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M. Sabbaghan, F. Ayatollah Zadeh Shirazi (College Sci., Univ. Tehran, Iran),
A. Hosseini (Univ. Guilan, Rasht, Iran)

CODECOMPOSITION OF A TRANSFORMATION SEMIGROUP

КОРОЗКЛАД ТРАНСФОРМАЦІЙНОЇ НАПІВГРУПИ

The following text deals with the concept of "codecomposition" of a transformation semigroup which interact with the phase semigroup. In this way we distinguish new classes of transformation semigroups with meaningful relations, e.g., we will show "the class of all distal transformation semigroups $\subset$ the class of all transformation semigroups decomposable to distal ones $\subset$ the class of all transformation semigroups" (where $\subset$ is strict inclusion).

Розглянуто концепцію „корозкладу" трансформаційної напівгрупи, що взаємодіє з фазовою напівгрупою. Таким чином, ми вирізняємо новий клас трансформаційних напівгруп зі змістовними співвідношеннями. Так, показано, що „клас всіх дистальних трансформаційних напівгруп $\subset$ клас всіх трансформаційних напівгруп, що розкладаються в дистальні напівгрупи$\subset$ клас всіх трансформаційних напівгруп" (де $\subset$ позначає строге включення).

1. Introduction. The approach of decomposition of a transformation group has been studied in several manuscripts, e.g., as it has been mentioned in [3] (Proposition 2.6) in transformation group (X, T) with X locally compact T_2, every point of X is an almost periodic point if and only if $\{\overline{xT}: x \in X\}$ is a partition (decomposition) of X consisting of compact sets; however about twenty five years before that text, i.e., in 1944, one may find this theorem "*In order that the homeomorphism h give an orbit-closure decomposition it is sufficient that h be pointwise almost periodic; and in case X is compact, this condition is also necessary*" [8] (Theorem 2) (for metric space X and homeomorphism $h\colon X \to X$), also the author has mentioned that a direct proof of this theorem can be found in [7]; or in [4] (Section 2) in a minimal transformation group (X, T) a relatively dense subgroup of T namely G has been considered and the elements of decomposition $\left\{\overline{xG}: x \in X\right\}$ has been studied (for abelian T and compact metric X ... one may find discussions on decomposition of a transformation semigroup in several other papers like [2].

In most of the above mentioned cases the emphasis is on simplifying the matter by dividing the phase space to some smaller subspaces, then study them and their interactions; in this text by codecomposition of a transformation semigroup we mean certain collection of transformation semigroups with smaller phase semigroup, and then study the interaction of these codecompositions with the original one.

2. First steps towards codecomposition approach: multitransformation semigroups. By a *right transformation semigroup* we mean a triple (X, S, π) or simply (X, S) where X is a topological space, S is a topological semigroup with identity e and $\pi\colon X \times S \to X$ ($\pi(x, s) = xs$) is a continuous map such that for each $x \in X$ and for each $s, t \in S$ we have $xe = x$ and $x(st) = (xs)t$. *Left transformation semigroup* (S, X) is defined in a similar way. By a *bitransformation semigroup* we mean (S_2, X, S_1), where (S_2, X) is a left transformation semigroup, (X, S_1) is a right transformation semigroup, and for each $s_1 \in S_1, s_2 \in S_2, x \in X$ we have $(s_2x)s_1 = s_2(xs_1)$ (which is denoted by s_2xs_1) (note to the fact that left and right transformation groups coincide, e.g., if (S, X) is a left transformation group, then (X, S) is a right transformation group, where $xs := s^{-1}x$ $(x \in X,\ s \in \in S)$). *But how much it will be interesting if instead of two sides* (*left and right*) *one consider several sides for a point $x \in X$?* This will be our primary motivation to introduce multitransformation semigroups.

However in the following subsection we do our best to cause more interest for readers through some examples.

2.1. A short motivation through examples. Consider a mill supplied with the following powers:

S_1: water	S_2: wind,
S_3: sun	S_4: manpower.

Regardless of the power source, all of these powers act on the grain in the mill. Furthermore, it does not matter which force is applied first. In this example one may have general approach or partial approach. Whenever we study (X, S), where X is the collection of our grain and S is the power supplied for the mill (hence S is generated by $S_1 \cup S_2 \cup S_3 \cup S_4$), we have general approach. Whenever we study (X, S_1), (X, S_2), (X, S_3) and (X, S_4) we have partial approach. As a matter of fact regarding our future topics, $((X, S_i)\colon i \in \{1, 2, 3, 4\})$ is a multitransformation semigroup and a codecomposition of (X, S).

We continue this subsection with some more specialized examples. Here we want to explain how codecomposition of a transformation semigroup is partial approach to a system, and how multitransformation semigroup's concept helps us in this process (suppose all phase semigroups are discrete).

