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SOLVABILITY FOR A COUPLED SYSTEM OF FRACTIONAL DIFFERENTIAL EQUATIONS WITH PERIODIC BOUNDARY CONDITIONS AT RESONANCE *

РОЗВ'ЯЗНІСТЬ ЗВ'ЯЗАНОЇ СИСТЕМИ ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ ДРОБОВОГО ПОРЯДКУ З ПЕРІОДИЧНИМИ ГРАНИЧНИМИ УМОВАМИ ПРИ РЕЗОНАНСІ

By using the coincidence degree theory, we study the existence of solutions for a coupled system of fractional differential equations with periodic boundary conditions. A new result on the existence of solutions for above fractional boundary-value problem is obtained.

Із використанням теорії збігу степенів досліджено існування розв'язків зв'язаних систем диференціальних рівнянь дробового порядку з періодичними граничними умовами. Встановлено новий результат щодо існування розв'язків граничної задачі дробового порядку.

1. Introduction. In recent years, the fractional differential equations have received more and more attention. The fractional derivative has been occurring in many physical applications such as a non-Markovian diffusion process with memory [1], charge transport in amorphous semiconductors [2], propagations of mechanical waves in viscoelastic media [3], etc. Phenomena in electromagnetics, acoustics, viscoelasticity, electrochemistry and material science are also described by differential equations of fractional order (see [4 – 9]).

Recently, boundary-value problems for fractional differential equations have been studied in many papers (see [10 – 19]). Moreover, the existence of solutions to a coupled systems of fractional differential equations have been studied by many authors (see [20 – 26]). But the existence of solutions for a coupled system of fractional differential equations with periodic boundary conditions at resonance has not been studied. We will fill this gap in the literature. In this paper, we consider the following periodic boundary-value problem (PBVP for short) for a coupled system of fractional differential equations given by:

$$
\begin{gathered}
D^{\alpha}_{0+}u(t) = f\big(t, v(t), v'(t)\big), \quad t \in (0,1), \\
D^{\beta}_{0+}v(t) = g\big(t, u(t), u'(t)\big), \quad t \in (0,1), \\
u(0) = u(1), \qquad u'(0) = u'(1), \qquad v(0) = v(1), \qquad v'(0) = v'(1),
\end{gathered}
\tag{1.1}
$$

where D^{α}_{0+}, D^{β}_{0+} are the standard Caputo fractional detivative, $1 < \alpha \leq 2$, $1 < \beta \leq 2$ and $f, g\colon [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous.

The rest of this paper is organized as follows. Section 2 contains some necessary notations, definitions and lemmas. In Section 3, we establish a theorem on existence of solutions for PBVP (1.1) under nonlinear growth restriction of f and g, basing on the coincidence degree theory due to Mawhin (see [27]). Finally, in Section 4, an example is given to illustrate the main result.

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2. Preliminaries. In this section, we will introduce some notations, definitions and preliminary facts which are used throughout this paper.

Let X and Y be real Banach spaces and let $L\colon \operatorname{dom} L \subset X \to Y$ be a Fredholm operator with index zero, and $P\colon X \to X$, $Q\colon Y \to Y$ be projectors such that

$$\operatorname{Im} P = \operatorname{Ker} L, \qquad \operatorname{Ker} Q = \operatorname{Im} L,$$

$$X = \operatorname{Ker} L \oplus \operatorname{Ker} P, \qquad Y = \operatorname{Im} L \oplus \operatorname{Im} Q.$$

It follows that

$$L|_{\operatorname{dom} L \cap \operatorname{Ker} P}\colon \operatorname{dom} L \cap \operatorname{Ker} P \to \operatorname{Im} L$$

is invertible. We denote the inverse by K_P.

If Ω is an open bounded subset of X, and $\operatorname{dom} L \cap \overline{\Omega} \neq \varnothing$, the map $N\colon X \to Y$ will be called L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_P(I-Q)N\colon \overline{\Omega} \to X$ is compact. Where I is identity operator.

Lemma 2.1 [27]. *Let $L\colon \operatorname{dom} L \subset X \to Y$ be a Fredholm operator of index zero and $N\colon X \to \to Y$ is L-compact on $\overline{\Omega}$. Assume that the following conditions are satisfied*

(1) *$Lx \neq \lambda Nx$ for every $(x,\lambda) \in \big[(\operatorname{dom} L \setminus \operatorname{Ker} L)\big] \cap \partial\Omega \times (0,1)$;*

(2) *$Nx \notin \operatorname{Im} L$ for every $x \in \operatorname{Ker} L \cap \partial\Omega$;*

(3) $\deg(QN|_{\operatorname{Ker} L}, KerL\cap\Omega, 0) \neq 0$, *where $Q\colon Y \to Y$ is a projection such that* $\operatorname{Im} L = \operatorname{Ker} Q$. *Then the equation $Lx = Nx$ has at least one solution in* $\operatorname{dom} L \cap \overline{\Omega}$.

