frac{1}{2}\big[\delta_i^+ z^{(m)}(t) + \delta_i^- z^{(m)}(t)\big], \qquad \delta_{ii} z^{(m)}(t) = \delta_i^+ \delta_i^- z^{(m)}(t), \quad 1 \le i \le n,$$

and

$$\delta_{ij} z^{(m)}(t) = \frac{1}{2}\big[\delta_i^+ \delta_j^- z^{(m)}(t) + \delta_i^- \delta_j^+ z^{(m)}(t)\big] \qquad \text{for} \quad (i, j) \in J_-,$$

$$\delta_{ij} z^{(m)}(t) = \frac{1}{2}\big[\delta_i^+ \delta_j^+ z^{(m)}(t) + \delta_i^- \delta_j^- z^{(m)}(t)\big] \qquad \text{for} \quad (i, j) \in J_+.$$

Let $T_h\colon \mathbb{F}_c(B_h, \mathbb{R}) \to C(B, \mathbb{R})$ be an interpolating operator. Write

$$\mathbf{F}_h[z]^{(m)}(t) = \sum_{i,j=1}^{n} F_{ij}(t, x^{(m)}, T_h z_{[t,m]}) \delta_{ij} z^{(m)}(t) +$$

$$+ \sum_{i=1}^{n} F_i(t, x^{(m)}, T_h z_{[t,m]}) \delta_i z^{(m)}(t) + G(t, x^{(m)}, T_h z_{[t,m]}) \delta_{ij} z^{(m)}(t)$$

and suppose that $\psi_h\colon E_{0.h} \cup E_h \to \mathbb{R}$ is a given function. We consider the functional differential equations

$$\frac{d}{dt} = \mathbf{F}_h[z]^{(m)}(t), \quad m \in \operatorname{Int} \mathbb{K}, \tag{42}$$

with the initial boundary condition

$$z^{(m)}(t) = \psi_h^{(m)}(t) \qquad \text{on} \quad E_{0.h} \cup E \partial_0 E_h. \tag{43}$$

We will approximate classical solutions of (3), (2) with solutions to (42), (43).

We claim that (42), (43) is a particular case of (4), (5). Write

$$\mathbb{F}_h(t, x, w, \zeta) = G(t, x, T_h w) + \sum_{i=1}^{n} F_i(t, x, T_h w) \delta_i \zeta^{(\theta)} + \sum_{i,j=1}^{n} F_{ij}(t, x, T_h w) \delta_{ij} \zeta^{(\theta)}, \tag{44}$$

where $(t, x, w) \in \Omega_h$, $\zeta \in F(A_h, \mathbb{R})$. The expressions

$$\delta \zeta^{(\theta)} = \big(\delta_1 \zeta^{(\theta)}, \ldots, \delta_n \zeta^{(\theta)}\big), \qquad \delta^{(2)} \zeta^{(\theta)} = \big[\, \delta_{ij} \zeta^{(\theta)} \,\big]_{i,j=1,\ldots,n}$$

are defined in the following way. Put

$$\delta_i^+ \zeta^{(\theta)} = \frac{1}{h_i}\big[\zeta^{(e_i)} - \zeta^{(\theta)}\big], \qquad \delta_i^- \zeta^{(\theta)} = \frac{1}{h_i}\big[\zeta^{(\theta)} - \zeta^{(-e_i)}\big], \quad 1 \le i \le n.$$

Write

$$\delta_i\zeta^{(\theta)} = \frac{1}{2}\big[\delta_i^+\zeta^{(\theta)} + \delta_i^-\zeta^{(\theta)}\big], \qquad \delta_{ii}\zeta^{(\theta)} = \delta_i^+\delta_i^-\zeta^{(\theta)}, \quad 1 \le i \le n,$$

and

$$\delta_{ij}\zeta^{(\theta)} = \frac{1}{2}\big[\delta_i^+\delta_j^-\zeta^{(\theta)} + \delta_i^-\delta_j^+\zeta^{(\theta)}\big] \qquad \text{for} \quad (i,j) \in J_-,$$

$$\delta_{ij}\zeta^{(\theta)} = \frac{1}{2}\big[\delta_i^+\delta_j^+\zeta^{(\theta)} + \delta_i^-\delta_j^-\zeta^{(\theta)}\big] \qquad \text{for} \quad (i,j) \in J_+.$$

It is easy to see that problem (42), (43) with the above given $\mathbb{F}_h$ is equivalent to (4), (5).

