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SOME PROPERTIES OF MULTIVALENT FUNCTIONS ASSOCIATED A CERTAIN OPERATOR*

ДЕЯКІ ВЛАСТИВОСТІ БАГАТОЗНАЧНИХ ФУНКЦІЙ, АСОЦІЙОВАНИХ З ОПЕРАТОРОМ

We obtain certain subordinations and superordinations results involving a new operator. By means of the new introduced operator $\mathcal{C}_{p,n}^{\lambda}(a,c)f(z)$, for certain multivalent functions in the open unit disc, we establish differential Sandwich Theorem.

Отримано деякі субординації і результати для суперординацій із використанням нового оператора. З допомогою введеного оператора $\mathcal{C}_{p,n}^{\lambda}(a,c)f(z)$ доведено диференціальну сендвіч-теорему для багатозначних функцій у відкритому одиничному крузі.

1. Introduction. Let Σ_p denote the class of functions $f(z)$ of the form

$$f(z) = z^p + \sum_{k=n}^{\infty} a_{p+k} z^{p+k}, \qquad p \in N = \{1, 2, 3, ...\}, \tag{1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z\colon z \in \mathbb{C},\ |z| < 1\}$.

For functions $f \in \Sigma_p$ given by (1) and $g \in \Sigma_p$ given by

$$g(z) = z^p + \sum_{k=n}^{\infty} b_{p+k} z^{p+k}.$$

We define the Hadamard product (or convolution) of f and g by

$$(f * g)(z) = z^p + \sum_{k=n}^{\infty} a_{p+k} b_{p+k} z^{p+k}. \tag{2}$$

Let $f(z)$ and $g(z)$ be analytic in $\mathbb{U}$. We say that the function $g(z)$ is subordinate to $f(z)$, if there exists a function $w(z)$ analytic in $\mathbb{U}$, with $w(0) = 0$ and $|w(z)| < 1$, and such that $g(z) = f(w(z))$. In such a case, we write $g(z) \prec f(z)$. If the function f is univalent in $\mathbb{U}$, then $g(z) \prec f(z)$ if and only if $g(0) = f(0)$ and $g(\mathbb{U}) \subset f(\mathbb{U})$.

Let $H(\mathbb{U})$ denote the class of analytic functions in $\mathbb{U}$ and let $H(a, n)$ denote the subclass of functions $f \in H(\mathbb{U})$ of the form:

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots\ .$$

Denote by Q, the set of all functions $f(z)$ that are analytic and injective on $\mathbb{U}\backslash E(f)$, where $E(f) = \{\xi \in \partial\mathbb{U}\colon \lim_{z\to\xi} f(z) = \infty\}$, and such that $f'(\xi) \neq 0$ for $\xi \in \partial\mathbb{U}\backslash E(f)$.

Let $\psi\colon \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$, let $h(z)$ be univalent in $\mathbb{U}$ and $q(z) \in Q$. Miller and Mocanu [1] considered the problem of determining conditions on admissible function ψ such that

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$$\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z) \tag{3}$$

implies $p(z) \prec q(z)$, for all functions $p(z) \in H(a,n)$ that satisfy the differential subordination (3). Moreover, they found conditions so that $q(z)$ is the smallest function with this property, called the best dominant of the subordination (3).

Let $\varphi\colon \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$, let $h(z) \in H$ and $q(z) \in H(a,n)$. Recently Miller and Mocanu [2] studied the dual problem and determined conditions on φ such that

$$h(z) \prec \varphi(p(z), zp'(z), z^2p''(z); z) \tag{4}$$

implies $q(z) \prec p(z)$, for all functions $p(z) \in Q$ that satisfy the above superordination. They also found conditions so that the function $q(z)$ is the largest function with this property, called the best subordinant of the superordination (4).

In [3], N. E. Cho, O. S. Kwon and H. M. Srivastava extended the multiplier transformation and defined the operator $\mathcal{J}_{p,n}^{\lambda}(a,c)f(z)$ by the following infinite series:

$$\mathcal{J}_{p,n}^{\lambda}(a,c)f(z) = z^p + \sum_{k=n}^{\infty} \frac{(\lambda+p)_k(c)_k}{k!(a)_k} a_{k+p} z^{k+p}. \tag{5}$$

In recent years, Aghalary [4], Patel [5], Patel et al. [6], Sokl and Trojnar-Spelina [7], Zeng et al. [8] and Wang et al. [9] obtained many interesting results associated with the Cho – Kwon – Srivastava operator.