1. Consider $X_1 := [0, 1]$ with induced topology of $\mathbb{R}$, $X_2 := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ with induced topology of $\mathbb{R}^2$, $X_3 := \{(x, y, z) \in \mathbb{R}^3\colon x^2 + y^2 + z^2 = 1\}$ with induced topology of $\mathbb{R}^3$, and X as disjoint union of X_1, X_2, X_3. For $i = 1, 2, 3$ suppose S_i is the collection of all homeomorphisms $f\colon X \to X$ such that $f(x) = x$ for all $x \in X \setminus X_i$. Also consider S as the group of all homeomorphisms of X. Then:

for all $i \neq j$, $f \in S_i$ and $g \in S_j$ we have $fg = gf$,

the semigroup S is generated by $S_1 \cup S_2 \cup S_3$.

In particular regarding our future topics, $((X, S_i)\colon i \in \{1, 2, 3\})$ is a multitransformation semigroup and a codecomposition of (X, S).

2. Consider $X := \{(x, y) \in \mathbb{R}^2\colon x^2 + y^2 = 1\}$ with induced topology of $\mathbb{R}^2$, and for prime natural number q, S_q is the group of all rotations under a integer multiple of $\dfrac{\pi}{q}$. Moreover suppose S is the group of rotations under a rational multiple of π. Then:

for all $i \neq j$, $s \in S_i$ and $t \in S_j$ we have $st = ts$,

the semigroup S is generated by $\bigcup\{S_q\colon q \text{ is a prime number}\}$.

In particular regarding our future topics, $((X, S_i)\colon i \text{ is a prime number})$ is a multitransformation semigroup and a codecomposition of (X, S).

3. Consider $X := [0, 1]$ with induced topology of $\mathbb{R}$, S as the group of all homeomorphisms of X, $n \geq 2$, and $0 \leq \theta_1 < \theta_2 < \ldots < \theta_n \leq 1$. Moreover for $i = 1, \ldots, n$ suppose $S_i := \{f \in \in S\colon f(\theta_i) = \theta_i\}$, then:

for all $i \neq j$ there exists $f \in S_i$ and $g \in S_j$ such that $fg \neq gf$,

the semigroup S is generated by $S_1 \cup \ldots \cup S_n$.

Then $((X, S_i)\colon i \in \{1, \ldots, n\})$ is not a multitransformation semigroup, since there are elements of different S_i's such that it is important which one act first on an element of X, although S is generated by $S_1 \cup \ldots \cup S_n$.

In the following text in the transformation semigroup (X, S), S *acts effectively* on X, i.e., for each $s, t \in S$ with $s \neq t$ there exists $x \in X$ such that $xs \neq xt$.

Definition 2.1. *If $\Gamma \neq \varnothing$ and for each $\alpha \in \Gamma$, (X, S_α) is a transformation semigroup with e_α as the identity element of S_α, such that for each $y \in X$, $\alpha_1, \ldots, \alpha_n \in \Gamma$ which are distinct, $s_{\alpha_1} \in S_{\alpha_1}, \ldots, s_{\alpha_n} \in S_{\alpha_n}$, and $\sigma \in \mathbf{S}_n$ (where $\mathbf{S}_n$ is the group of all permutations on $\{1, \ldots, n\}$) we have $(\ldots((ys_{\alpha_1})s_{\alpha_2})\ldots)s_{\alpha_n} = (\ldots((ys_{\alpha_{\sigma(1)}})s_{\alpha_{\sigma(2)}})\ldots)s_{\alpha_{\sigma(n)}}$. We call $(X, (S_\alpha \colon \alpha \in \Gamma))$ a multitransformation semigroup (it is sufficient to consider this definition for $n = 2$).*

Now we have the main tool to introduce codecomposition of a transformation semigroup, since codecompositions are certain multitransformation semigroups.

3. Codecompositions in transformation semigroups with discrete phase semigroup and compact Hausdorff phase space. Codecompositions of transformation semigroup (X, S) are multitransformation semigroups $(X, (S_\alpha \colon \alpha \in \Gamma))$ with certain properties. Since the idea of codecomposition of a transformation semigroup, when phase semigroup is discrete brings the best motivation, so in this section all phase semigroups considered discrete. Moreover in this section all phase spaces are compact Hausdorff.