Assumption H_0 $[\mathbf{F}, F, G, \psi]$. The functions $\mathbf{F}\colon \Omega \to M_{n\times n}$, $F\colon \Omega \to \mathbb{R}^n$, $G\colon \Omega \to \mathbb{R}$ and $\psi\colon E_0 \cup \partial_0 E \to \mathbb{R}$ satisfy the conditions

1) $\mathbf{F}$, F, G are continuous and they satisfy condition (V),

2) there is $\Upsilon\colon [0,a] \times C([-r,0],\mathbb{R}_+) \to \mathbb{R}_+$ such that Assumption $H[\Upsilon]$ is satisfied and

$$\big|G(t,x,w)\big| \le \Upsilon(t, Ww) \qquad \text{on} \quad \Omega,$$

3) the matrix $\mathbf{F}$ is symmetric and for $P = (t,x,w) \in \Omega$ we have

$$F_{ij}(t,x,w) \le 0 \qquad \text{for} \quad (i,j) \in J_-, \qquad F_{ij}(t,x,w) \ge 0 \qquad \text{for} \quad (i,j) \in J_+, \tag{45}$$

and

$$\sum_{i,j=1}^{n} F_{ij}(t,x,w) y_i y_j \ge 0 \qquad \text{for} \quad y = (y_1,\ldots,y_n) \in \mathbb{R}^n, \tag{46}$$

4) $h \in \tilde{H}$ and for $P = (t,x,w) \in \Omega$ we have

$$F_{ii}(P) - \sum_{\substack{j=1\\ j\ne i}}^{n} \frac{h_i}{h_j}\big|F_{ij}F(P)\big| - \frac{h_i}{2}\big|F_i(P)\big| \ge 0, \quad 1 = 1,\ldots,n,$$

5) $\psi \in C(E_0 \cup \partial_0 E, \mathbb{R})$, $\psi_h \in F_c(E_{0.h} \cup \partial_0 E_h, \mathbb{R})$ and there is $\alpha_0\colon \tilde{H} \to \mathbb{R}_+$ such that

$$\big|\psi^{(m)}(t) - \psi_h^{(m)}(t)\big| \le \alpha_0(t) \qquad \text{on} \quad E_{0.h} \cup \partial_0 E_h \qquad \text{and} \qquad \lim_{h\to 0} \alpha_0(h0 = 0,$$

6) $\eta \in C([-r,0],\mathbb{R}_+)$ and

$$|\psi^{(m)}(t)| \le \eta(t), \qquad |\psi_h^{(m)}(t)| \le \eta(t) \quad \text{on} \quad E_{0.h}$$

and the maximal solution $\omega(\,\cdot\,,\eta)$ to (6) with $\mu = \eta$ satisfies the conditions:

$$|\psi^{(m)}(t)| \le \omega(t,\eta), \qquad |\psi_h^{(m)}(t)| \le \omega(t,\eta) \qquad \text{on} \quad \partial_0 E_{0.h}.$$

***Remark* 4.1.** We have assumed that $\mathbf{F}$ satisfies the condition: for each $(i,j) \in J$ the function

$$\hat{F}_{ij}(t,x,w) = \text{sign}\, F_{ij}(t,x,w), \quad (t,x,w) \in \Omega,$$

is constant. Inequalities (45) can be considered as definitions of J_- and J_+.

Suppose that there is $\hat{a} > 0$ such that

$$F_{ii}(t,x,w) - \sum_{\substack{j=1\\ j\neq i}}^{n} \bigl|F_{ij}(P)\bigr| \geq \hat{a} \qquad \text{on} \quad \Omega \qquad \text{for} \quad 1 \leq i \leq n.$$

Then condition (46) is satisfied and there is $\varepsilon_0 > 0$ such that for $\|h\| < \varepsilon_0$ and for $h_1 = h_2 = \ldots \ldots = h_n$ condition 4 of Assumption $H_0[\mathbf{F}, F, G, \psi]$ is satisfied.