Definition 2.1. *The Riemann–Liouville fractional integral operator of order $\alpha > 0$ of a function x is given by*

$$I_{0+}^{\alpha} x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s)\,ds,$$

provided that the right-hand side integral is pointwise defined on $(0,+\infty)$.

Definition 2.2. *The Caputo fractional derivative of order $\alpha > 0$ of a continuous function x is given by*

$$D_{0+}^{\alpha} x(t) = I_{0+}^{n-\alpha} \frac{d^n x(t)}{dt^n} = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} x^{(n)}(s)\,ds,$$

where n is the smallest integer greater than or equal to α, provided that the right-hand side integral is pointwise defined on $(0,+\infty)$.

Lemma 2.2 [28]. *Assume that $x \in C(0,1) \cap L(0,1)$ with a Caputo fractional derivative of order $\alpha > 0$ that belongs to $C(0,1) \cap L(0,1)$. Then*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} x(t) = x(t) + c_0 + c_1 t + c_2 t^2 + \ldots + c_{n-1} t^{n-1},$$

where $c_i \in \mathbb{R}$, $i = 0,1,2,\ldots,n-1$, here n is the smallest integer greater than or equal to α.

In this paper, we denote $X = C^1[0,1]$ with the norm $\|x\|_X = \max\{\|x\|_\infty, \|x'\|_\infty\}$ and $Y = C[0,1]$ with the norm $\|y\|_Y = \|y\|_\infty$, where $\|x\|_\infty = \max_{t\in[0,1]}|x(t)|$. Then we denote $\overline{X} = X \times X$ with the norm $\|(u,v)\|_{\overline{X}} = \max\{\|u\|_X, \|v\|_X\}$ and $\overline{Y} = Y \times Y$ with the norm $\|(x,y)\|_{\overline{Y}} = \max\{\|x\|_Y, \|y\|_Y\}$ Obviously, both $\overline{X}$ and $\overline{Y}$ are Banach spaces.

Define the operator $L_1\colon \operatorname{dom} L \subset X \to Y$ by

$$L_1 u = D^\alpha_{0+} u,$$

where

$$\operatorname{dom} L_1 = \{u \in X | D^\alpha_{0+} u(t) \in Y, u(0) = u(1), u'(0) = u'(1)\}.$$

Define the operator $L_2\colon \operatorname{dom} L_2 \subset X \to Y$ by

$$L_2 v = D^\beta_{0+} v,$$

where

$$\operatorname{dom} L_2 = \{v \in X | D^\beta_{0+} v(t) \in Y, v(0) = v(1), v'(0) = v'(1)\}.$$

Define the operator $L\colon \operatorname{dom} L \subset \overline{X} \to \overline{Y}$ by

$$L(u,v) = (L_1 u, L_2 v), \tag{2.1}$$

where

$$\operatorname{dom} L = \{(u,v) \in \overline{X} | u \in \operatorname{dom} L_1, v \in \operatorname{dom} L_2\}.$$

Let $N\colon \overline{X} \to \overline{Y}$ be the Nemytski operator

$$N(u,v) = (N_1 v, N_2 u),$$

where $N_1\colon Y \to X$

$$N_1 v(t) = f\big(t, v(t), v'(t)\big),$$

and $N_2\colon Y \to X$

$$N_2 u(t) = g\big(t, u(t), u'(t)\big).$$

Then PBVP (1.1) is equivalent to the operator equation

$$L(u,v) = N(u,v), \quad (u,v) \in \operatorname{dom} L.$$

3. Main result. In this section, a theorem on existence of solutions for PBVP (1.1) will be given.

Theorem 3.1. *Let $f, g\colon [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ be continuous. Assume that*