***Remark* 3.1.** In the transformation semigroup (X, S) for compact Hausdorff X and discrete S, for each $s \in S$, define the continuous map $\pi^s \colon X \to X$ by $x\pi^s = xs$ ($\forall x \in X$), then $\mathrm{E}(X, S)$ (or simply $\mathrm{E}(X)$) is the closure of $\{\pi^s | \, s \in S\}$ in X^X with pointwise convergence topology, moreover it is called the *enveloping semigroup* (or *Ellis semigroup*) of (X, S). We used to write s instead of π^s ($s \in S$). $\mathrm{E}(X, S)$ has a semigroup structure [3] (Chapter 3). A nonempty subset Z of X is called *invariant* if $ZS \subseteq Z$. Let $\mathrm{P}(X, S) = \{(x, y) \in X \times X : \exists p \in E(X) \colon xp = yp\}$. (X, S) is called *distal* if $\mathrm{E}(X)$ is a group (or equivalently $\mathrm{P}(X, S) = \Delta_X$ [3] (Chapter 5)), and it is called *equicontinuous* if for all element α of the uniformity on X, there exists an element β of the uniformity on X with $\beta S \subseteq \alpha$ and $\mathrm{E}(X)$ is a group of continuous functions (see [3] (Proposition 4.4)). (X, S) is called *proximal* if $\mathrm{P}(X, S) = X \times X$ (for more about proximal transformation groups see [6]). (X, S) is called *point transitive* if $\exists w \in X \; \overline{wS} = X$, it is called *minimal* if $\forall w \in X \; \overline{wS} = X$.

Definition 3.1 (Discrete phase semigroup case)**.** *Multitransformation semigroup $(X, (S_\alpha \colon \alpha \in$ $\in \Gamma))$ is a codecomposition of transformation semigroup (X, S), where S is the semigroup generated by $\bigcup_{\alpha \in \Gamma} S_\alpha$ and S_αs are distinct subsemigroups of S.*

3.1. Elementary properties.

Lemma 3.1. *Let multitransformation semigroup $(X, (S_\alpha \colon \alpha \in \Gamma))$ be a codecomposition of transformation semigroup (X, S), then we have*

$\mathrm{E}(X, S_\alpha) \subseteq \mathrm{E}(X, S)$, $\alpha \in \Gamma \ldots$;

$ps = sp$, $p \in \mathrm{E}(X, S_\alpha)$, $s \in S_\beta$, $\alpha \neq \beta$.

Proof. If $\alpha \neq \beta$, $p \in \mathrm{E}(X, S_\alpha)$, and $s \in S_\beta$, then there exists a net $(t^\lambda)_{\lambda \in \Lambda}$ in S_α converges to p (in $\mathrm{E}(X, S_\alpha)$), thus for each $x \in X$, $(xt^\lambda)_{\lambda \in \Lambda}$ converges to xp and by continuity of s, $(xt^\lambda s)_{\lambda \in \Lambda}$ converges to xps. Now replacing x by xs leads us to the fact that $(xst^\lambda)_{\lambda \in \Lambda}$ converges to xsp, but for each λ, $t^\lambda s = st^\lambda$; thus $xsp = xps$ for all $x \in X$ and $sp = ps$.

Theorem 3.1. *Let multitransformation semigroup $(X, (S_\alpha \colon \alpha \in \Gamma))$ be a codecomposition of transformation semigroup (X, S), then we have:*

1. *If (X, S) is distal (resp. equicontinuous), then for each $\alpha \in \Gamma$, (X, S_α) is distal (resp. equicontinuous).*

2. *If for some $\alpha \in \Gamma$, (X, S_α) is minimal (resp. point transitive, proximal), then (X, S) is minimal (resp. point transitive, proximal).*

Proof. 1. Use Lemma 3.1, and the fact that every transformation semigroup is distal if and only if its enveloping semigroup has at most one idempotent element [3] (Propositions 3.5 and 5.3).

2. If (X, S_β) is minimal, then for each $x \in X$, $\overline{xS_\beta} = X$, thus $X = \overline{xS_\beta} \subseteq \overline{xS} \subseteq X$, and (X, S) is minimal.

Definition 3.2. *Transformation semigroup* (X, S) *is called codecomposable to distal* (*resp. proximal, equicontinuous, minimal, point transitive*) *transformation semigroups if there exists a codecomposition of* (X, S) *like* $(X, (S_\alpha : \alpha \in \Gamma))$ *such that for each* $\alpha \in \Gamma$, (X, S_α) *is distal* (*resp. proximal, equicontinuous, minimal, point transitive*).

Theorem 3.2. *In the transformation semigroup* (X, S), (X, S) *is codecomposable to distal* (*resp. minimal, equicontinuous, proximal*) *transformation semigroups if and only if for each closed nonempty invariant subset* Z *of* X, (Z, S) *is codecomposable to distal* (*resp. minimal, equicontinuous, proximal*) *transformation semigroups.*

Proof. Use the fact that for each nonempty closed invariant subset Z of X, if $(X, (S_\alpha : \alpha \in \Gamma))$ is a codecomposition of (X, S), then $(Z, (S_\alpha : \alpha \in \Gamma))$ is a codecomposition of (Z, S), and in distal (resp. equicontinuous, proximal) transformation semigroup (Y, T) if W is a nonempty closed invariant subset of Y then (W, T) is distal (resp. equicontinuous, proximal).

