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VALUE-SHARING PROBLEM FOR p-ADIC MEROMORPHIC FUNCTIONS AND THEIR DIFFERENCE OPERATORS AND DIFFERENCE POLYNOMIALS *

ЗАДАЧА ПРО СПІЛЬНІ ЗНАЧЕННЯ ДЛЯ p-АДИЧНИХ МЕРОМОРФНИХ ФУНКЦІЙ ТА ЇХ РІЗНИЦЕВИХ ОПЕРАТОРІВ І РІЗНИЦЕВИХ ПОЛІНОМІВ

We discuss the value-sharing problem, versions of the Hayman conjecture, and the uniqueness problem for p-adic meromorphic functions and their difference operators and difference polynomials.

Досліджено питання про спільні значення і єдиність та аналоги гіпотези Хеймена для p-адичних мероморфних функцій та їх різницевих операторів і різницевих поліномів.

1. Introduction. The problem of determining a meromorphic (or entire) function on $\mathbb{C}$ by its single pre-images, counting multiplicities, of finite sets is an important one and it has been studied by many mathematicians. For instance, in 1921 G. Polya showed that an entire function on $\mathbb{C}$ is determined by the inverse images, counting multiplicities, of three distinct non-omitted values. In 1926, R. Nevanlinna showed that a meromorphic function on the complex plane is uniquely determined by the inverse images, ignoring multiplicities, of 5 distinct values.

In [16] Hayman proved the following well-known result:

Theorem 1.1. *Let f be a meromorphic function on $\mathbb{C}$. If $f(z) \neq 0$ and $f^{(k)}(z) \neq 1$ for some fixed positive integer k and for all $z \in \mathbb{C}$, then f is constant.*

Hayman also proposed the following conjecture (see [16]).

Hayman Conjecture. *If an entire function f satisfies $f^n(z)f'(z) \neq 1$ for a positive integer n and all $z \in \mathbb{C}$, then f is a constant.*

It has been verified for transcendental entire functions by Hayman himself for $n > 1$ [16], and by Clunie for $n \geq 1$ [5]. These results and some related problems caused increasingly attentions to the value-sharing problem of meromorphic functions and their derivatives (see [2, 4, 19, 21]).

In 1997 Yang and Hua [23] studied the unicity problem for meromorphic functions and differential monomials of the form $f^n f'$, when they share only one value, and obtained the following theorem.

Theorem 1.2. *Let f and g be two non-constant meromorphic functions, let $n \geq 11$ be an integer, and $a \in \mathbb{C}$ be a non-zero finite value. If $f^n f'$ and $g^n g'$ share the value a CM, then either $f \equiv dg$ for some $(n+1)$-th root of unity d, or $f = c_1 e^{cz}$ and $g = c_2 e^{-cz}$ for three non-zero constants c_1, c_2 and c such that $(c_1 c_2)^{n+1} c^2 = -a^2$.*

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Recently, there has been an increasing interest in studying value-sharing and uniqueness for meromorphic functions and their difference operators and difference polynomials. Halburd and Korhonen [14] established a version of Nevanlinna theory based on difference operators. For an analog of Hayman Conjecture for difference, Laine and Yang [20] investigated the value distribution of difference products of entire functions, and obtained the following theorem.

Theorem 1.3. *Let $f(z)$ be a transcendental entire function of finite order, and c be a non-zero complex constant. Then $n \geq 2$, $f(z)^n f(z+c)$ assumes every non-zero value $a \in \mathbb{C}$ infinitely often.*

In recent years the similar problems are investigated for functions in a non-Archimedean fields (see, for example, [3, 5]). In [22] J. Ojeda proved that for a transcendental meromorphic function f in an algebraically closed fields of characteristic zero, complete for a non-Archimedean absolute value $\mathbb{K}$, the function $f'f^n - 1$ has infinitely many zeros, if $n \geq 2$.

Ha Huy Khoai and Vu Hoai An [12] established a similar results for a differential monomial of the form $f^n\big(f^{(k)}\big)^m$, where f is a meromorphic function in $\mathbb{C}_p$.

Now let $\mathbb{K}$ be an algebraically closed field of characteristic zero, complete for a non-Archimedean absolute value. We denote by $\mathcal{A}(\mathbb{K})$ the ring of entire functions in $\mathbb{K}$, by $\mathcal{M}(\mathbb{K})$ the field of meromorphic functions, i.e., the field of fractions of $\mathcal{A}(\mathbb{K})$, and $\widehat{\mathbb{K}} = \mathbb{K} \cup \{\infty\}$. The value-sharing problem for meromorphic functions in $\mathbb{K}$ was investigated first in [1] and [8]. In recent years, many interesting results on the value-sharing problem for meromorphic functions in $\mathbb{K}$ were obtained (see [17, 13]).

Let us first recall some basic definitions. For $f \in \mathcal{M}(\mathbb{K})$ and $S \subset \widehat{\mathbb{K}}$, we define

$$E_f(S) = \bigcup_{a \in S} \{(z,m) | f(z) = a \quad \text{with multiplicity} \quad m\}.$$

Let $\mathcal{F}$ be a nonempty subset of $\mathcal{M}(\mathbb{K})$. Two functions f, g of $\mathcal{F}$ are said to *share* S, *counting multiplicity* (share S CM), if $E_f(S) = E_g(S)$.

Now for $f \in \mathcal{M}(\mathbb{K})$. We define difference operators of f as

$$\Delta_c f = f(z+c) - f(z), \qquad \Delta_c^1 f = \Delta_c f$$

and

$$\Delta_c^{n+1} f = \Delta^n f(z+c) - \Delta^n f(z), \quad n = 1, 2, \ldots,$$

where $c \in \mathbb{C}_p$ is a non-zero constant; and difference polynomial of f as

$$A(z,f) = \sum_{\Lambda \in I} a_\Lambda(z) f(z)^{\Lambda_0} f(z)^{\Lambda_1} \ldots f(z)^{\Lambda_n},$$

where I be a finite set of multiindex $\Lambda = (\Lambda_0, \Lambda_1, \ldots, \Lambda_n)$ and the coefficients $a_\Lambda(z)$ are *small* with respect to $f(z)$ in the sence that $T_{a_\Lambda}(r) = o(T_f(r))$.

From now on, we assume $P(z)$ is a non-zero polynomial on $\mathbb{C}_p$ of degree n. Write $P(z) = a_0(z-a_1)^{m_1}(z-a_2)^{m_2} \ldots (z-a_s)^{m_s}$, $a_0 \neq 0$.