(H_1) *there exist nonnegative functions $p_i, q_i, r_i \in C[0,1]$, $i = 1,2$, with*

$$\frac{\Gamma(\alpha+1)\Gamma(\beta+1) - (\alpha+1)(\beta+1)(Q_1+R_1)(Q_2+R_2)}{\Gamma(\alpha+1)\Gamma(\beta+1)} > 0$$

such that for all $(u,v)\in\mathbb{R}^2$, $t\in[0,1]$

$$\big|f(t,u,v)\big|\le p_1(t)+q_1(t)|u|+r_1(t)|v|,$$

and

$$\big|g(t,u,v)\big|\le p_2(t)+q_2(t)|u|+r_2(t)|v|,$$

where $P_i=\|p_i\|_\infty$, $Q_i=\|q_i\|_\infty$, $R_i=\|r_i\|_\infty$, $i=1,2$;

(H_2) *there exists a constant* $B>0$ *such that for all* $t\in[0,1]$, $|u|>B$, $v\in\mathbb{R}$ *either*

$$uf(t,u,v)>0,\qquad ug(t,u,v)>0,$$

or

$$uf(t,u,v)<0,\qquad ug(t,u,v)<0;$$

(H_3) *there exists a constant* $D>0$ *such that for every* $c_1,c_2\in\mathbb{R}$ *satisfying* $\min\{c_1,c_2\}>D$ *either*

$$c_1N_1(c_2)>0,\qquad c_2N_2(c_1)>0$$

or

$$c_1N_1(c_2)<0,\qquad c_2N_2(c_1)<0.$$

Then PBVP (1.1) *has at least one solution.*

Now, we begin with some lemmas below.

Lemma 3.1. *Let* L *be defined by* (2.1), *then*

$$\operatorname{Ker}L=(\operatorname{Ker}L_1,\operatorname{Ker}L_2)=\big\{(u,v)\in\overline{X}|(u,v)=(a,b),a,b\in\mathbb{R}\big\},\tag{3.1}$$

$$\operatorname{Im}L=(\operatorname{Im}L_1,\operatorname{Im}L_2)=\big\{(x,y)\in\overline{Y}|T_1=0,T_2=0\big\},\tag{3.2}$$

where $T_1=\int_0^1(1-s)^{\alpha-2}x(s)ds$, $T_2=\int_0^1(1-s)^{\beta-2}y(s)ds$.

Proof. By Lemma 2.2, $L_1u=D_{0+}^{\alpha}u(t)=0$ has solution

$$u(t)=c_0+c_1t,\quad c_0,c_1\in\mathbb{R}.$$

Combining with the boundary-value conditions of PBVP (1.1), one has

$$\operatorname{Ker}L_1=\big\{u\in X|u=a,\ a\in\mathbb{R}\big\}.$$

For $x\in\operatorname{Im}L_1$, there exists $u\in\operatorname{dom}L_1$ such that $x=L_1u\in Y$. By Lemma 2.2, we have

$$u(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}x(s)ds+c_0+c_1t.$$

Then, we obtain

$$u'(t)=\frac{1}{\Gamma(\alpha-1)}\int_0^t(t-s)^{\alpha-2}x(s)ds+c_1.$$

By conditions of PBVP (1.1), we can get that x satisfies

$$T_1 = \int\limits_0^1 (1-s)^{\alpha-2} x(s)ds = 0.$$

On the other hand, suppose $x \in Y$ and satisfies $\int_0^1 (1-s)^{\alpha-2} x(s)ds = 0$. Let $u(t) = I_{0+}^{\alpha} x(t) - \mu t$, where $\mu = I_{0+}^{\alpha} x(t)|_{t=1}$, then $u \in \operatorname{dom} L_1$ and $D_{0+}^{\alpha} u(t) = x(t)$. So that, $x \in \operatorname{Im} L_1$. Then we have

$$\operatorname{Im} L_1 = \big\{x \in Y | T_1 = 0\big\}.$$

Similarly, we can show that

$$\operatorname{Ker} L_2 = \big\{v \in X | v = b,\ b \in \mathbb{R}\big\},$$

$$\operatorname{Im} L_2 = \big\{y \in Y | T_2 = 0\big\}.$$

Lemma 3.1 is proved.

Lemma 3.2. *Let L be defined by* (2.1), *then L is a Fredholm operator of index zero, and the linear continuous projector operators $P\colon \overline{X} \to \overline{X}$ and $Q\colon \overline{Y} \to \overline{Y}$ can be defined as*

$$P(u,v) = (P_1 u, P_2 v) = \big(u(0), v(0)\big),$$

$$Q(x,y) = (Q_1 x, Q_2 y) = \big((\alpha-1)T_1, (\beta-1)T_2\big).$$

Furthermore, the operator $K_P\colon \operatorname{Im} L \to \operatorname{dom} L \cap \operatorname{Ker} P$ can be written by

$$K_P(x,y) = \big(I_{0+}^{\alpha} x(t) - \mu t,\ I_{0+}^{\beta} y(t) - \nu t\big),$$

where $\mu = I_{0+}^{\alpha} x(t)|_{t=1}$, $\nu = I_{0+}^{\beta} y(t)|_{t=1}$.