Lemma 4.1. *If Assumptions $H_0[\mathbf{F}, F, G, \psi]$, $H[T_h]$ are satisfied then for each $h \in \tilde{H}$ there exists a solution $\tilde{z}_h \colon E_{0.h} \cup E_h \to \mathbb{R}$ to* (42), (43) *and*

$$|\tilde{z}_h^{(m)}(t)| \leq \omega(t,\eta) \qquad \text{on} \quad E_h.$$

Proof. We apply Lemma 2.1. Let us define $F_h \colon \Omega_h \to \mathbb{R}^\kappa$, $F_h = \{F_{h.\lambda}\}_{\lambda\in\Lambda}$ and $G_h \colon \Omega_h \to \mathbb{R}$ in the following way. Write

$$\Lambda_0 = \bigl\{\lambda \in \Lambda\colon \text{ there is } i, \ 1 \leq i \leq n, \text{ such that } \lambda = e_i \text{ or } \lambda = -e_i\bigr\},$$

$$\Lambda_I = \bigl\{\lambda \in \Lambda\colon \text{ there is } (i,j) \in J_+ \text{ such that } \lambda = e_i + e_j \text{ or } \lambda = -e_i - e_j\bigr\},$$

$$\Lambda_{II} = \bigl\{\lambda \in \Lambda\colon \text{ there is } (i,j) \in J_- \text{ such that } \lambda = e_i - e_j \text{ or } \lambda = -e_i + e_j\bigr\}$$

and

$$\tilde{\Lambda} = \Lambda \setminus \bigl[\Lambda_0 \cup \Lambda_I \cup \Lambda_{II} \cup \{\,\theta\,\}\bigr].$$

Write

$$G_h(t,x,w) = G(t,x,T_h w),$$

$$F_{h.\theta}(t,x,w) = -2\sum_{i=1}^{n} \frac{1}{h_i^2} F_{ii}(t,x,T_h w) + \sum_{(i,j)\in J} \frac{1}{h_i h_j}\bigl|F_{ij}(t,x,T_h w)\bigr|$$

and

$$F_{h.e_i}(t,x,w) = \frac{1}{h_i^2} F_{ii}(t,x,T_h w) - \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{1}{h_i h_j}\bigl|F_{ij}(t,x,T_h w)\bigr| + \frac{1}{2h_i} F_i(t,x,T_h w),$$

$$F_{h.-e_i}(t,x,w) = \frac{1}{h_i^2} F_{ii}(t,x,T_h w) - \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{1}{h_i h_j}\bigl|F_{ij}(t,x,T_h w)\bigr| - \frac{1}{2h_i} F_i(t,x,T_h w),$$

and we put $i = 1,\ldots,n$ in the above formulas. Moreover we put

$$F_{h.e_i+e_j}(t,x,w) = F_{h.-e_i-e_j}(t,x,w) = \frac{1}{2h_i h_j} F_{ij}(t,x,T_h w) \qquad \text{for} \quad (i,j) \in J_+,$$

$$F_{h.e_i-e_j}(t,x,w) = F_{h.-e_i+e_j}(t,x,w) = -\frac{1}{2h_i h_j} F_{ij}(t,x,T_h w) \qquad \text{for} \quad (i,j) \in J_-,$$

$$F_{h,\lambda}(t,x,w)=0 \qquad \text{for} \quad \lambda\in\tilde{\Lambda},$$

and $\mathbf{F}_h(t,x^{(m)},w,\zeta) = G_h(t,x^{(m)},w) + F_h(t,x^{(m)}w)\circ\zeta$. It follows that all the assumption of Lemma 2.1 are satisfied and the assertion follows.

Now we construct estimates of solutions to (3), (2). We say that $z\in C(E_0\cup E,\mathbb{R})$ is of class $C^{1.2}$ if $z(\,\cdot\,,x)\colon[-b_0,a]\to\mathbb{R}$ is of class C^1 for $x\in[-b,b]$ and $z(t,\cdot\,)\colon[-b,b]\to\mathbb{R}$ is of class C^2 for $t\in[-b_0,a]$.