We now introduce the following family of linear operators:

$$\mathcal{L}_{p,n}^{\lambda}(a,c)f(z) = z^p + \sum_{k=n}^{\infty} \frac{k!(a)_k}{(\lambda+p)_k(c)_k} a_{k+p} z^{k+p}. \tag{6}$$

It is readily verified from the definition (6) that

$$z(\mathcal{L}_{p,n}^{\lambda}(a,c+1)f(z))' = c\mathcal{L}_{p,n}^{\lambda}(a,c)f(z) - (c-p)\mathcal{L}_{p,n}^{\lambda}(a,c+1)f(z) \tag{7}$$

and

$$z(\mathcal{L}_{p,n}^{\lambda}(a,c)f(z))' = (c-1)\mathcal{L}_{p,n}^{\lambda}(a,c-1)f(z) - (c-1-p)\mathcal{L}_{p,n}^{\lambda}(a,c)f(z). \tag{8}$$

We also note that $\mathcal{L}_{p,n}^{1}(p+1,1)f(z) = f(z)$ and $\mathcal{L}_{p,n}^{0}(p,1)f(z) = f(z)$. In this paper, we will derive several subordination results, superordination results and sandwich results involving the operator $\mathcal{L}_{p,n}^{\lambda}(a,c)f(z)$ and some of its special operators.

2. Some lemmas. In order to prove our main results, we need the following lemmas.

Lemma 1 [10]. *Let $q(z)$ be univalent in $\mathbb{U}$, $\gamma \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}$ and suppose that*

$$\operatorname{Re}\left\{1 + \frac{zq''(z)}{q'(z)}\right\} > \max\left\{0, -\operatorname{Re}\frac{1}{\gamma}\right\}.$$

If $p(z)$ is analytic in $\mathbb{U}$ and

$$p(z) + \gamma z p'(z) \prec q(z) + \gamma z q'(z),$$

then $p(z) \prec q(z)$, and $q(z)$ is the best dominant.

Lemma 2 [10]. *Let $q(z)$ be convex in $\mathbb{U}$, $q(0) = a$ and $\gamma \in \mathbb{C}$, $\operatorname{Re}\gamma > 0$. If $p \in H(a,1)$ and $p(z) + \gamma z p'(z)$ is univalent in $\mathbb{U}$, then*

$$q(z) + \gamma z q'(z) \prec p(z) + \gamma z p'(z),$$

where $q(z) \prec p(z)$ and $q(z)$ is the best subordinant.

3. Main results. We shall assume in the reminder of this paper that $p, n \in \mathbb{N}$ and $z \in \mathbb{U}$.

Theorem 1. *Let $q(z)$ be univalent in $\mathbb{U}$ with $q(0) = 1$, $\alpha \in \mathbb{C}^*$, and suppose that*

$$\operatorname{Re}\left\{1 + \frac{zq''(z)}{q'(z)}\right\} > \max\left\{0, -\operatorname{Re}\frac{1}{\alpha}\right\}. \tag{9}$$

If $f(z) \in \Sigma_p$ satisfies the subordination

$$\mathcal{R}(\alpha, n, p, \lambda, a, c) \prec q(z) + \alpha z q'(z), \tag{10}$$

where $\mathcal{R}(\alpha, n, p, \lambda, a, c)$ is given by

$$\mathcal{R}(\alpha, n, p, \lambda, a, c) =$$

$$= (1-\alpha)\frac{\mathcal{L}_{p,n}^{\lambda}(a, c+1)\ f(z)}{\mathcal{L}_{p,n}^{\lambda}(a, c)\ f(z)} + \alpha\left\{c - (c-1)\frac{\mathcal{L}_{p,n}^{\lambda}(a, c+1)f(z)\ \mathcal{L}_{p,n}^{\lambda}(a, c-1)\ f(z)}{(\mathcal{L}_{p,n}^{\lambda}(a, c)\ f(z))^2}\right\}, \tag{11}$$

then

$$\frac{\mathcal{L}_{p,n}^{\lambda}(a, c+1)f(z)}{\mathcal{L}_{p,n}^{\lambda}(a, c)f(z)} \prec q(z)$$

and $q(z)$ is the best dominant.

Proof. Let

$$p(z) = \frac{\mathcal{L}_{p,n}^{\lambda}(a, c+1)f(z)}{\mathcal{L}_{p,n}^{\lambda}(a, c)f(z)}, \tag{12}$$

differentiating (12) with respect to z and using the identity (7) and (8) in the resulting equation, we have

$$zp'(z) = c - (c-1)\frac{\mathcal{L}_{p,n}^{\lambda}(a, c+1)f(z) \cdot \mathcal{L}_{p,n}^{\lambda}(a, c-1)f(z)}{(\mathcal{L}_{p,n}^{\lambda}(a, c)f(z))^2} - \frac{\mathcal{L}_{p,n}^{\lambda}(a, c+1)f(z)}{\mathcal{L}_{p,n}^{\lambda}(a, c)f(z)}.$$

Therefore, we have

$$\mathcal{R}(\alpha, n, p, \lambda, a, c) = p(z) + \alpha z p'(z).$$

By (10), we obtain

$$p(z) + \alpha z p'(z) \prec q(z) + \alpha z q'(z).$$

By Lemma 1, $\dfrac{\mathcal{L}_{p,n}^{\lambda}(a, c+1)f(z)}{\mathcal{L}_{p,n}^{\lambda}(a, c)f(z)} \prec q(z)$, and the proof of Theorem 1 is completed.

Taking the convex function $q(z) = \dfrac{1 + Az}{1 + Bz}$ in Theorem 1, we have the following corollary.