Proof. Obviously, $\operatorname{Im} P = \operatorname{Ker} L$ and $P^2(u,v) = P(u,v)$. It follows from $(u,v) = \big((u,v) - P(u,v)\big) + P(u,v)$ that $\overline{X} = \operatorname{Ker} P + \operatorname{Ker} L$. By simple calculation, we can get that $\operatorname{Ker} L \cap \operatorname{Ker} P = \big\{(0,0)\big\}$. Then we get

$$\overline{X} = \operatorname{Ker} L \oplus \operatorname{Ker} P.$$

For $(x,y) \in \overline{Y}$, we have

$$Q^2(x,y) = Q\big(Q_1 x, Q_2 y)\big) = (Q_1^2 x, Q_2^2 y).$$

By the definition of Q_1, we can get

$$Q_1^2 x = Q_1 x \cdot (\alpha-1) \int\limits_0^1 (1-s)^{\alpha-2} ds = Q_1 x.$$

Similar proof can show that $Q_2^2 y = Q_2 y$. Thus, we obtain $Q^2(x,y) = Q(x,y)$.

Let $(x, y) = \big((x, y) - Q(x, y)\big) + Q(x, y)$, where $(x, y) - Q(x, y) \in \operatorname{Ker} Q = \operatorname{Im} L,\ Q(x, y) \in \in \operatorname{Im} Q$. It follows from $\operatorname{Ker} Q = \operatorname{Im} L$ and $Q^2(x, y) = Q(x, y)$ that $\operatorname{Im} Q \cap \operatorname{Im} L = \{(0, 0)\}$. Then, we have

$$\overline{Y} = \operatorname{Im} L \oplus \operatorname{Im} Q.$$

Thus

$$\dim \operatorname{Ker} L = \dim \operatorname{Im} Q = \operatorname{codim} \operatorname{Im} L.$$

This means that L is a Fredholm operator of index zero.

Now, we will prove that K_P is the inverse of $L|_{\operatorname{dom} L \cap \operatorname{Ker} P}$. In fact, for $(x, y) \in \operatorname{Im} L$, we have

$$LK_P(x, y) = \Big(D^{\alpha}_{0+}(I^{\alpha}_{0+}x - \mu t), D^{\beta}_{0+}(I^{\beta}_{0+}y - \nu t)\Big) = (x, y). \tag{3.3}$$

Moreover, for $(u, v) \in \operatorname{dom} L \cap \operatorname{Ker} P$, we get $u(0) = 0$, $v(0) = 0$ and

$$K_P L(u, v) = \left(I^{\alpha}_{0+}D^{\alpha}_{0+}u(t) - \left\{I^{\alpha}_{0+}D^{\alpha}_{0+}u(t)\right\}\Big|_{t=1} t, I^{\beta}_{0+}D^{\beta}_{0+}v(t) - \left\{I^{\beta}_{0+}D^{\beta}_{0+}v(t)\right\}\Big|_{t=1} t\right) =$$

$$= \left(u(t) + c_0 + c_1 t - \left\{I^{\alpha}_{0+}D^{\alpha}_{0+}u(t)\right\}\Big|_{t=1} t, v(t) + c_0 + c_1 t - \left\{I^{\beta}_{0+}D^{\beta}_{0+}v(t)\right\}\Big|_{t=1} t\right),$$

which together with $u(0) = u(1)$ and $v(0) = v(1)$ yields that

$$K_P L(u, v) = (u, v). \tag{3.4}$$

Combining (3.3) with (3.4), we know that $K_P = (L|_{\operatorname{dom} L \cap \operatorname{Ker} P})^{-1}$.

Lemma 3.2 is proved.

Lemma 3.3. *Assume $\Omega \subset \overline{X}$ is an open bounded subset such that dom$L \cap \overline{\Omega} \neq \varnothing$, then N is L-compact on $\overline{\Omega}$.*

Proof. By the continuity of f and g, we can get that $QN(\overline{\Omega})$ and $K_P(I - Q)N(\overline{\Omega})$ are bounded. So, in view of the Arzelà–Ascoli theorem, we need only prove that $K_P(I - Q)N(\overline{\Omega}) \subset \overline{X}$ is equicontinuous.