Theorem 3.3. *In the transformation group* (X, S), *if* (X, S) *is codecomposable to distal* (*resp. point transitive, minimal, proximal*) *transformation groups and* $\Re$ *is a clopen invariant equivalence relation on* X, *then* $(X/\Re, S)$ *is codecomposable to distal* (*resp. point transitive, minimal, proximal*) *transformation groups.*

Proof. Use Definition 3.1 [1] (Section 1.3.2) or [9] (Exercise 1.11), and similar methods described in Theorem 3.2.

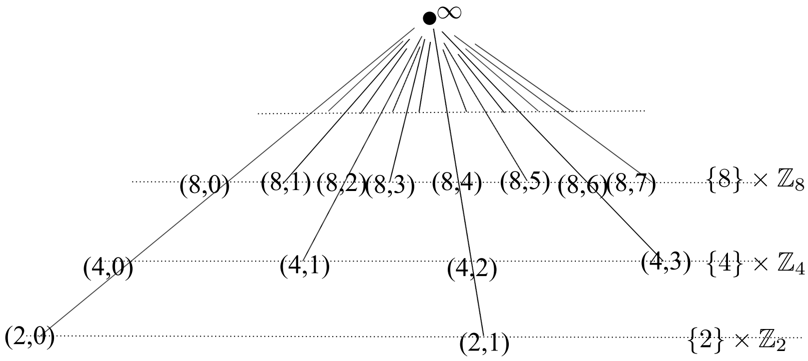
Theorem 3.4. *Let* $((X_\theta, S) : \theta \in \Theta)$ *be a nonempty collection of transformation semigroups, such that* $\left(\prod_{\theta \in \Theta} X_\theta, S\right)$ *is codecomposable to distal* (*resp. point transitive, minimal, proximal*) *transformation semigroups. Then for each* $\theta \in \Theta$, (X_θ, S) *is codecomposable to distal* (*resp. point transitive, minimal, proximal*) *transformation semigroups.*

Proof. Use Definition 3.1, and similar methods described in Theorem 3.2.

3.2. Counterexamples. Since the following counterexamples play important roles to find out main results during discussion on compact Hausdorff phase space with discrete phase semigroup case, so we devote them a special subsection.

***Counterexample* 3.1.** Let $X = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}$ (with induced topology of $\mathbb{R}$) and $S = \{\varphi^n : n \geq 0\}$ (with the discrete topology) where $0\varphi = 0$, $\frac{1}{n}\varphi = \frac{1}{n+1}$ (for $n \in \mathbb{N}$). Let $(X, (S_\alpha : \alpha \in \Gamma))$ be a codecomposition of (X, S), thus there exists $\alpha \in \Gamma$ such that S_α is infinite, which leads to the fact that constant function 0 belongs to $\mathrm{E}(X, S_\alpha)$ and $\mathrm{E}(X, S_\alpha)$ is not a group, therefore (X, S_α) is not distal. Therefore transformation semigroup (X, S) is not codecomposable to distal transformation semigroups.

***Counterexample* 3.2.** Let $Y = \bigcup_{n \in \mathbb{N}} \{2^n\} \times \mathbb{Z}_{2^n}$. For each $(2^n, x), (2^m, y) \in Y$ define $(2^n, x) \sim (2^m, y)$ if and only if $(n \geq m \wedge 2^{n-m}y = x) \vee (m \geq n \wedge 2^{m-n}x = y)$, $\sim$ is an equivalence relation on Y. Consider quotient space $\frac{Y}{\sim}$ with discrete topology and let $X := \frac{Y}{\sim} \cup \{\infty\}$ be its one point compactification, moreover denote the equivalence class of $(2^m, y) \in Y$ under $\sim$ by $[2^m, y]$.



An imagination of X: vertical bold lines show equivalence classes in Y, however X is homeomorph with $\left\{\frac{1}{n} : n \in \in \mathbb{N}\right\} \cup \{0\}$ (with induced topology of $\mathbb{R}$), i.e., ∞ is the unique limit point of X, X is infinite countable, and each open neighborhood of ∞ contains all points of X except finitely.

Let $n \in \mathbb{N}$. $(X, \mathbb{Z}_{2^n})$ is a transformation group, where for each $l \in \mathbb{Z}_{2^n} = \{0, 1, \ldots, 2^n - 1\}$ and $[2^m, y] \in X$ with $m \geq n$ we have $[2^m, y]l = [2^m, y + l2^{m-n}]$ and $\infty l = \infty$ (note to the fact that for each $[2^k, y] \in X$ we have $[2^k, y] = [2^{k+n}, 2^n y]$ and $k + n > n$ so $[2^k, y]l = [2^{k+n}, 2^n y + l2^k]$).