In this paper we discuss the value-sharing and versions of the Hayman Conjecture and uniqueness for p-adic meromorphic functions and their difference operators and difference polynomials, and prove a p-adic analog of Laine – Yang's result. Namely, we prove the following theorems.

Theorem 1.4 (A version of the Hayman Conjecture for p-adic meromorphic functions and their difference operators). *Let f be a meromorphic function on $\mathbb{C}_p$ and n, k_i, s, q, $i = 1, \ldots, q$, be are integers*,

$$s \geq 1, \quad q \geq 1, \quad k_i \geq 1, \qquad n \geq \sum_{i=1}^{q} (2k_i + 1)2^i + q + s + 1 - 3\sum_{i=1}^{q} k_i,$$

and $\Delta^q f$ is not identically zero. Then $P(f)(\Delta_c^1 f)^{k_1} \ldots (\Delta_c^q f)^{k_q} - a$ has zeros, where $a \in \mathbb{C}_p$ is a non-zero.

Theorem 1.5 (A version of the Hayman Conjecture for p-adic meromorphic functions and their difference polynomials). *Let f be a meromorphic function on $\mathbb{C}_p$ and n, q_i, s, k, $i = 1, \ldots, k$, be are integers, and*

$$s \geq 1, \quad k \geq 1, \quad q_i \geq 1, \qquad n \geq \sum_{i=1}^{k} q_i + 2k + s + 1.$$

Then $P(f)(f(z+c))^{q_1} \ldots (f(z+kc))^{q_k} - a$ has zeros, where $a \in \mathbb{C}_p$ is a non-zero.

Theorem 1.6 (A version of the Yang and Hua's Theorem 1.2 for p-adic meromorphic functions and their difference polynomials). *Let f and g be two non-constant p-adic meromorphic functions.*

(1) *If $E_{f^n f(z+c) \ldots f(z+kc)}(1) = E_{g^n g(z+c) \ldots g(z+kc)}(1)$, with $k \geq 1$ and $n \geq 5k+8$ be are integers, then $f = hg$ with $h^{n+k} = 1$ or $fg = l$ with $l^{n+k} = 1$.*

(2) *If $E_{f^n (f(z+c))^{q_1} \ldots (f(z+kc))^{q_k}}(1) = E_{g^n (g(z+c))^{q_1} \ldots (g(z+kc))^{q_k}}(1)$, with*

$$q_i > 1, \quad i = 1, \ldots, k, \quad k \geq 1, \quad n \geq \sum_{i=1}^{k} q_i + 8k + 8$$

be are integers, then $f = hg$ with $h^{n+q_1+\ldots+q_k} = 1$ or $fg = l$ with $l^{n+q_1+\ldots+q_k} = 1$.

(3) *If*

$$E_{f^n (f(z+e_1c) \ldots f(z+e_mc)(f(z+t_1c))^{q_1} \ldots (f(z+t_kc))^{q_k}}(1) =$$

$$= E_{g^n (g(z+e_1c) \ldots g(z+e_mc)(g(z+t_1c))^{q_1} \ldots (g(z+t_kc))^{q_k}}(1),$$

with

$$e_j \geq 1, \quad j = 1, \ldots, m, \quad t_i \geq 1, \quad q_i > 1, \quad i = 1, \ldots, k, \quad k \geq 1,$$

$$n \geq 5m + \sum_{i=1}^{k} q_i + 8k + 8$$

be are integers, then $f = hg$ with $h^{n+m+q_1+\ldots+q_k} = 1$ or $fg = l$ with $l^{n+m+q_1+\ldots+q_k} = 1$.

The main tool of the proof is the p-adic Nevanlinna theory ([8 – 3, 17]). Therefore, in the next section we first establish some properties of the height function (a p-adic analog of the Nevanlinna characteristic functionrm) for p-adic meromorphic functions and their difference operator and difference polynomials for later use.

2. Height of p-adic meromorphic functions. Let f be a non-constant holomorphic function on $\mathbb{C}_p$. For every $a \in \mathbb{C}_p$, expanding f around a as $f = \sum P_i(z-a)$ with homogeneous polynomials P_i of degree i, we define

$$v_f(a) = \min\{i \colon P_i \not\equiv 0\}.$$

For a point $d \in \mathbb{C}_p$ we define the function $v_f^d \colon \mathbb{C}_p \to \mathbb{N}$ by

$$v_f^d(a) = v_{f-d}(a).$$

Fix a real number ρ with $0 < \rho \leq r$. Define

$$N_f(a,r) = \frac{1}{\ln p} \int\limits_{\rho}^{r} \frac{n_f(a,x)}{x} dx,$$

where $n_f(a,x)$, as usually, is the number of the solutions of the equation $f(z) = a$ (counting multiplicity) in the disk $D_x = \{z \in \mathbb{C}_p \colon |z| \leq x\}$.

If $a = 0$, then set $N_f(r) = N_f(0,r)$.

For l a positive integer, set

$$N_{l,f}(a,r) = \frac{1}{\ln p} \int\limits_{\rho}^{r} \frac{n_{l,f}(a,x)}{x} dx,$$

where

$$n_{l,f}(a,r) = \sum_{|z| \leq r} \min\left\{v_{f-a}(z), l\right\}.$$

Let k be a positive integer. Define the function $v_f^{\leq k}$ from $\mathbb{C}_p$ into $\mathbb{N}$ by

$$v_f^{\leq k}(z) = \begin{cases} 0 & \text{if } \ v_f(z) > k, \\ v_f(z) & \text{if } \ v_f(z) \leq k, \end{cases}$$

and

$$n_f^{\leq k}(r) = \sum_{|z| \leq r} v_f^{\leq k}(z), \qquad n_f^{\leq k}(a,r) = n_{f-a}^{\leq k}(r).$$

Define

$$N_f^{\leq k}(a,r) = \frac{1}{\ln p} \int\limits_{\rho}^{r} \frac{n_f^{\leq k}(a,x)}{x} dx.$$

If $a = 0$, then set $N_f^{\leq k}(r) = N_f^{\leq k}(0,r)$.