From the continuity of f and g, there exist constant $A_i, B_i > 0$, $i = 1, 2$, such that $\forall (u, v) \in \overline{\Omega}$

$$\big|(I - Q_1)N_1 v\big| \leq A_1, \qquad \big|I^{\alpha}_{0+}(I - Q_1)N_1 v\big| \leq B_1,$$

$$\big|(I - Q_2)N_2 u\big| \leq A_2, \qquad \big|I^{\alpha}_{0+}(I - Q_2)N_2 u\big| \leq B_2.$$

Furthermore for $0 \leq t_1 < t_2 \leq 1$, $(u, v) \in \overline{\Omega}$, we have

$$\left|K_P(I - Q)N\big(u(t_2), v(t_2)\big) - \Big(K_P(I - Q)N\big(u(t_1), v(t_1)\big)\Big)\right| =$$

$$= \Big(I^{\alpha}_{0+}(I - Q_1)N_1 v(t_2) - \mu t_2,\ I^{\beta}_{0+}(I - Q_2)N_2 u(t_2) - \nu t_2\Big) -$$

$$- \Big(I^{\alpha}_{0+}(I - Q_1)N_1 v(t_1) - \mu t_1,\ I^{\beta}_{0+}(I - Q_2)N_2 u(t_1) - \nu t_1\Big) =$$

$$= \Big(I_{0+}^{\alpha}(I-Q_1)N_1v(t_2) - I_{0+}^{\alpha}(I-Q_1)N_1v(t_1) - \mu(t_2-t_1),$$

$$I_{0+}^{\beta}(I-Q_2)N_2u(t_2) - I_{0+}^{\beta}(I-Q_2)N_2u(t_1) - \nu(t_2-t_1)\Big),$$

where $\mu = \big\{I_{0+}^{\alpha}(I-Q_1)N_1v(t)\big\}\big|_{t=1}$, $\nu = \big\{I_{0+}^{\alpha}(I-Q_2)N_2u\big\}\big|_{t=1}$.

By

$$\Big| I_{0+}^{\alpha}(I-Q_1)N_1v(t_2) - I_{0+}^{\alpha}(I-Q_1)N_1v(t_1) - \mu(t_2-t_1)\Big| \le$$

$$\le \frac{1}{\Gamma(\alpha)} \left| \int\limits_0^{t_2} (t_2-s)^{\alpha-1}(I-Q_1)N_1v(s)ds - \right.$$

$$\left. - \int\limits_0^{t_1} (t_1-s)^{\alpha-1}(I-Q_1)N_1v(s)ds \right| + B_1|t_2-t_1| \le$$

$$\le \frac{A_1}{\Gamma(\alpha)} \left[\int\limits_0^{t_1} (t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}ds + \int\limits_{t_1}^{t_2} (t_2-s)^{\alpha-1}ds \right] + B_1|t_2-t_1| =$$

$$= \frac{A_1}{\Gamma(\alpha+1)}(t_2^{\alpha} - t_1^{\alpha}) + B_1|t_2-t_1|$$

and

$$\Big| \big(I_{0+}^{\alpha}(I-Q_1)N_1v\big)'(t_2) - \big(I_{0+}^{\alpha}(I-Q_1)N_1v\big)'(t_1)\Big| =$$

$$= \frac{\alpha-1}{\Gamma(\alpha)} \left| \int\limits_0^{t_2} (t_2-s)^{\alpha-2}(I-Q_1)N_1v(s)ds - \right.$$

$$\left. - \int\limits_0^{t_1} (t_1-s)^{\alpha-2}(I-Q_1)N_1v(s)ds \right| \le$$

$$\le \frac{A_1}{\Gamma(\alpha-1)} \left[\int\limits_0^{t_1} (t_1-s)^{\alpha-2} - (t_2-s)^{\alpha-2}ds + \int\limits_{t_1}^{t_2} (t_2-s)^{\alpha-2}ds \right] \le$$

$$\le \frac{A_1}{\Gamma(\alpha)} \Big[t_2^{\alpha-1} - t_1^{\alpha-1} + 2(t_2-t_1)^{\alpha-1} \Big].$$

Similar proof can show that

$$\Big| I_{0+}^{\beta}(I-Q_2)N_2u(t_2) - I_{0+}^{\beta}(I-Q_2)N_2u(t_1) - \nu(t_2-t_1)\Big| \le$$

$$\leq \frac{A_2}{\Gamma(\beta+1)}(t_2^{\beta}-t_1^{\beta})+B_2|t_2-t_1|,$$

$$\left|\left(I_{0+}^{\beta}(I-Q_2)N_2u\right)'(t_2)-\left(I_{0+}^{\beta}(I-Q_2)N_2u\right)'(t_1)\right|\leq$$

$$\leq \frac{A_2}{\Gamma(\beta)}\Big[t_2^{\beta-1}-t_1^{\beta-1}+2(t_2-t_1)^{\beta-1}\Big].$$

Since t^{α}, $t^{\alpha-1}$, t^{β} and $t^{\beta-1}$ are uniformly continuous on $[0,1]$, we can get that $K_P(I-Q)N(\overline{\Omega})\subset\overline{X}$ is equicontinuous.