For each $n \in \mathbb{N}$, let $T_n = \left\{(x_i)_{i\in\mathbb{N}} \in \prod_{i\in\mathbb{N}} \mathbb{Z}_{2^i}\ \forall i \neq n (x_i = 0)\right\}$. (X, T_n) is a transformation semigroup where for each $(x_i)_{i\in\mathbb{N}} \in T_n$ and $z \in X$ we define $z(x_i)_{i\in\mathbb{N}} := zx_n$. $(X, (T_n : n \in \mathbb{N}))$ is a multitransformation semigroup and a codecomposition of (X, T) when T is the subgroup of $\prod_{i\in\mathbb{N}} \mathbb{Z}_{2^i}$ generated by $\bigcup\{T_n : n \in \mathbb{N}\}$ $\left(\text{thus } T = \bigoplus_{n\in\mathbb{N}} \mathbb{Z}_{2^n} \text{ (with discrete topology)}\right)$.

Claim. We have the following facts:

1) (X, T) is niether distal nor equicontinuous;

2) (X, T) is point transitive;

3) For each $n \in \mathbb{N}$, (X, T_n) is equicontinuous, and distal;

4) For each $n \in \mathbb{N}$, (X, T_n) is not point transitive.

Proof. 1. For each $n \in \mathbb{N}$ let $t_n = (\delta_{mn})_{m\in\mathbb{N}} \in T_n$ with $\delta_{nn} = 1$ and $\delta_{mn} = 0$ for $m \neq n$. The sequence $\{t_n\}_{n\in\mathbb{N}}$ in $\mathrm{E}(X, T)$ converges to constant function ∞; since for each $x \in X - \{\infty\}$ and $n \neq m$ we have $xt_n \neq xt_m$, and if U is an open neighborhood of ∞ then $X - U$ is finite and there exists $m \in \mathbb{N}$ such that $xt_n \in U$ for all $n \geq m$, i.e., $\{t_n\}_{n\in\mathbb{N}}$ converges to ∞, moreover for each $n \in \mathbb{N}$ we have $\infty t_n = \infty$. So the constant function ∞ belongs to $\mathrm{E}(X, T)$ (with no inverse) thus $\mathrm{E}(X, T)$ is not a group, therefore (X, T) is neither distal nor equicontinuous (see [3], Proposition 4.4).

2. (X, T) is point transitive since $\overline{[2, 1]T} = X$.

3. T_n is a finite group and $\mathrm{E}(X, T_n) = T_n$ is a (finite) group of continuous functions on X, which shows distality and equicontinuity of (X, T_n).

4. For each $x \in X$, $\overline{xT_n} = xT_n$ is a finite subspace of infinite space X, thus $\overline{xT_n} \neq X$ and (X, T_n) is not point transitive.

Therefore in transformation semigroup (X, T) we have

(X, T) is a non-distal transformation semigroup codecomposable to distal ones;

(X, T) is a non-equicontinuous transformation semigroup codecomposable to equicontinuous ones;

(X, T) is a point transitive transformation semigroup codecomposable to non-point transitive ones.

***Counterexample* 3.3.** Let $X := \{e^{i\theta}\colon \theta \in [0, 2\pi]\}$ with the induced topology of $\mathbb{R}^2$, and $S_n = \{z \in X\colon z^n = 1\}$ ($n \in \mathbb{N}$), then S_n acts on X as one of its subsemigroups. For each $n \in \mathbb{N}$, let $T_n = \left\{(x_i)_{i\in\mathbb{N}} \in \prod_{i\in\mathbb{N}} S_n\colon \forall i \neq n(x_i = 1)\right\}$. (X, T_n) is a transformation semigroup where for each $(x_i)_{i\in\mathbb{N}} \in T_n$ and $z \in X$ we define $z(x_i)_{i\in\mathbb{N}} := zx_n$. $(X, (T_n\colon n \in \mathbb{N}))$ is a multitransformation semigroup and a codecomposition of (X, T) when T is the subgroup of $\prod\limits_{i\in\mathbb{N}} S_n$ generated by $\bigcup\{T_n\colon n \in \mathbb{N}\}$ $\left(\text{thus } T = \bigoplus_{n\in\mathbb{N}} S_n \text{ (with discrete topology)}\right)$. (X, T) is minimal, but (X, T_n) is not minimal ($\forall n \in \mathbb{N}$), thus (X, T) is a minimal transformation semigroup codecomposable to non-minimal ones.

***Counterexample* 3.4.** In this counterexample we show that there are examples of multitransformation semigroup $(X, (S_\alpha\colon \alpha \in \Gamma))$ in which there exist $\alpha \neq \beta$, $p \in \mathrm{E}(X, S_\alpha)$ and $q \in \mathrm{E}(X, S_\beta)$ with $pq \neq qp$ (compare with the second item in Theorem 3.1). Let $X = \left\{(-1)^i\left(1 - \dfrac{1}{n}\right)\colon n \in \mathbb{N}, i = \pm 1\right\} \cup \{1, -1\}$ with induced topology from $\mathbb{R}$. Define $q, p, \varphi : X \to X$ with:

$$x\varphi := \begin{cases} x, & x = \pm 1, \\ 1 - \dfrac{1}{n}, & x = 1 - \dfrac{1}{n+1}, \quad n \in \mathbb{N}, \\ -1 + \dfrac{1}{n+1}, & x = -1 + \dfrac{1}{n}, \quad n \in \mathbb{N}, \end{cases}$$

$$xp := \begin{cases} -1, & x \neq 1, \\ 1, & x = 1, \end{cases} \qquad xq := \begin{cases} 1, & x \neq -1, \\ -1, & x = -1. \end{cases}$$