Lemma 4.2. *If Assumption $H_0[\mathbf{F},F,G,\psi]$ is satisfied and $v\colon E_0\cup E\to\mathbb{R}$ is a solution to* (3), (2) *and v is of class $C^{1.2}$ then*

$$|v(t,x)|\le\omega(t,\eta) \qquad on \quad E. \tag{47}$$

Proof. For $\varepsilon>0$ we denote by $\omega(\,\cdot\,,\eta,\varepsilon)$ the maximal solution to (27). There is $\varepsilon_0>0$ such that for $0<\varepsilon<\varepsilon_0$ the solution $\omega(\,\cdot\,,\eta,\varepsilon)$ is defined on $[-b_0,a]$ and

$$\lim_{\varepsilon\to 0}\omega(t,\eta,\varepsilon)=\omega(t,\eta) \qquad \text{uniformly on} \quad [-b_0,a].$$

Write

$$\hat{\omega}(t)=\max\big\{|v(t,x)|\colon x\in[-b,b]\big\}, \quad t\in[-b_0,a].$$

We prove that

$$\hat{\omega}(t)<\omega(t,\eta,\varepsilon) \qquad \text{for} \quad t\in[-b_0,a], \tag{48}$$

where $0<\varepsilon<\varepsilon_0$. It is clear that $\hat{\omega}(t)<\omega(t,\eta,\varepsilon)$ for $t\in[-b_0,0]$. Suppose by contradiction that (48) fails to be true. Then there is $t\in(0,a]$ such that

$$\hat{\omega}(\tau)<\omega(\tau,\eta,\varepsilon) \qquad \text{for} \quad \tau\in[-b_0,t) \qquad \text{and} \qquad \hat{\omega}(t)=\omega(t,\eta,\varepsilon).$$

This gives

$$D_-\hat{\omega}(t)\ge\omega'(t,\eta,\varepsilon). \tag{49}$$

There is $x\in[-b,b]$ such that $\hat{\omega}(t)=|v(t,x)|$. It follows from condition 6 of Assumption $H_0[\mathbf{F},F,G,\psi]$ that $(t,x)\notin\partial_0 E$. Let us consider the case when $\hat{\omega}(t)=v(t,x)$. Then we have

$$\partial_x v(t,x)=\theta \qquad \text{and} \qquad \sum_{i,j=1}^{n}\partial_{x_ix_j}v(t,x)y_iy_j\le 0 \qquad \text{for} \quad y=(y_1,\ldots,y_n)\in\mathbb{R}^n.$$

The above relations and (46) imply

$$\sum_{i,j=1}^{n}F_{ij}(t,x,v_{(t,x)})\partial_{x_ix_j}v(t,x)\le 0$$

and consequently

$$D_-\hat{\omega}(t)\le\partial_t v(t,x)\le\Upsilon(t,\hat{\omega}_t)<\omega'(t,\eta,\varepsilon),$$

which contradicts (49). The case $\hat{\omega}(t)=-v(t,x)$ can be treated in a similar way. This completes the proof of (48). From this we obtain in the limit, letting ε tend to 0, estimate (47).

Lemma 4.2 is proved.

Now we prove that the method of lines (42), (43) is convergent. Suppose that Assumption $H_0[\mathbf{F}, F, G, \psi]$ and $H[T_h]$ are satisfied. Write $\hat{c} = \omega(a, \eta)$ and

$$X[\hat{c}] = \big\{ w \in C(B, \mathbb{R})\colon \|w\|_B \leq \hat{c} \big\}.$$

Assumption $H[\mathbf{F}, F, G, \psi]$. The functions $\mathbf{F}\colon \Omega \to M_{n\times n}$, $F\colon \omega \to \mathbb{R}^n$, $G\colon \Omega \to \mathbb{R}$, $\psi\colon E_0 \cup \cup\, \partial_0 E$ satisfy Assumption $H_0[\mathbf{F}, F, G, \psi]$ and there is $\sigma\colon [0, a] \times C([-r, 0], \mathbb{R}_+) \to \mathbb{R}_+$ such that Assumption $H[\sigma]$ is satisfied and the expressions

$$\|\mathbf{F}(t, x, w) - \mathbf{F}(t, x, \tilde{w})\|_{n\times n}, \qquad \|F(t, x, w) - F(t, x, \tilde{w})\|, \qquad \big|G(t, x, w) - G(t, x, \tilde{w})\big|$$

for are bounded from above by $\sigma(t, W[w - \tilde{w}])$, where $(t, x) \in E$, $w, \tilde{w} \in X[\hat{c}]$.