Thus, we get that $K_P(I-Q)N\colon\overline{\Omega}\to\overline{X}$ is compact.

Lemma 3.3 is proved.

Lemma 3.4. *Suppose $(H_1),(H_2)$ hold, then the set*

$$\Omega_1=\Big\{(u,v)\in\operatorname{dom}L\setminus\operatorname{Ker}L\mid L(u,v)=\lambda N(u,v),\lambda\in(0,1)\Big\}$$

is bounded.

Proof. Take $(u,v)\in\Omega_1$, then $N(u,v)\in\operatorname{Im}L$. By (3.2), we have

$$\int_0^1(1-s)^{\alpha-2}f\big(s,v(s),v'(s)\big)ds=0,$$

$$\int_0^1(1-s)^{\beta-2}g\big(s,u(s),u'(s)\big)ds=0.$$

Then, by the integral mean value theorem, there exists constants $\xi,\eta\in(0,1)$ such that $f\big(\xi,v(\xi),v'(\xi)\big)=$ $=0$ and $g\big(\eta,u(\eta),u'(\eta)\big)=0$. So, from (H_2), we get $\big|v(\xi)\big|\leq B$ and $\big|u(\eta)\big|\leq B$. Hence

$$|u(t)|=\left|u(\eta)+\int_{\eta}^{t}u'(s)ds\right|\leq B+\|u'\|_{\infty}. \tag{3.5}$$

That is

$$\|u\|_{\infty}\leq B+\|u'\|_{\infty}. \tag{3.6}$$

Similar proof can show that

$$\|v\|_{\infty}\leq B+\|v'\|_{\infty}. \tag{3.7}$$

By $L(u,v)=\lambda N(u,v)$, we have

$$u(t)=\frac{\lambda}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f\big(s,v(s),v'(s)\big)ds+u(0)-\lambda\mu t$$

and

$$v(t) = \frac{\lambda}{\Gamma(\beta)} \int\limits_0^t (t-s)^{\beta-1} g\big(s, u(s), u'(s)\big) ds + v(0) - \lambda \nu t,$$

where $\mu = I_{0+}^{\alpha} f\big(t, v(t), v'(t)\big)|_{t=1}$, $\nu = I_{0+}^{\beta} g\big(t, u(t), u'(t)\big)|_{t=1}$.

Then we obtain

$$u'(t) = \frac{\lambda}{\Gamma(\alpha-1)} \int\limits_0^t (t-s)^{\alpha-2} f\big(s, v(s), v'(s)\big) ds - \lambda\mu$$

and

$$v'(t) = \frac{\lambda}{\Gamma(\beta-1)} \int\limits_0^t (t-s)^{\beta-2} g\big(s, u(s), u'(s)\big) ds - \lambda\nu.$$

From (H_1) and (3.7), we get that

$$|\mu| = \Big| I_{0+}^{\alpha} f\big(t, v(t), v'(t)\big)\big|_{t=1} \Big| =$$

$$= \frac{1}{\Gamma(\alpha)} \int\limits_0^1 (1-s)^{\alpha-1} \Big| f\big(s, v(s), v'(s)\big) \Big| ds \leq$$

$$\leq \frac{1}{\Gamma(\alpha)} \int\limits_0^1 (1-s)^{\alpha-1} \Big[p_1(s) + q_1(s)\big|v(s)\big| + r_1(s)\big|v'(s)\big| \Big] ds \leq$$

$$\leq \frac{1}{\Gamma(\alpha)} \Big[P_1 + Q_1 B + (Q_1 + R_1)\|v'\|_\infty \Big] \int\limits_0^1 (t-s)^{\alpha-1} ds \leq$$

$$\leq \frac{1}{\Gamma(\alpha+1)} \Big[P_1 + Q_1 B + (Q_1 + R_1)\|v'\|_\infty \Big].$$