Set

$$N_{l,f}^{\le k}(a,r) = \frac{1}{\ln p} \int\limits_{\rho}^{r} \frac{n_{l,f}^{\le k}(a,x)}{x} dx,$$

where

$$n_{l,f}^{\le k}(a,r) = \sum_{|z| \le r} \min \left\{ v_{f-a}^{\le k}(z), l \right\}.$$

In a like manner we define

$$N_f^{<k}(a,r), \quad N_{l,f}^{<k}(a,r), \quad N_f^{>k}(a,r), \quad N_f^{\ge k}(a,r), \quad N_{l,f}^{\ge k}(a,r), \quad N_{l,f}^{>k}(a,r).$$

Recall that for a holomorphic function $f(z)$ in $\mathbb{C}_p$, represented by the power series

$$f(z) = \sum_{0}^{\infty} a_n z^n,$$

for each $r > 0$, we define $|f|_r = \max\{|a_n| r^n, 0 \le n < \infty\}$.

Now let $f = \dfrac{f_1}{f_2}$ be a non-constant meromorphic function on $\mathbb{C}_p$, where $f_1,\ f_2$ be holomorphic functions on $\mathbb{C}_p$ having no common zeros, we set $|f|_r = \dfrac{|f_1|_r}{|f_2|_r}$. For a point $d \in \mathbb{C}_p \cup \{\infty\}$ we define the function $v_f^d \colon \mathbb{C}_p \to \mathbb{N}$ by

$$v_f^d(a) = v_{f_1 - d f_2}(a)$$

with $d \ne \infty$, and

$$v_f^{\infty}(a) = v_{f_2}(a).$$

For a point $a \in \mathbb{C}$ define

$$m_f(\infty, r) = \max \left\{ 0, \log |f|_r \right\}, \qquad qm_f(a,r) = m_{1/f-a}(\infty, r),$$

$$N_f(a,r) = N_{f_1 - a f_2}(r), \qquad N_f(\infty, r) = N_{f_2}(r),$$

$$T_f(r) = \max_{1 \le i \le 2} \log |f_i|_r.$$

In a like manner we define

$$N_{l,f}(a,r), \quad N_f^{\le k}(a,r), \quad N_{l,f}^{\le k}(a,r), \quad N_f^{<k}(a,r), \quad N_{l,f}^{<k}(a,r), \quad N_f^{>k}(a,r),$$

$$N_f^{\ge k}(a,r), \quad N_{l,f}^{\ge k}(a,r), \quad N_{l,f}^{>k}(a,r),$$

with $a \in \mathbb{C}_p \cup \{\infty\}$.

Then we have (see [9])

$$N_f(a,r) + m_f(a,r) = T_f(r) + O(1)$$

with $a \in \mathbb{C}_p \cup \{\infty\}$,

$$T_f(r) = T_{1/f}(r) + O(1),$$

$$|f^{(k)}|_r \leq \frac{|f|_r}{r^k},$$

$$m_{f^{(k)}/f}(\infty, r) = O(1).$$

The following two lemmas were proved in [9].

Lemma 2.1. *Let f be a non-constant holomorphic function on $\mathbb{C}_p$. Then*

$$T_f(r) - T_f(\rho) = N_f(r),$$

where $0 < \rho \leq r$.

Lemma 2.2. *Let f be a non-constant meromorphic function on $\mathbb{C}_p$ and let $a_1, a_2, \ldots, a_q$ be distinct points of $\mathbb{C}_p$. Then*

$$(q-1)T_f(r) \leq N_{1,f}(\infty, r) + \sum_{i=1}^{q} N_{1,f}(a_i, r) - N_{0,f'}(r) - \log r + O(1),$$

where $N_{0,f'}(r)$ is the counting function of the zeros of f' which occur at points other than roots of the equations $f(z) = a_i$, $i = 1, \ldots, q$, and $0 < \rho \leq r$.

3. Two versions of the Hayman Conjecture for p-adic meromorphic functions and their difference operators and difference polynomials. We are going to prove Theorems 1.4 – 1.6. We need the following lemmas.

Lemma 3.1. *Let f be a non-constant p-adic meromorphic function and Δf is not identically zero and k, q be a positive integer. Then*:

(1) $m_{f(z+c)/f(z)}(\infty, r) = O(1)$;

(2) $m_{f(z+kc)/f(z)}(\infty, r) = O(1)$;

(3) $m_{\Delta_c f/f}(\infty, r) = O(1)$;

(4) $m_{(\Delta_c f)^q/f}(\infty, r) = O(1)$;

(5) $T_{f(z+c)}(r) = T_{f(z)}(r) + O(1)$;

(6) $T_{f(z+qc)}(r) = T_{f(z)}(r) + O(1)$;

(7) $T_{\Delta_c f/f}(r) \leq 2T_f(r) + O(1)$.

Proof. Set $A_c = \dfrac{f(z+c)}{f(z)}$. Then:

(1) If $|c| < r$. Notice that the set of $r \in \mathbb{R}_+$ such that there exist $z \in \mathbb{C}_p$ with $|z| = r$ is dense in $\mathbb{R}_+$. Therefore, without loss of generality one may assume that there exist $z \in \mathbb{C}_p$ such that $|z| = r$. Then $|c + z| = |z| = r$. So $|f(z)|_r = |f(z+c)|_r$ and $|A_c| = 1$. If $r \leq |c|$, then $|c + z| \leq$ $\leq \max\big\{|c|, |z|\big\} \leq |c|$. Thus $|A_c| = O(1)$. Therefore $m_{A_c}(\infty, r) = \max\big\{0, \log|A_c|_r\big\} = O(1)$.

(2) Similarly as the arguments of (1), we obtain $m_{f(z+kc)/f(z)}(\infty, r) = O(1)$.

(3) By $m_{f(z+c)/f(z)}(\infty, r) = O(1)$, $m_{\Delta_c f/f}(\infty, r) \leq \max\big\{m_{f(z+c)/f(z)}(\infty, r), 0\big\}$, we have $m_{\Delta_c f/f}(\infty, r) = O(1)$.

(4) By $m_{\Delta_c f/f}(\infty,r)=O(1)$, $m_{(\Delta_c f)^q/f}(\infty,r)=qm_{\Delta_c f/f}(\infty,r)$, we have $m_{(\Delta_c f)^q/f}(\infty,r)=$ $=O(1)$.

(5) Let $f=\dfrac{f_1}{f_2}$ be a non-constant meromorphic function on $\mathbb{C}_p$, where $f_1,\ f_2$ be holomorphic functions on $\mathbb{C}_p$ having no common zeros. Similarly as the arguments of (1), we have:

If $|c|<r$, then $|f_1(z)|_r=|f_1(z+c)|_r$ and $|f_2(z)|_r=|f_2(z+c)|_r$. If $r\le|c|$, then $|f_1(z)|_r\le$ $\le|f_1(z)|_c, |f_1(z+c)|_r\le|f_1(z)|_c|$, and $|f_2(z)|_r\le|f_2(z)|_c, |f_2(z+c)|_r\le|f_2(z)|_c$. Moreover, $T_f(r)=\max_{1\le i\le 2}\log|f_i|_r$. So $T_{f(z+c)}(r)=T_{f(z)}(r)+O(1)$.