For $S = \{\varphi^n\colon n \in \mathbb{Z}\}$, $S_1 = \{\varphi^n\colon n \in \mathbb{N} \cup \{0\}\}$, $S_2 = \{\varphi^{-n}\colon n \in \mathbb{N} \cup \{0\}\}$, and $S_3 = \{\varphi^{2n} : n \in \mathbb{Z}\}$; $(X, (S_i\colon i = 1, 2, 3))$ is a codecomposition of (X, S). $p \in \mathrm{E}(X, S_1)$ and $q \in \mathrm{E}(X, S_2)$ moreover $pq \neq qp$.

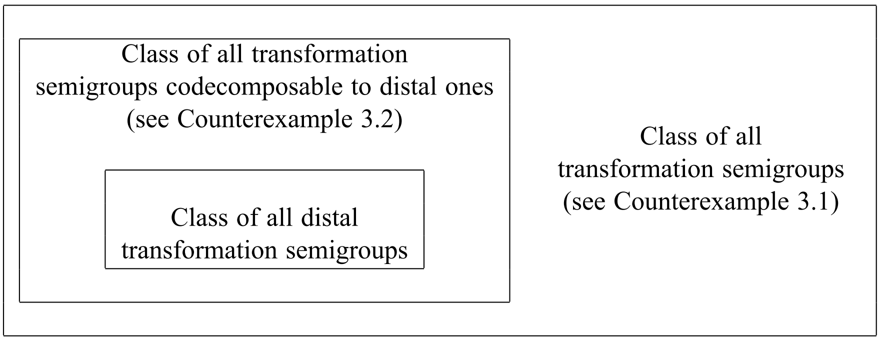
3.3. Main achievements.

Theorem 3.5 (Main theorem). *We have the following classifications* (*where "$\subset$" means strict inclusion*):

1. *Class of all distal transformation semigroups $\subset$ Class of all transformation semigroups codecomposable to distal ones $\subset$ Class of all transformation semigroups.*

2. *Class of all equicontinuous transformation semigroups $\subset$ Class of all transformation semigroups codecomposable to equicontinuous ones $\subset$ Class of all transformation semigroups.*

Proof. Use Counterexamples 3.1 and 3.2. One may consider the following diagram for (1):



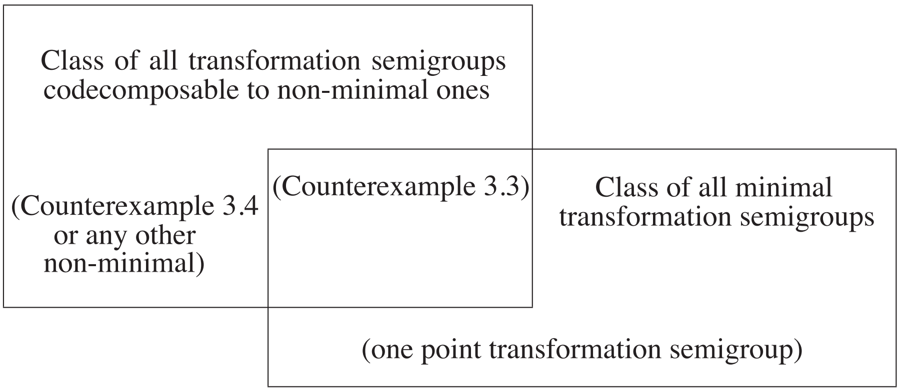
and a similar diagram for (2).

Theorem 3.6. *We have the following classifications* (*where "⊂" means strict inclusion*):

1. *Class of all point transitive transformation semigroups codecomposable to non-point transitive ones ⊂ Class of all point transitive transformation semigroups.*

2. *Class of all minimal transformation semigroups codecomposable to non-minimal ones ⊂ Class of all minimal transformation semigroups.*

Proof. Use Counterexamples 3.2 and 3.3. One may consider the following diagram for (2):



and a similar diagram for (1).

***Corollary* 3.1.** *We have the following diagram in the class of all transformation groups* (*with compact Hausdorff phase space and discrete phase semigroup*) (*use* [3] (*Proposition* 4.4)):

$$\begin{array}{ccc} \textit{EQUICONTINUOUS} & \Rightarrow & \textit{codecomposABLE TO EQUICONTINUOUS ONES} \\ \Downarrow & & \Downarrow \\ \textit{DISTAL} & \Rightarrow & \textit{codecomposABLE TO DISTAL ONES.} \end{array}$$

In order to complete the reverse implications of the above diagram see Counterexample 3.2 (*which shows the negative answer for horizontal reverse implications*), *see also* [5] (*which shows the negative answer for first vertical reverse implication by an example*).