So, we have

$$\|u'\|_\infty \leq \frac{1}{\Gamma(\alpha-1)} \int\limits_0^t (t-s)^{\alpha-2} \Big| f\big(s, v(s), v'(s)\big) \Big| ds + |\mu| \leq$$

$$\leq \frac{1}{\Gamma(\alpha-1)} \int\limits_0^t (t-s)^{\alpha-2} \Big[p_1(s) + q_1(s)\big|v(s)\big| + r_1(s)\big|v'(s)\big| \Big] ds + |\mu| \leq$$

$$\leq \frac{1}{\Gamma(\alpha-1)} \Big[P_1 + Q_1 B + (Q_1 + R_1)\|v'\|_\infty \Big] \int\limits_0^t (t-s)^{\alpha-2} ds + |\mu| \leq$$

$$\leq \left(\frac{1}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha+1)}\right)\Big[P_1 + Q_1B + (Q_1+R_1)\|v'\|_\infty\Big] =$$

$$= \frac{\alpha+1}{\Gamma(\alpha+1)}\Big[P_1 + Q_1B + (Q_1+R_1)\|v'\|_\infty\Big]. \tag{3.8}$$

Similarly, we can get

$$\|v'\|_\infty \leq \frac{\beta+1}{\Gamma(\beta+1)}\Big[P_2 + Q_2B + (Q_2+R_2)\|u'\|_\infty\Big]. \tag{3.9}$$

Together with (3.8), (3.9), we have

$$\|u'\|_\infty \leq$$

$$\leq \frac{\alpha+1}{\Gamma(\alpha+1)}\left\{P_1 + Q_1B + (Q_1+R_1)\frac{\beta+1}{\Gamma(\beta+1)}\Big[P_2 + Q_2B + (Q_2+R_2)\|u'\|_\infty\Big]\right\}.$$

Thus, from $\dfrac{\Gamma(\alpha+1)\Gamma(\beta+1) - (\alpha+1)(\beta+1)(Q_1+R_1)(Q_2+R_2)}{\Gamma(\alpha+1)\Gamma(\beta+1)} > 0$ and (3.9), we obtain that

$$\|u'\|_\infty \leq \frac{(\alpha+1)\Big[\Gamma(\beta+1)(P_1+Q_1B) + (\beta+1)(Q_1+R_1)(P_2+Q_2B)\Big]}{\Gamma(\alpha+1)\Gamma(\beta+1) - (\alpha+1)(\beta+1)(Q_1+R_1)(Q_2+R_2)} := M_1$$

and

$$\|v'\|_\infty \leq \frac{\beta+1}{\Gamma(\beta+1)}\Big[P_2 + Q_2B + (Q_2+R_2)M_1\Big] := M_2.$$

Together with (3.6), (3.7), we get

$$\big\|(u,v)\big\|_{\overline{X}} \leq \max\{M_1+B, M_2+B\} := M.$$

So Ω_1 is bounded.

Lemma 3.4 is proved.

Lemma 3.5. *Suppose* (H_3) *holds, then the set*

$$\Omega_2 = \big\{(u,v) | (u,v) \in \operatorname{Ker} L, N(u,v) \in \operatorname{Im} L\big\}$$

is bounded.

Proof. For $(u,v) \in \Omega_2$, we have $(u,v) = (c_1, c_2)$, $c_1, c_2 \in \mathbb{R}$. Then from $N(u,v) \in \operatorname{Im} L$, we obtain

$$\int_0^1 (1-s)^{\alpha-2} f(s, c_2, 0)\,ds = 0,$$

$$\int_0^1 (1-s)^{\beta-2} g(s, c_1, 0)\,ds = 0,$$

which together with (H_3) implies $|c_1|$, $|c_2| \leq D$. Thus, we get

$$\|(u,v)\|_{\overline{X}} \leq D.$$

Hence, Ω_2 is bounded.

Lemma 3.5 is proved.

Lemma 3.6. *Suppose the first part of* (H_3) *holds, then the set*

$$\Omega_3 = \Big\{(u,v) \in \operatorname{Ker} L | \lambda(u,v) + (1-\lambda)QN(u,v) = (0,0), \lambda \in [0,1]\Big\}$$

is bounded.