(6) Similarly as the arguments of (5), we obtain $T_{f(z+qc)}(r)=T_{f(z)}(r)+O(1)$.

(7) We have

$$T_{\frac{\Delta_c f}{f}}(r)=m_{\frac{f(z+c)-f(z)}{f(z)}}(\infty,r)+N_{\frac{f(z+c)-f(z)}{f(z)}}(\infty,r)\le$$

$$\le m_{\frac{f(z+c)}{f(z)}}(\infty,r)+N_{\frac{f(z+c)}{f(z)}}(\infty,r)+O(1)\le$$

$$\le m_{f(z)}(\infty,r)+N_{f(z)}(\infty,r)+m_{f(z+c)}(\infty,r)+N_{f(z+c)}(\infty,r)+O(1)=$$

$$=T_{f(z+c)}(r)+T_{f(z)}(r)+O(1)\le 2T_f(r)+O(1).$$

Lemma 3.1 is proved.

Lemma 3.2. *Let f be a non-constant p-adic meromorphic function and $\Delta^q f$ is not identically zero and $k,\ q,\ m$ be a positive integer and $P(z)$ is the above. Then*:

(1) $T_{\Delta_c^q f}(r)\le 2^q T_f(r)+O(1)$;

(2) $T_{\Delta_c^q f/f}(r)\le 2(2^q-1)T_f(r)+O(1)$;

(3) $\left(n+3\sum_{i=1}^q k_i-\sum_{i=1}^q k_i 2^{i+1}\right)T_f(r)\le T_{P(f)(\Delta_c^1 f)^{k_1}\ldots(\Delta_c^q f)^{k_q}}(r)+O(1)$;

(4) $\left(n-\sum_{i=1}^k q_i\right)T_f(r)\le T_{P(f)(f(z+c))^{q_1}\ldots(f(z+kc))^{q_k}}(r)+O(1)$.

Proof. We will prove (1) by induction on j, $1\le j\le q-1$. With $j=1$, we have $T_{\Delta_c f}(\infty,r)\le$ $\le T_{f(z+c)}(r)+T_{f(z)}(r)+O(1)$. By $T_{f(z+c)}(r)=T_{f(z)}(r)+O(1)$, $T_{\Delta_c f}(r)\le 2T_f(r)+O(1)$. Now assume that $T_{\Delta_c^j f}(r)\le 2^j T_f(r)+O(1)$. Moreover we have $\Delta_c^{j+1}f=\Delta_c(\Delta_c^j f)$. From this and by induction, $T_{\Delta_c^{j+1}f}(r)=T_{\Delta_c(\Delta_c^j f)}(r)\le T_{\Delta_c^j f(z+c)}(r)+T_{\Delta_c^j f(z)}(r)+O(1)\le 2.2^j T_f(r)+O(1)=$ $=2^{j+1}T_f(r)+O(1)$.

We will prove (2) by induction on j, $1\le j\le q-1$. With $j=1$, by 3.1 (7) we have $T_{\Delta_c f/f}(r)\le$ $\le 2T_f(r)+O(1$. Now assume that $T_{\Delta_c^j f/f}(r)\le 2(2^j-1)T_f(r)+O(1)$. Moreover we have

$$T_{\frac{\Delta_c^{j+1}f}{f}}(r)=T_{\frac{\Delta_c^{j+1}f}{\Delta_c^j f}\frac{\Delta_c^j f}{f}}(r)\le T_{\frac{\Delta_c^{j+1}f}{\Delta_c^j f}}+T_{\frac{\Delta_c^j f}{f}}+O(1)\le$$

$$\le T_{\frac{\Delta_c(\Delta_c^j f)}{\Delta_c^j f}}+T_{\frac{\Delta_c^j f}{f}}+O(1)\le 2T_{\Delta_c^j}+2(2^j-1)T_f(r)+O(1).$$

By (1), $T_{\Delta_c^q f}(r)\le 2^q T_f(r)+O(1)$. Thus $2T_{\Delta_c^j}+2(2^j-1)T_f(r)\le 2(2^j+2^j-1)T_f(r)+O(1)\le$ $\le 2(2^{j+1}-1)T_f(r)+O(1)$. Therefore

$$T_{\frac{\Delta_c^{j+1} f}{f}}(r) \leq 2(2^{j+1}-1)T_f(r) + O(1).$$

(3) Set $G = P(f)(\Delta_c^1 f)^{k_1} \ldots (\Delta_c^q f)^{k_q}$. We have

$$f^{k_1} \ldots f^{k_q} G = f^{k_1+\ldots+k_q} P(f)(\Delta_c^1 f)^{k_1} \ldots (\Delta_c^q f)^{k_q}$$

and

$$\frac{1}{f^{k_1+\ldots+k_q} P(f)} = \frac{1}{G} \left(\frac{\Delta_c^1 f}{f}\right)^{k_1} \ldots \left(\frac{\Delta_c^q f}{f}\right)^{k_q}.$$

From this and (2), we obtain

$$\left(n + \sum_{i=1}^{q} k_i\right) T_f(r) = T_{\frac{1}{f^{k_1+\ldots+k_q} P(f)}}(r) + O(1) = T_{\frac{1}{G}\left(\frac{\Delta_c^1 f}{f}\right)^{k_1} \ldots \left(\frac{\Delta_c^q f}{f}\right)^{k_q}}(r) + O(1) \leq$$

$$\leq T_{1/G}(r) + \sum_{i=1}^{q} T_{(\Delta_c^i f/f)^{k_i}}(r) + O(1) \leq T_{1/G}(r) + \sum_{i=1}^{q} k_i T_{\Delta_c^i f/f}(r) + O(1) \leq$$

$$\leq T_G(r) + \sum_{i=1}^{q} k_i 2(2^i - 1) T_f(r) + O(1).$$

So

$$\left(n + 3\sum_{i=1}^{q} k_i - \sum_{i=1}^{q} k_i 2^{i+1}\right) T_f(r) \leq T_{P(f)(\Delta_c^1 f)^{k_1} \ldots (\Delta_c^q f)^{k_q}}(r) + O(1).$$