***Remark* 4.2.** It is important that we have assumed nonlinear estimates of the Perron-type for $\|w\|_B$, $\|\tilde{w}\|_B \leq \hat{c}$. There are differential equations with deviated variables and differential integral equations such that Assumption $H[\mathbf{F}, F, G, \psi]$ holds and global estimates are not satisfied. Example given in Section 3 for first order partial functional differential equations can be extended on parabolic problems.

Theorem 4.1. *Suppose that Assumption $H[\mathbf{F}, F, G, \psi]$ and $H[T_h]$ are satisfied and*

1) $v\colon E_0 \cup E \to \mathbb{R}$ *is a solution to* (3), (2) *and v is of class $C^{1.2}$*,

2) $h \in \tilde{H}$ *and* $z_h\colon E_{0.h} \cup E_h \to \mathbb{R}$ *is a solution to* (42), (43).

Then there is $\alpha\colon \tilde{H}\colon \to \mathbb{R}_+$ such that

$$\big|v_h^{(m)}(t) - z_h^{(m)}(t)\big| \leq \alpha(h) \qquad \text{on} \quad E_h \qquad \text{and} \qquad \lim_{h\to 0} \alpha(h) = 0. \tag{50}$$

Proof. We apply Theorem 2.1 to prove (50). We start with the observation that

$$\delta_{ij} v_h^{(m)}(t) = \frac{1}{2} \int_0^1 \int_0^1 \partial_{x_j x_j} v(t, x^{(m)} + \tau h_i e_i + \nu h_j e_j)\, d\tau\, d\nu \times$$

$$\times \frac{1}{2} \int_0^1 \int_0^1 \partial_{x_j x_j} v(t, x^{(m)} - \tau h_i e_i - \nu h_j e_j)\, d\tau\, d\nu,$$

where $(i, j) \in J_+$ and

$$\delta_{ij} v_h^{(m)}(t) = \frac{1}{2} \int_0^1 \int_0^1 \partial_{x_j x_j} v(t, x^{(m)} + \tau h_i e_i - \nu h_j e_j)\, d\tau\, d\nu \times$$

$$\times \frac{1}{2} \int_0^1 \int_0^1 \partial_{x_j x_j} v(t, x^{(m)} - \tau h_i e_i + \nu h_j e_j)\, d\tau\, d\nu,$$

where $(i, j) \in J_+$. Let Γ_h be defined by the relations

$$\frac{d}{dt} v_h^{(m)}(t) = \mathbb{F}_h[v_h]^{(m)}(t) + \Gamma_h^{(m)}(t), \qquad t \in [0, a], \quad m \in \operatorname{Int}\mathbb{K}.$$

It follows from the above relation and from Assumption $H[T_h]$ that there is $\gamma\colon \tilde{H} \to \mathbb{R}_+$ such that

$$\big|\Gamma_h^{(m)}(t)\big| \le \gamma(h) \qquad \text{for} \quad t \in [0,a], \quad m \in \operatorname{Int}\mathbb{K} \qquad \text{and} \qquad \lim_{h\to 0}\gamma(h) = 0.$$

We conclude that v_h satisfies (12)–(14). There is $\bar{c} \in \mathbb{R}_+$ such that

$$\|\partial_x v(t,x)\| \le \bar{c}, \qquad \|\partial_{xx} v(t,x)\|_{n\times n} \le \bar{c} \qquad \text{on} \quad E.$$

Let us denote by Y_h the class of all $\zeta \in F(A_h,\mathbb{R})$ satisfying the conditions:

$$\|\delta\zeta^{(\theta)}\| \le \bar{c}, \qquad \|\delta^{(2)}\zeta^{(\theta)}\|_{n\times n} \le \bar{c}.$$