Proof. For $(u,v) \in \Omega_3$, we have $(u,v) = (c_1, c_2)$, $c_1, c_2 \in \mathbb{R}$ and

$$\lambda c_1 + (1-\lambda)(\alpha-1)\int\limits_0^1 (1-s)^{\alpha-2} f(s,c_2,0)ds = 0, \tag{3.10}$$

$$\lambda c_2 + (1-\lambda)(\beta-1)\int\limits_0^1 (1-s)^{\beta-2} g(s,c_1,0)ds = 0. \tag{3.11}$$

If $\lambda = 0$, then $|c_1|$, $|c_2| \leq D$ because of the first part of (H_3). If $\lambda = 1$, then $c_1 = c_2 = 0$. For $\lambda \in (0,1]$, we can obtain $|c_1|$, $|c_2| \leq D$. Otherwise, if $|c_1|$ or $|c_2| > D$, in view of the first part of (H_3), one has

$$\lambda c_1^2 + (1-\lambda)(\alpha-1)\int\limits_0^1 (1-s)^{\alpha-2} c_1 f(s,c_2,0)ds > 0$$

or

$$\lambda c_2^2 + (1-\lambda)(\beta-1)\int\limits_0^1 (1-s)^{\beta-2} c_2 g(s,c_1,0)ds > 0,$$

which contradict to (3.10) or (3.11). Therefore, Ω_3 is bounded.

Lemma 3.6 is proved.

Remark **3.1.** If the second part of (H_3) holds, then the set

$$\Omega_3' = \big\{(u,v) \in \operatorname{Ker} L | -\lambda(u,v) + (1-\lambda)QN(u,v) = (0,0), \lambda \in [0,1]\big\}$$

is bounded.

Proof of Theorem **3.1.** Set $\Omega = \big\{(u,v) \in \overline{X} | \|(u,v)\|_{\overline{X}} < \max\{M, D\} + 1\big\}$. It follows from Lemmas 3.2 and 3.3 that L is a Fredholm operator of index zero and N is L-compact on $\overline{\Omega}$. By Lemmas 3.4 and 3.5, we get that the following two conditions are satisfied:

(1) $L(u,v) \neq \lambda N(u,v)$ for every $\big((u,v),\lambda\big) \in \big[(\operatorname{dom} L \setminus \operatorname{Ker} L) \cap \partial\Omega\big] \times (0,1)$;

(2) $Nx \notin \operatorname{Im} L$ for every $(u,v) \in \operatorname{Ker} L \cap \partial\Omega$.

Take

$$H\big((u,v),\lambda\big) = \pm\lambda(u,v) + (1-\lambda)QN(u,v).$$

According to Lemma 3.6 (or Remark 3.1), we know that $H\big((u,v),\lambda\big) \neq 0$ for $(u,v) \in \operatorname{Ker} L \cap \partial\Omega$. Therefore

$$\deg\Big(QN|_{\operatorname{Ker} L}, \Omega \cap \operatorname{Ker} L, (0,0)\Big) = \deg\Big(H(\cdot,0), \Omega \cap \operatorname{Ker} L, (0,0)\Big) =$$

$$= \deg\Big(H(\cdot,1), \Omega \cap \operatorname{Ker} L, (0,0)\Big) = \deg\Big(\pm I, \Omega \cap \operatorname{Ker} L, (0,0)\Big) \neq 0.$$

So that, the condition (3) of Lemma 2.1 is satisfied. By Lemma 2.1, we can get that $L(u,v) = N(u,v)$ has at least one solution in $\operatorname{dom} L \cap \overline{\Omega}$. Therefore PBVP (1.1) has at least one solution.

Theorem 3.1 is proved.

4. Example.

***Example* 4.1.** Consider the following PBVP:

$$\begin{aligned}
&D_{0^+}^{\frac{3}{2}} u(t) = \frac{1}{16}\Big[v(t) - 10\Big] + \frac{t^2}{16} e^{-|v'(t)|}, \quad t \in [0,1],\\
&D_{0^+}^{\frac{5}{4}} v(t) = \frac{1}{12}\Big[u(t) - 8\Big] + \frac{t^3}{12} \sin^2\big(u'(t)\big), \quad t \in [0,1],\\
&u(0) = u(1), \quad u'(0) = u'(1), \quad v(0) = v(1), \quad v'(0) = v'(1).
\end{aligned} \tag{4.1}$$

Choose $p_1(t) = \dfrac{11}{16}$, $p_2(t) = \dfrac{3}{4}$, $q_1(t) = \dfrac{1}{16}$, $q_2(t) = \dfrac{1}{12}$, $r_1(t) = r_2(t) = 0$, $B = D = 10$.

By simple calculation, we can get that $(H_1), (H_2)$ and the first part of (H_3) hold.

By Theorem 3.1, we obtain that the problem PBVP (4.1) has at least one solution.

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