(4) Set $F = P(f)(f(z+c))^{q_1} \ldots (f(z+kc))^{q_k}$. We have $f^{q_1} \ldots f^{q_k} F = f^{q_1+\ldots+q_k} P(f)(f(z+ +c))^{q_1} \ldots (f(z+kc))^{q_k}$ and $f^{q_1+\ldots+q_k} P(f) = F. \left(\frac{f(z)}{f(z+c)}\right)^{q_1} \ldots \left(\frac{f(z)}{f(z+kc)}\right)^{q_k}$. From this and 3.1 (5), 3.1 (6), we obtain

$$\left(n + \sum_{i=1}^{k} q_i\right) T_f(r) = T_{f^{q_1+\ldots+q_k} P(f)}(r) + O(1) =$$

$$= T_{F.\left(\frac{f(z)}{f(z+c)}\right)^{q_1} \ldots \left(\frac{f(z)}{f(z+kc)}\right)^{q_k}}(r) + O(1) \leq T_F(r) + \sum_{i=1}^{k} T_{\left(\frac{f(z)}{f(z+ic)}\right)^{q_i}}(r) + O(1) \leq$$

$$\leq T_F(r) + \sum_{i=1}^{k} q_i T_{\frac{f(z)}{f(z+ic)}}(r) + O(1) \leq T_F(r) + \sum_{i=1}^{k} q_i (T_f(r) + T_{f(z+ic)}(r)) + O(1).$$

Therefore

$$\left(n + \sum_{i=1}^{k} q_i\right) T_f(r) \leq T_F(r) + 2\sum_{i=1}^{k} q_i T_f(r) + O(1).$$

So

$$\left(n - \sum_{i=1}^{k} q_i\right) T_f(r) \leq T_{P(f)(f(z+c))^{q_1} \ldots (f(z+kc))^{q_k}}(r) + O(1).$$

Lemma 3.2 is proved.

Lemma 3.3. *Let f and g be non-constant p-adic meromorphic functions. If $E_f(1) = E_g(1)$, then one of the following three cases holds*:

(1) $T_f(r) \leq N_{1,f}(\infty,r) + N^{\geq 2}_{1,f}(\infty,r) + N_{1,f}(0,r) + N^{\geq 2}_{1,f}(0,r) + N_{1,g}(\infty,r) + N^{\geq 2}_{1,g}(\infty,r) +$ $+ N_{1,g}(0,r) + N^{\geq 2}_{1,g}(0,r) - \log r + O(1)$, *the same inequality holding for* $H_g(r)$;

(2) $f \equiv g$;

(3) $fg \equiv 1$.

Proof. Set

$$F = \frac{1}{f-1}, \qquad G = \frac{1}{g-1},$$
$$L = \frac{f''}{f'} - 2\frac{f'}{f-1} - \frac{g"}{g'} + 2\frac{g'}{g-1}. \tag{3.1}$$

Then

$$L = \frac{F''}{F'} - \frac{G''}{G'}. \tag{3.2}$$

Next we consider the following two cases:

Case 1: $L \not\equiv 0$. Since $E_f(1) = E_g(1)$, if $f(a) = 1, g(a) = 1$ and $v^1_f(a) = v^1_g(a)$, then $L(a) = 0$. We now consider the poles of L. It is clear that all poles of L are of order 1. We can easily see from (3.1) that any simple pole of f and g is not a pole of L and the poles of L only occur at zeros of f' and g' and the multiple poles of f and g.

From (3.1) we have

$$m_L(\infty,r) = O(1),$$

and

$$N^{\leq 1}_f(1,r) = N^{\leq 1}_g(1,r) \leq N_L(0,r) \leq T_L(r) + O(1) \leq N_L(\infty,r) + O(1). \tag{3.3}$$

On the other hand, by Lemma 2.2,

$$T_f(r) \leq N_{1,f}(\infty,r) + N_{1,f}(0,r) + N_{1,f}(1,r) - N_{0,f'}(r) - \log r + O(1),$$

where $N_{0,f'}(r)$ denotes the counting function of those zeros of f' but not that of $f(f-1)$. Also, $N_{1,0,f'}(r)$ is defined similarly, where in counting, each zero of f' is counted with multiplicity 1. From (3.1), (3.2) and (3.3) we deduce that

$$N^{\leq 1}_f(1,r) \leq N^{\geq 2}_{1,f}(\infty,r) +$$
$$+ N^{\geq 2}_{1,g}(\infty,r) + N_{1,0,f'}(r) + N_{1,0,g'}(r) + N^{\geq 2}_{1,f}(0,r) + N^{\geq 2}_{1,g}(0,r) + O(1). \tag{3.4}$$

Since $E_f(1) = E_g(1)$,

$$N_{1,f}(1,r) = N_f^{\leq 1}(1,r) + N_{1,g}^{\geq 2}(1,r).$$

Then

$$T_f(r) \leq N_{1,f}(\infty,r) + N_{1,f}(0,r) + N_f^{\leq 1}(1,r) + N_{1,g}^{\geq 2}(1,r) - N_{0,f'}(r) - \log r + O(1). \tag{3.5}$$

Now we consider $N_{1,g}^{\geq 2}(1,r)$.

By Lemma 2.1,

$$N_{g'}(0,r) - N_g(0,r) + N_{1,g}(0,r) = N_{\frac{g'}{g}}(0,r) \leq T_{\frac{g'}{g}}(r) + O(1) =$$

$$= N_{\frac{g'}{g}}(\infty,r) + m_{\frac{g'}{g}}(\infty,r) + O(1) =$$

$$= N_{1,g}(\infty,r) + N_{1,g}(0,r) + O(1).$$

Therefore

$$N_{g'}(0,r) \leq N_{1,g}(\infty,r) + N_g(0,r) + O(1).$$

Moreover

$$N_{0,g'}(r) + N_{1,g}^{\geq 2}(1,r) + N_g^{\geq 2}(0,r) - N_{1,g}^{\geq 2}(0,r) \leq N_{g'}(0,r).$$

The above two inequalities yield

$$N_{0,g'}(r) + N_{1,g}^{\geq 2}(1,r) \leq N_{1,g}(\infty,r) + N_{1,g}(0,r) + O(1).$$

Combining this inequality and (3.4) and (3.5), we obtain (1).

Case 2: $L \equiv 0$. Then

$$\frac{f''}{f'} - 2\frac{f'}{f-1} \equiv \frac{g''}{g'} - 2\frac{g'}{g-1}. \tag{3.6}$$

By (3.6) we have

$$\frac{F''}{F'} \equiv \frac{G''}{G'}.$$

Thus

$$f \equiv \frac{ag+b}{cg+d},$$

where a, b, c, $d \in \mathbb{C}_p$ satisfying $ad - bc \neq 0$. Then $T_f(r) = T_g(r) + O(1)$.