Now the next question arises.

***Question*.** Is there any transformation group decomposable to distal ones, and non-decomposable to equicontinuous ones?

4. General case. In this section our main aim is to define codecomposition of a transformation semigroup in general case.

In multi transformation semigroup $(X,(S_\alpha\colon \alpha\in\Gamma))$ if

$$\bigoplus_{\alpha\in\Gamma} S_\alpha = \big\{(s_\alpha)_{\alpha\in\Gamma}\colon \exists\alpha_1,\ldots,\alpha_n\in\Gamma\ \forall\alpha\neq\alpha_1,\ldots,\alpha_n(s_\alpha=e_\alpha)\big\},$$

then $\bigoplus_{\alpha\in\Gamma} S_\alpha$ acts on X (not necessarily continuously), by

$$w(s_\alpha)_{\alpha\in\Gamma} = ws_{\alpha_1}\ldots s_{\alpha_n}$$

$\Big($for all $w\in X$ and $(s_\alpha)_{\alpha\in\Gamma}\in\bigoplus_{\alpha\in\Gamma} S_\alpha$ with distinct $\alpha_1,\ldots,\alpha_n\in\Gamma$ and $s_\alpha=e_\alpha$ for all $\alpha\in\Gamma-\{\alpha_1,\ldots,\alpha_n\}\Big)$.

In this section we will use the above defined action, moreover for more convenient the element $(s_\alpha)_{\alpha\in\Gamma}\in\bigoplus_{\alpha\in\Gamma} S_\alpha$ with $s_\alpha=e_\alpha$ for all $\alpha\neq\beta$, is denoted by $s_\beta x^\beta$.

Moreover if $\Big(X,\bigoplus_{\alpha\in\Gamma} S_\alpha\Big)$ is a transformation semigroup and for each $\alpha\in\Gamma$ the semigroup S_α has at least two elements, also for each $\beta\in\Gamma$, $T_\beta := \Big\{(s_\alpha)_{\alpha\in\Gamma}\in\bigoplus_{\alpha\in\Gamma} S_\alpha\colon \forall\alpha\neq\beta s_\alpha=e_\alpha\Big\}$, then $(X,(T_\alpha\colon\alpha\in\Gamma))$ is a multitransformation semigroup.

The above statements leads us to the following definition.

Definition 4.1 (General case)**.** *Multi transformation semigroup $(X,(S_\alpha\colon\alpha\in\Gamma))$ is called a codecomposition of transformation semigroup (X,S) if S is the semigroup generated by $\bigcup_{\alpha\in\Gamma} S_\alpha$, S_αs are distinct subsemigroups of S, and there exists a topology on $\bigoplus_{\alpha\in\Gamma} S_\alpha$ which makes a topological semigroup and:*

$\Big(X,\bigoplus_{\alpha\in\Gamma} S_\alpha\Big)$ *is a (topological) transformation semigroup*;

for each $\beta\in\Gamma$ the map $\iota_\beta\colon S_\beta\to\bigoplus_{\alpha\in\Gamma} S_\alpha$ with $\iota_\beta(s)=sx^\beta$, be an embedding;

under the map $s_{\alpha_1}x^{\alpha_1}+\ldots+s_{\alpha_n}x^{\alpha_n}\mapsto s_{\alpha_1}\ldots s_{\alpha_n}$, S is a continuous semigroup homomorphism image of $\bigoplus_{\alpha\in\Gamma} S_\alpha$.

***Note* 4.1** (Compatibility of Definitions 3.1 and 4.1)**.** In Definition 4.1, whenever S is discrete, discrete topology on $\bigoplus\limits_{\alpha\in\Gamma} S_\alpha$ shows that if $(X,(S_\alpha\colon\alpha\in\Gamma))$ is a codecomposition of (X,S) in the sense of Definition 3.1, then it is a codecomposition of (X,S) in the sense of Definition 4.1 too.

On the other hand if S is discrete and $(X,(S_\alpha\colon\alpha\in\Gamma))$ is a codecomposition of (X,S) in the sense of Definition 4.1, then it is clear that $(X,(S_\alpha\colon\alpha\in\Gamma))$ is a codecomposition of (X,S) in the sense of Definition 3.1 too. In multitransformation semigroup $(X,(S_\alpha\colon\alpha\in\Gamma))$, let $T:=\bigoplus_{\alpha\in\Gamma} S_\alpha$ with the induced product topology from $\prod_{\alpha\in\Gamma} S_\alpha$, then T is a topological semigroup, however in the following example T does not act continuously on X.