Set $X_h = X[\hat{c}]$. It follows from Lemmas 4.1 and 4.2 that

$$\big((v_h)_{[t,m]}, (v_h)_{\langle t,m\rangle}\big) \in X_h \times Y_h, \quad (z_h)_{[t,m]} \in X_h.$$

We conclude from Assumption $H[\mathbf{F},F,G,\psi]$ and $H[T_h]$ that there is $c_\star > 0$ such that the operator $\mathbf{F}_h$ given by (44) satisfies the condition

$$\big|\mathbf{F}_h(t,x,w,\zeta) - \mathbf{F}_h(t,x,\tilde{w},\zeta)\big| \le (1+c_\star)\sigma(t, W_h[w-\tilde{w}]), \tag{51}$$

where $(t,x) \in E_h$, $w,\tilde{w} \in X_h$, $\zeta \in Y_h$.

Then all the assumptions of Theorem 2.1 are satisfied and condition (50) follows.

***Remark* 4.3.** Note that estimate (51) is not satisfied for all $\zeta \in F(A_h,\mathbb{R})$.

If all the assumptions of Theorem 4.1 are satisfied and there is $\tilde{L} \in \mathbb{R}_+$ such that the expressions

$$\|\mathbf{F}(t,x,w) - \mathbf{F}(t,x,\tilde{w})\|_{n\times n}, \qquad \|F(t,x,w) - F(t,x,\tilde{w})\|, \qquad |G(t,x,w) - G(t,x,\tilde{w})|$$

are bounded from above by $\tilde{L}\|w-\tilde{w}\|_B$ where $(t,x) \in E$ and $w,\tilde{w} \in C[\hat{c}]$ then there is $L \in \mathbb{R}_+$ such that

$$\big|v_h^{(m)}(t) - z_h^{(m)}(t)\big| \le \tilde{\alpha}(h) \qquad \text{on} \quad E_h,$$

where $\tilde{\alpha}\colon \hat{H} \to \mathbb{R}_+$ is given by (20), (21).

We apply the results on the numerical method of lines to differential equations with deviated variables and to differential integral equations. We have transformed initial boundary-value problems into systems of ordinary functional differential equations. The system such obtained is solved by using the explicit Euler method. Let us denote by ε_h the maximal error of the difference method. In the tables we give experimental values for ε_h.

***Example* 4.1.** Put $n = 2$ and $E = [0,\, 0.25] \times [-0.5, 0.5] \times [-0.5, 0.5]$. Consider the differential equation

$$\partial_t z(t,x,y) = \partial_{xx} z(t,x,y) + \partial_{yy} z(t,x,y) + \frac{1}{\pi^2}\partial_{xy} z(t,x,y) -$$

$$-\pi^2 \int_0^x \int_0^y z(t,\mu,\nu)\,d\nu\,d\mu + \int_0^t z(\tau,x,y)d\tau + 2\pi^2 z(t,x,y) + (t+1)\cos\pi x\,\cos\pi y$$

with the initial boundary conditions (35)–(37).

The solution of the above problem is known. It is $v(t,x,y) = \big(e^t - 1\big) \cos \pi x \ \cos \pi y$ (see Table 3.3).

Table 3.3

(h_0, h_1, h_2)	ε_h	Time $[s]$
$(2^{-10}, 2^{-4}, 2^{-4})$	$2.38495068 \cdot 10^{-3}$	0.099
$(2^{-12}, 2^{-5}, 2^{-5})$	$6.05392401 \cdot 10^{-4}$	0.458
$(2^{-14}, 2^{-6}, 2^{-6})$	$1.55723669 \cdot 10^{-4}$	2.851
$(2^{-16}, 2^{-7}, 2^{-7})$	$5.51208597 \cdot 10^{-5}$	27.469
$(2^{-18}, 2^{-8}, 2^{-8})$	$2.39516870 \cdot 10^{-5}$	732.380