Next we consider the following subcases:

Subcase 1: $ac \neq 0$. Then

$$f - \frac{a}{c} \equiv \frac{b - \dfrac{ad}{c}}{cg + d}.$$

By Lemma 2.3

$$T_f(r) \leq N_{1,f}(\infty, r) + N_{1,f-\frac{a}{c}}(0, r) + N_{1,f}(0, r) + O(1) =$$

$$= N_{1,f}(\infty, r) + N_{1,g}(\infty, r) + N_{1,f}(0, r) + O(1).$$

We get (1).

Subcase 2: $a \neq 0$, $c = 0$. Then $f \equiv \dfrac{ag + b}{d}$. If $b \neq 0$, by Lemma 2.2,

$$T_f(r) \leq N_{1,f}(\infty, r) + N_{1,f-\frac{b}{d}}(0, r) + N_{1,f}(0, r) + O(1) =$$

$$= N_{1,f}(\infty, r) + N_{1,g}(0, r) + N_{1,f}(0, r) + O(1).$$

We get (1). If $b = 0$, then $f \equiv \dfrac{ag}{d}$. If $\dfrac{a}{d} = 1$, then $f \equiv g$. We obtain (2). If $\dfrac{a}{d} \neq 1$, then by $E_f(1) = E_g(1)$ and Lemma 2.3

$$f \neq 1, \qquad f \neq \frac{a}{d},$$

$$T_f(r) \leq N_{1,f}(\infty, r) + N_{1,f}(\frac{a}{d}, r) + N_{1,f}(1, r) + O(1) = N_{1,f}(\infty, r) + O(1).$$

We get (1).

Subcase 3: $a = 0$, $c \neq 0$. Then $f \equiv \dfrac{b}{cg + d}$. If $d \neq 0$, by Lemma 2.2,

$$T_f(r) \leq N_{1,f}(\infty, r) + N_{1,f-\frac{b}{d}}(0, r) + N_{1,f}(0, r) + O(1) =$$

$$= N_{1,f}(\infty, r) + N_{1,g}(0, r) + N_{1,f}(0, r) + O(1).$$

We obtain (1).

If $d = 0$, then $f \equiv \dfrac{b}{cg}$. If $\dfrac{b}{c} = 1$, then $fg \equiv 1$. We obtain (3).

If $\dfrac{b}{c} \neq 1$, then by $E_f(1) = E_g(1)$ and Lemma 2.2,

$$f \neq 1, \qquad f \neq \frac{b}{c},$$

$$T_f(r) \leq N_{1,f}(\infty, r) + N_{1,f}\left(\frac{b}{c}, r\right) + N_{1,f}(1, r) + O(1) = N_{1,f}(\infty, r) + O(1).$$

We get (1).

Lemma 3.3 is proved.

Now we use the above lemmas to prove the main result of the paper.

***Proof of Theorem* 1.4.** From $P(z) = a_0(z-a_1)^{m_1}(z-a_2)^{m_2}\ldots(z-a_s)^{m_s}$, $a_0 \neq 0$, $P(f) = = a_0(f-a_1)^{m_1}(f-a_2)^{m_2}\ldots(f-a_s)^{m_s}$. Set $G = P(f)(\Delta_c^1 f)^{k_1}\ldots(\Delta_c^q f)^{k_q}$. We see that any pole of G can occur only at poles of f, $f(z+c)$, $f(z+2c),\ldots, f(z+qc)$, and any zero of G can occur only at zeros of $f-a_1$, $f-a_2,\ldots, f-a_s$, $\Delta_c^1 f,\ldots,\Delta_c^q f$. From this and by Lemmas 2.1, 2.2, 3.1 (5), 3.1 (6), 3.2 (1), 3.2 (3) we have

$$\left(n + 3\sum_{i=1}^{q} k_i - \sum_{i=1}^{q} k_i 2^{i+1}\right) T_f(r) \leq T_G(r) + O(1) \leq$$

$$\leq N_{1,G}(\infty, r) + N_{1,G}(0,r) + N_{1,G}(a,r) - \log r + O(1) \leq$$

$$\leq N_{1,f}(\infty,r) + \sum_{i=1}^{q} N_{1,f(z+ic)}(\infty,r) + \sum_{i=1}^{s} N_{1,f}(a_i,r) +$$

$$+\sum_{i=1}^{q} N_{1,\Delta_c^i f} + N_{1,G}(a,r) - \log r + O(1) \leq$$

$$\leq T_f(r) + qT_f(r) + sT_f(r) + \sum_{i=1}^{q} 2^i T_f(r) + N_{1,G}(a,r) - \log r + O(1) =$$

$$= \left(\sum_{i=1}^{q} 2^i + q + s + 1\right) T_f(r) + N_{1,G}(a,r) - \log r + O(1).$$

Therefore

$$\left(n + 3\sum_{i=1}^{q} k_i - \sum_{i=1}^{q}(2k_i+1)2^i - q - s - 1\right) T_f(r) + \log r \leq N_{1,G}(a,r) + O(1).$$

Since and

$$n \geq \sum_{i=1}^{q}(2k_i+1)2^i + q + s + 1 - 3\sum_{i=1}^{q} k_i,$$

we obtain

$$P(f)(\Delta_c^1 f)^{k_1}\ldots(\Delta_c^q f)^{k_q} - a$$

has zeros.

Theorem 1.4 is proved.