***Corollary* 1.** *Let* A, B, $\alpha \in \mathbb{C}$, $A \neq B$, $|B| < 1$, $\operatorname{Re}\alpha > 0$. *If* $f(z) \in \Sigma_p$ *satisfies the subordination*

$$\mathcal{R}(\alpha, n, p, \lambda, a, c) \prec \frac{1 + Az}{1 + Bz} + \alpha\frac{(A - B)z}{(1 + Bz)^2},$$

where $\mathcal{R}(\alpha, n, p, \lambda, a, c)$ *is given by* (11), *then*

$$\frac{\mathcal{L}_{p,n}^{\lambda}(a, c + 1)f(z)}{\mathcal{L}_{p,n}^{\lambda}(a, c)f(z)} \prec \frac{1 + Az}{1 + Bz},$$

and the function $\dfrac{1 + Az}{1 + Bz}$ *is the best dominant.*

Theorem 2. *Let* $q(z)$ *be convex in* $\mathbb{U}$, $q(0) = 1$ *and* $\alpha \in \mathbb{C}$, $\operatorname{Re}\alpha > 0$. *If* $f(z) \in \Sigma_p$ *such that* $\dfrac{\mathcal{L}_{p,n}^{\lambda}(a, c + 1)f(z)}{\mathcal{L}_{p,n}^{\lambda}(a, c)f(z)} \in H(q(0), 1) \bigcap Q$, *and* $\mathcal{R}(\alpha, n, p, \lambda, a, c)$ *is univalent in* $\mathbb{U}$ *and satisfies the superordination*

$$q(z) + \alpha z q'(z) \prec \mathcal{R}(\alpha, n, p, \lambda, a, c), \tag{13}$$

where $\mathcal{R}(\alpha, n, p, \lambda, a, c)$ *is given by* (11), *then*

$$q(z) \prec \frac{\mathcal{L}_{p,n}^{\lambda}(a, c + 1)f(z)}{\mathcal{L}_{p,n}^{\lambda}(a, c)f(z)},$$

and $q(z)$ *is the best subordinant.*

Proof. Let $p(z)$ be given by (12) and proceeding as in the proof of Theorem 1, the subordination (13) becomes

$$q(z) + \alpha z q'(z) \prec p(z) + \alpha z p'(z).$$

The proof follows by an application of Lemma 2.

***Corollary* 2.** *Let* A, B, $\alpha \in \mathbb{C}$, $A \neq B$, $|B| < 1$, $\operatorname{Re}\alpha > 0$. *If* $f(z) \in \Sigma_p$ *such that* $\dfrac{\mathcal{L}_{p,n}^{\lambda}(a, c + 1)f(z)}{\mathcal{L}_{p,n}^{\lambda}(a, c)f(z)} \in H(q(0), 1) \bigcap Q$, *and* $\mathcal{R}(\alpha, n, p, \lambda, a, c)$ *is univalent in* $\mathbb{U}$ *and satisfies the superordination*

$$\frac{1 + Az}{1 + Bz} + \alpha\frac{(A - B)z}{(1 + Bz)^2} \prec \mathcal{R}(\alpha, n, p, \lambda, a, c),$$

then

$$\frac{1 + Az}{1 + Bz} \prec \frac{\mathcal{L}_{p,n}^{\lambda}(a, c + 1)f(z)}{\mathcal{L}_{p,n}^{\lambda}(a, c)f(z)},$$

and the function $\dfrac{1 + Az}{1 + Bz}$ *is the best subordinant.*

Combining Theorems 1 and 2, we have the following sandwich theorem.

Theorem 3. *Let* $q_1(z)$ *and* $q_2(z)$ *be convex in* $\mathbb{U}$, $q_1(0) = q_2(0) = 1$ *and* $q_2(z)$ *satisfies* (9), *and* $\alpha \in \mathbb{C}$, $\operatorname{Re}\alpha > 0$. *If* $f(z) \in \Sigma_p$ *such that* $\dfrac{\mathcal{L}_{p,n}^{\lambda}(a, c+1)f(z)}{\mathcal{L}_{p,n}^{\lambda}(a, c)f(z)} \in H(q(0), 1) \bigcap Q$, *and* $\mathcal{R}(\alpha, n, p, \lambda, a, c)$ *is univalent in* $\mathbb{U}$ *and satisfies*

$$q_1(z) + \alpha z q_1'(z) \prec \mathcal{R}(\alpha, n, p, \lambda, a, c) \prec q_2(z) + \alpha z q_2'(z),$$

where $\mathcal{R}(\alpha, n, p, \lambda, a, c)$ *is given by* (11), *then*

$$q_1(z) \prec \frac{\mathcal{L}_{p,n}^{\lambda}(a, c+1)f(z)}{\mathcal{L}_{p,n}^{\lambda}(a, c)f(z)} \prec q_2(z)$$

and $q_1(z)$, $q_2(z)$ *are the best subordinant and the best dominant, respectively.*

Remark. Combining Corollaries 1, 2, we obtain the corresponding sandwich results for the operators $\dfrac{\mathcal{L}_{p,n}^{\lambda}(a, c+1)f(z)}{\mathcal{L}_{p,n}^{\lambda}(a, c)f(z)}$.

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