***Example* 4.1.** Let $\{p_n\colon n\in\mathbb{N}\}$ be a $\mathbb{Q}$-linearly independent sequence in $\mathbb{R}$ such that $\lim\limits_{n\to\infty} p_n=2$ and $p_1=1$. In the multitransformation semigroup $(\mathbb{R},(p_n\mathbb{Q}: n\in\mathbb{N}))$ ($p_n\mathbb{Q}$s act as subgroups of $\mathbb{R}$ on topological group $(\mathbb{R},+)$) as above let $T:=\bigoplus_{n\in\mathbb{N}} p_n\mathbb{Q}$ with the induced product topology from $\prod_{n\in\mathbb{N}} p_n\mathbb{Q}$. The sequence $(1,2x^1+p_nx^{n+1})_{n\in\mathbb{N}}$ (where $2x^1+p_nx^{n+1}=(a_n^m)_{m\in\mathbb{N}}$ for $a_n^1=2$, $a_n^{n+1}=p_n$ and $a_n^m=0$ for $m\neq n+1,1$) converges to $(1,2x^1)$ in $\mathbb{R}\times T$, but $(1(2x^1+p_nx^{n+1}))_{n\in\mathbb{N}}$

converges to 5 (since for all $n \in \mathbb{N}$ we have $1(2x^1 + p_n x^{n+1}) = 3 + p_n$) and it does not converges to $3(= 1(2x^1))$, thus $(\mathbb{R}, T)$ is not a topological transformation semigroup.

Theorem 4.1. *If $((X_\theta, S)\colon \theta \in \Theta)$ is a nonempty collection of transformation semigroups, then $\left(\prod_{\theta\in\Theta} X_\theta, (S_\alpha\colon \alpha \in \Gamma)\right)$ is a codecomposition of $\left(\prod_{\theta\in\Theta} X_\theta, S\right)$ if and only if for each $\theta \in \Theta$, $(X_\theta, (S_\alpha\colon \alpha \in \Gamma))$ is a codecomposition of (X_θ, S).*

Proof. Use Definitions 2.1 and 4.1.

Theorem 4.2. *In the transformation group (X, S) if $\Re$ is an open invariant equivalence relation on X, Z is a nonempty invariant subset of X, and $(X, (S_\alpha\colon \alpha \in \Gamma))$ is a codecomposition of (X, S), then $(X/\Re, (S_\alpha\colon \alpha \in \Gamma))$ is a codecomposition of $(X/\Re, S)$ and $(Z, (S_\alpha\colon \alpha \in \Gamma))$ is a codecomposition of (Z, S).*

Proof. Use Definitions 2.1, 4.1 and [1] (Section 1.3.2).

***Remark* 4.1.** It is possible to define multitransformation group, and codecomposition of a transformation group in a similar way.

***Remark* 4.2.** In all sections of this paper except the last one, we dealt with discrete phase semigroup. However one may be interested in the following cases: We call a codecomposition $(X, (S_\alpha\colon \alpha \in \Gamma))$ of (X, S) a *degenerated codecomposition* if there exists $\alpha \in \Gamma$ with $S_\alpha = S$. It is clear that if $S = \mathbb{Z}_p$, when p is a prime, then all codecompositions of (X, S) are degenerated. Is there any transformation semigroup (X, S) with just degenerated codecompositions, but (X, S_d) has non-degenerated codecompositions, where S_d is S with discrete topology?

Under which topologies on $\bigoplus_{\alpha\in\Gamma} S_\alpha$, the multitransformation semigroup $(X, (S_\alpha\colon \alpha \in \Gamma))$ is a codecomposition of $\left(X, \bigoplus_{\alpha\in\Gamma} S_\alpha\right)$ (is there any?) moreover if Λ denotes the set of such topologies, what are the properties of minimal and maximal elements of $(\Lambda, \subseteq)$?

1. *Bronstein I. U.* Extensions of minimal transformation groups. – Sijthoff & Noordhoff, 1979.
2. *Chu H.* Some inheritance theorems in topological dynamics // J. London Math. Soc. – 1968. – **43**. – P. 168 – 170.
3. *Ellis R.* Lectures on topological dynamics. – New York: W. A. Benjamin, 1969.
4. *Garcia M., Hedlund G. H.* The structure of minimal sets // Bull. Amer. Math. Soc. – 1948. – **54**. – P. 954 – 964.
5. *Glasner E., Megrelishvili M., Uspenskij V. V.* On metrizable enveloping semigroups // Isr. J. Math. – 2008. – **164**. – P. 317 – 332.
6. *Glasner Sh.* Proximal flows // Lect. Notes Math. – Berlin: Springer-Verlag, 1976. – Vol. 517.
7. *Gottschalk W. H.* Powers of homeomorphisms with almost periodic properties // Bull. Amer. Math. Soc. – 1944. – **50**. – P. 222 – 227.
8. *Gottschalk W. H.* Orbit-closure decompositions and almost periodic properties // Bull. Amer. Math. Soc. – 1944. – **50**. – P. 915 – 919.
9. *Rotman J. J.* An introduction to algebraic topology // Grad. Texts Math. – New York: Springer-Verlag, 1988. – **119**.

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