***Proof of Theorem* 1.5.** From $P(z) = a_0(z-a_1)^{m_1}(z-a_2)^{m_2}\ldots(z-a_s)^{m_s}, a_0 \neq 0$, $P(f) = = a_0(f-a_1)^{m_1}(f-a_2)^{m_2}\ldots(f-a_s)^{m_s}$. Set $F = P(f)(f(z+c))^{q_1}\ldots(f(z+kc))^{q_k}$. We see that any pole of F can occur only at poles of f, $f(z+c)$, $f(z+2c),\ldots, f(z+kc)$, and any zero of G can occur only at zeros of $f-a_1$, $f-a_2,\ldots,$ $f-a_s$, $f(z+c)$, $f(z+2c),\ldots,$ $f(z+kc)$. From this and by Lemmas 2.1, 2.2, 3.1 (5), 3.1 (6), 3.2 (4) we have

$$\left(n - \sum_{i=1}^{k} q_i\right) T_f(r) \leq T_F(r) + O(1) \leq$$

$$\leq N_{1,F}(\infty, r) + N_{1,F}(0, r) + N_{1,F}(a, r) - \log r + O(1) \leq$$

$$\leq N_{1,f}(\infty, r) + \sum_{i=1}^{k} N_{1,f(z+ic)}(\infty, r) + \sum_{i=1}^{s} N_{1,f}(a_i, r) + \sum_{i=1}^{k} N_{1,f(z+ic)}(0, r) +$$

$$+ N_{1,F}(a, r) - \log r + O(1) \leq T_f(r) + kT_f(r) + sT_f(r) + kT_f(r) + N_{1,F}(a, r) -$$

$$- \log r + O(1) = (2k + s + 1)T_f(r) + N_{1,F}(a, r) - \log r + O(1).$$

Therefore

$$\left(n - \sum_{i=1}^{k} q_i - 2k - s - 1\right) T_f(r) + \log r \leq N_{1,F}(a, r) + O(1).$$

Since and

$$n \geq \sum_{i=1}^{k} q_i + 2k + s + 1$$

we obtain

$$P(f)(f(z + c))^{q_1} \ldots (f(z + kc))^{q_k} - a$$

has zeros.

Theorem 1.5 is proved.

***Proof of Theorem* 1.6.** (1) Set $A = f^n f(z + c) \ldots f(z + kc)$, $B = g^n g(z + c) \ldots g(z + kc)$.

It suffices to consider the following cases:

Case 1:

$$T_A(r) + O(1) \leq N_{1,A}(\infty, r) + N_{1,A}^{\geq 2}(\infty, r) + N_{1,A}(0, r) + N_{1,A}^{\geq 2}(0, r) +$$

$$+ N_{1,B}(\infty, r) + N_{1,B}^{\geq 2}(\infty, r) + N_{1,B}(0, r) + N_{1,B}^{\geq 2}(0, r) - \log r + O(1).$$

By Lemmas 3.2 (4), 3.3,

$$\begin{gathered}(n - k)T_f(r) \leq T_A(r) + O(1) \leq N_{1,A}(\infty, r) + N_{1,A}^{\geq 2}(\infty, r) + N_{1,A}(0, r) + N_{1,A}^{\geq 2}(0, r) +\\ + N_{1,B}(\infty, r) + N_{1,B}^{\geq 2}(\infty, r) + N_{1,B}(0, r) + N_{1,B}^{\geq 2}(0, r) - \log r + O(1),\\ (n - k)T_g(r) \leq T_B(r) + O(1) \leq N_{1,A}(\infty, r) + N_{1,A}^{\geq 2}(\infty, r) + N_{1,A}(0, r) +\\ + N_{1,A}^{\geq 2}(0, r) + N_{1,B}(\infty, r) + N_{1,B}^{\geq 2}(\infty, r) + N_{1,B}(0, r) + N_{1,B}^{\geq 2}(0, r) - \log r + O(1).\end{gathered} \tag{3.7}$$

We see that any pole of A can occur only at poles of

$$f, f(z+c), f(z+2c), \ldots, f(z+kc).$$

From this and by Lemmas 2.1, 3.1 (5), 3.1 (6) we have

$$N_{1,A}(\infty, r) + N_{1,A}^{\geq 2}(\infty, r) \leq$$

$$\leq 2N_{1,f}(\infty, r) + \sum_{i=1}^{k}(N_{1,f(z+ic)}(\infty, r) + N_{1,f(z+ic)}^{\geq 2}(\infty, r)) + O(1) \leq$$

$$\leq 2N_f(\infty, r) + \sum_{i=1}^{k} N_{f(z+ic)}(\infty, r) + O(1) \leq$$

$$\leq 2T_f(r) + \sum_{i=1}^{k} T_{f(z+ic)}(r) + O(1) \leq (k+2)T_f(r) + O(1).$$

So

$$N_{1,A}(\infty, r) + N_{1,A}^{\geq 2}(\infty, r) \leq (k+2)T_f(r) + O(1). \tag{3.8}$$

Similarly, and note that any zero of A can occur only at zeros of

$$f, f(z+c), f(z+2c), \ldots, f(z+kc),$$

we obtain

$$N_{1,A}(0, r) + N_{1,A}^{\geq 2}(0, r) \leq (k+2)T_f(r) + O(1). \tag{3.9}$$

Similarly we obtain

$$\begin{gathered} N_{1,B}(\infty, r) + N_{1,B}^{\geq 2}(\infty, r) \leq (k+2)T_g(r) + O(1), \\ N_{1,B}(0, r) + N_{1,B}^{\geq 2}(0, r) \leq (k+2)T_g(r) + O(1). \end{gathered} \tag{3.10}$$

From (3.7) – (3.10) we have

$$(n-k)T_f(r) \leq (2k+4)(T_f(r) + T_g(r)) - \log r + O(1).$$

Similarly

$$(n-k)T_g(r) \leq (2k+4)(T_f(r) + T_g(r)) - \log r + O(1).$$

So

$$(n-k)(T_f(r) + T_g(r)) \leq (4k+8)(T_f(r) + T_g(r)) - 2\log r + O(1),$$

$$(n-5k-8)(T_f(r) + T_g(r)) + 2\log r \leq O(1).$$

By $n \geq 5k+8$ we obtain a contradiction.

Case 2: $A = f^n f(z+c)\ldots f(z+kc) \equiv B = g^n g(z+c)\ldots g(z+kc)$. Set $h = \dfrac{f}{g}$. Assume that h is not a constant. Then we get

$$h^n = \frac{1}{h(z+c)\ldots h(z+kc)}.$$

Thus, by Lemma 3.1 (5), we get

$$nT_h(r) = T_{h^n} = T_{\frac{1}{h(z+c)\ldots h(z+kc)}} \leq \sum_{i=1}^{k} T_{h(z+ic)}(r) + O(1) \leq kT_h(r),$$

which is a contradiction with $n \geq 5k+8$. Hence h must be a constant, which implies that $h^{n+k} = 1$, thus $f = hg$ with $h^{n+k} = 1$.

Case 3: $f^n f(z+c)\ldots f(z+kc).g^n g(z+c)\ldots g(z+kc) \equiv 1$. From this we have $(fg)^n(f(z+$ $+c)g(z+c))\ldots(f(z+kc)g(z+kc)) = 1$. Set $l = fg$. Assume that l is not a constant. Then we get

$$l^n = \frac{1}{l(z+c)\ldots l(z+kc)}.$$

Similar as above, l must be a constant. Thus $fg = l$ with $l^{n+k} = 1$.