***Example* 4.2.** Put $n = 2$ and $E = [0,\, 0.25] \times [-1, 1] \times [-1, 1]$. Consider the differential equation

$$\partial_t z(t,x,y) = 2\partial_{xx} z(t,x,y) + 2\partial_{yy} z(t,x,y) + \left[1 - \frac{z(t, \sin xy, \cos xy)}{1 + z^2(t, \sin xy, \cos xy)}\right] \partial_{xy} z(t,x,y) +$$

$$+ z(t, 0.5(x+y), 0.5(x-y)) + f(t,x,y) z(t,x,y),$$

$$f(t,x,y) = x^2 + y^2 - 1 - 8t - 2t^2(xy + 4x^2 + 4y^2) - \exp\left\{-t\left(\frac{x^2}{2} + \frac{y^2}{2}\right)\right\}$$

with the initial boundary conditions (38)–(41).

The solution of the above problem is known. It is $v(t,x,y) = \exp\big\{t\big(x^2 + y^2 - 1\big)\big\}$ (see Table 3.4).

Table 3.4

(h_0, h_1, h_2)	ε_h	Time $[s]$
$(2^{-9}, 2^{-3}, 2^{-3})$	$2.51412371 \cdot 10^{-4}$	0.086
$(2^{-11}, 2^{-4}, 2^{-4})$	$6.33848401 \cdot 10^{-5}$	0.449
$(2^{-13}, 2^{-5}, 2^{-5})$	$1.58817990 \cdot 10^{-5}$	3.877
$(2^{-17}, 2^{-7}, 2^{-7})$	$3.97273960 \cdot 10^{-6}$	55.038
$(2^{-18}, 2^{-8}, 2^{-8})$	$2.39516870 \cdot 10^{-5}$	1120.800

Difference methods described in Section 4 have the following property: a large number of previous values $z^{(i,m)}$ must be preserved, because they are needed to compute an approximate solution with $t = t^{(r+1)}$.

***Remark* 4.4.** Suppose that we apply a difference method to (23), (24) or (42), (43). The superposition of the numerical method of lines and the difference method for ordinary functional differential equations leads to difference schemes for original problems. The above examples show that there are explicit difference schemes which are convergent. It is not our aim to show theoretical results on such difference schemes.

***Remark* 4.5.** All the theorems on the numerical method of lines presented in the paper can be extended on weakly coupled functional differential systems.

5. Conclusions. A new theorem, useful for proving convergence of difference schemes for first order or second order parabolic PDEs, is given. The theory embraces initial boundary problems with functional dependence, namely integro-differential and deviating variable ones.

A wider class of these dependences has been made treatable, thanks to removing the requirement of **globality** on the estimate of growth of coefficients in functional variable. This is also illustrated by numerical examples of Section 3.

This is the last published work, written jointly by the authors. Professor Zdzisław Kamont passed away on the 3^{rd} of September 2012, in Gdańsk, Poland. The first author would like to express his deep gratitude for the years of mentoring and cooperation in mathematics. *Requiescat in pace.*

1. *Baranowska A., Kamont Z.* Numerical method of lines for first order partial differential functional equations // Z. Anal. Anwend. – 2002. – **21**. – S. 949 – 962.
2. *Brandi P., Kamont Z., Salvadori A.* Approximate solutions of mixed problems for first order partial differential functional equations // Atti semin. mat. e fis. Univ. Modena. – 1991. – **39**. – P. 277 – 302.
3. *Bahuguna D., Dabas J.* Existence and uniqueness of a solution to a partial integro-dufferential equation by the method of lines // Electron. J. Qual. Theor. Different. Equat. – 2008. – **4**. – 12 p.
4. *Bahuguna M., Dabas J., Shukla R. K.* Method of lines to hyperbolic integro-differential equations in $\mathbb{R}^n$ // Nonlinear Dynam. Syst. Theory. – 2008. – **8**. – P. 317 – 328.
5. *Brzychczy S.* Existence solution for nonlinear systems of differential functional equations of parabolic-type in an arbitrary domain // Ann. pol. math. – 1987. – **47**. – P. 309 – 317.
6. *Czernous W.* Generalized solutions of mixed problems for first order partial functional differential equations // Ukr. Mat. Zh. – 2006. – **58**, № 6. – P. 813 – 828.
7. *Gőri I.* On the method of lines for the solutions of nonlinear partial differential equations (in Russian). – Akad. Nauk SSSR, 1987. – **166**.
8. *Gőri I.* The method of lines for the solution of some nonlinear partial differential equations // Comput. Math. Appl. – 1988. – **15**. – P. 635 – 648.
9. *Hundsdorfer W., Verwer J.* Numerical solution of time-dependent advection-diffusion-reaction equations. – Berlin: Springer-Verlag, 2003.
10. *Jăger W., Simon L.* On a system of quasilinear parabolic functional differential equations // Acta Math. hung. – 2006. – **112**. – P. 39 – 55.
11. *Kamont Z.* Hyperbolic functional differential inequalities and applications. – Dordrecht etc.: Kluwer Acad. Publ., 1999.
12. *Kamont Z., Kropielnicka K.* Differential difference inequalities related to hyperbolic functional differential systems and applications // Different. Inequal. Appl. – 2005. – **8**. – P. 655 – 674.
13. *Kaughten J. P.* The method of lines for parabolic partial integro-differential equations // J. Integral Equat. and Appl. – 1992. – **4**. – P. 69 – 81.
14. *Łojasiewicz S.* Sur le probléme de Cauchy pour systémes d' équations aux dérivées partielles du premier ordre dans le cas hyperbolique de deux variables indépendantes // Ann. pol. math. – 1956. – **3**. – P. 87 – 117.
15. *Netka M.* Differential difference inequalities related to parabolic functional differential equations and applications // Opusc. math. – 2010. – **30**. – P. 95 – 115.
16. *Redlinger R.* Existence theorems for semilinear parabolic systems with functionals // Nonlinear Anal., TMA. – 1984. – **8**. – P. 667 – 682.

17. *Shakeri F., Dehghan M.* The method of lines for solution of the m-dimensional wave equation subject to an integral conservation conditions // Comput. Math. Appl. – 2008. – **56**. – P. 2175 – 2188.
18. *Simon L.* Application of monotone-type operators to parabolic and functional parabolic PDE's // Evolutionary Equations / Eds C. M. Dafermos, M. Pokorny. – Amsterdam: Elsevier, 2008. – P. 267 – 321.
19. *Schmitt K., Thompson R. C., Walter W.* Existence of solutions of a nonlinear boundary-values problem via method of lines // Nonlinear Anal. – 1978. – **2**. – P. 519 – 535.
20. *Topolski K.* On the existence of classical solutions for differential-functional IBVP // Abstrs and Appl. Anal. – 1998. – **3**. – P. 363 – 375.
21. *Vande Wouwer A., Saucez Ph., Schiesser W. E.* Adaptative method of lines. – Roca Baton: Chapman & Hall-CRC, 2001.
22. *Voigt A.* Line method approximations to the Cauchy problem for nonlinear parabolic differential equations // Numer. Math. – 1974. – **23**. – P. 23 – 36.
23. *Voigt A.* The method of lines for nonlinear parabolic differential equations with mixed derivatives // Numer. Math. – 1979. – **32**. – P. 197 – 207.
24. *Walter W.* Differential and integral inequalities. – Berlin: Springer-Verlag, 1970.
25. *Walter W., Acker A.* On the global existence of solutions of parabolic differential equations with a singular nonlinear term // Nonlinear Anal. – 1978. – **2**. – P. 499 – 504.
26. *Wei Dongming.* Existence, uniqueness and numerical analysis of solutions of a quasilinear parabolic problem // SIAM J. Numer. Anal. – 1992. – **29**. – P. 484 – 497.
27. *Wu J.* Theory and applications of partial functional differential equations. – Berlin: Springer, 1996.
28. *Yuan S.* The finite method of lines. – Beijing: Sci. Press, 1993.
29. *Zubik-Kowal B.* The method of lines for first order partial differential functional equations // Stud. Sci. Math. hung. – 1998. – **34**. – P. 413 – 428.
30. *Zubik-Kowal B.* The method of lines for parabolic differential functional equations // J. Numer. Anal. – 1997. – **17**. – P. 103 – 123.

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