(2) Set $C = f^n(f(z+c))^{q_1}\ldots(f(z+kc))^{q_k}, D = g^n(g(z+c))^{q_1}\ldots(g(z+kc))^{q_k}$.

It suffices to consider the following cases:

Case 1:

$$T_C(r) + O(1) \leq N_{1,C}(\infty, r) + N^{\geq 2}_{1,C}(\infty, r) + N_{1,C}(0, r) + N^{\geq 2}_{1,C}(0, r) +$$
$$+ N_{1,D}(\infty, r) + N^{\geq 2}_{1,D}(\infty, r) + N_{1,D}(0, r) + N^{\geq 2}_{1,D}(0, r) - \log r + O(1).$$

By Lemmas 3.2 (4),

$$\begin{gathered}\left(n - \sum_{i=1}^{k} q_i\right) T_f(r) \leq T_C(r) + O(1),\\ \left(n - \sum_{i=1}^{k} q_i\right) T_g(r) \leq T_D(r) + O(1).\end{gathered} \tag{3.11}$$

By $q_i \geq 2,\ i = 1,\ldots,k$,

$$N_{1,(f(z+ic))^{q_i}}(\infty, r) + N^{\geq 2}_{1,(f(z+ic))^{q_i}}(\infty, r)) \leq 2N_{f(z+ic)}(\infty, r).$$

From this and similar as (3.8) we obtain

$$\begin{gathered}N_{1,C}(\infty, r) + N^{\geq 2}_{1,C}(\infty, r) \leq (2k+2)T_f(r) + O(1),\\ N_{1,C}(0, r) + N^{\geq 2}_{1,C}(0, r) \leq (2k+2)T_f(r) + O(1),\\ N_{1,D}(\infty, r) + N^{\geq 2}_{1,D}(\infty, r) \leq (2k+2)T_g(r) + O(1),\\ N_{1,D}(0, r) + N^{\geq 2}_{1,D}(0, r) \leq (2k+2)T_g(r) + O(1).\end{gathered} \tag{3.12}$$

Since (3.11), (3.12), and similar as in (1) we obtain

$$T_C(r) \leq (4k+4)(T_f(r)+T_g(r)) - \log r + O(1),$$

$$T_D(r) \leq (4k+4)(T_f(r)+T_g(r)) - \log r + O(1),$$

$$\left(n - \sum_{i=1}^{k} q_i - 8k - 8\right)(T_f(r)+T_g(r)) + 2\log r \leq O(1).$$

By

$$n \geq \sum_{i=1}^{k} q_i + 8k + 8$$

we obtain a contradiction.

Case 2: $C = f^n(f(z+c))^{q_1}\ldots(f(z+kc))^{q_k} \equiv D = g^n(g(z+c))^{q_1}\ldots(g(z+kc))^{q_k}$.

Similar as Case 2 of (1) we get $f = hg$ with $h^{n+q_1+\ldots+q_k} = 1$.

Case 3: $f^n(f(z+c))^{q_1}\ldots(f(z+kc))^{q_k}.g^n(g(z+c))^{q_1}\ldots(g(z+kc))^{q_k} \equiv 1$.

Similar as Case 3 of (1) we get $fg = l$ with $l^{n+q_1+\ldots+q_k} = 1$.

(3) Set $E = f^n f(z+e_1c)\ldots f(z+e_mc)(f(z+t_1c))^{q_1}\ldots(f(z+t_kc))^{q_k}, H = g^n g(z+ + e_1c)\ldots g(z+e_mc)(g(z+t_1c))^{q_1}\ldots(g(z+t_kc))^{q_k}$.

It suffices to consider the following cases:

Case 1:

$$T_E(r) + O(1) \leq N_{1,E}(\infty,r) + N_{1,E}^{\geq 2}(\infty,r) + N_{1,E}(0,r) + N_{1,E}^{\geq 2}(0,r) +$$

$$+ N_{1,H}(\infty,r) + N_{1,H}^{\geq 2}(\infty,r) + N_{1,H}(0,r) + N_{1,H}^{\geq 2}(0,r) - \log r + O(1).$$

By Lemma 3.2 (4)

$$\left(n - m - \sum_{i=1}^{k} q_i\right) T_f(r) \leq T_E(r) + O(1),$$
$$\left(n - m - \sum_{i=1}^{k} q_i\right) T_g(r) \leq T_D(r) + O(1). \qquad (3.13)$$

Similar as Case 1 of (1) and (2) we have

$$N_{1,E}(\infty, r) + N_{1,E}^{\geq 2}(\infty, r) \leq$$

$$\leq 2N_f(\infty, r) + \sum_{i=1}^{m} N_{f(z+e_i c)}(\infty, r) + 2\sum_{i=1}^{k} N_{f(z+t_i c)}(\infty, r)O(1) \leq$$

$$\leq (m + 2k + 2)T_f(r) + O(1), \tag{3.14}$$

$$N_{1,E}(0, r) + N_{1,E}^{\geq 2}(0, r) \leq (m + 2k + 2)T_f(r) + O(1),$$

$$N_{1,H}(\infty, r) + N_{1,H}^{\geq 2}(\infty, r) \leq (m + 2k + 2)T_g(r) + O(1),$$

$$N_{1,H}(0, r) + N_{1,H}^{\geq 2}(0, r) \leq (m + 2k + 2)T_g(r) + O(1).$$

Since (3.13), (3.14), and similar as in (1), (2), we obtain

$$\left(n - m - \sum_{i=1}^{k} q_i\right) T_f(r) \leq 2(m + 2k + 2)(T_f(r) + T_g(r)) - \log r + O(1),$$

$$\left(n - m - \sum_{i=1}^{k} q_i\right) T_g(r) \leq 2(m + 2k + 2)(T_f(r) + T_g(r)) - \log r + O(1),$$

$$\left(n - m - \sum_{i=1}^{k} q_i\right) (T_f(r) + T_g(r)) \leq 4(m + 2k + 2)(T_f(r) + T_g(r)) - 2\log r + O(1),$$

$$\left(n - 5m - \sum_{i=1}^{k} q_i - 8k - 8\right) (T_f(r) + T_g(r)) + 2\log r \leq +O(1).$$

Which is a contradiction with

$$n \geq 5m + \sum_{i=1}^{k} q_i + 8k + 8.$$

Case 2: Prove is similarly as in Case 2 of (1) and (2) we get $f = hg$ with $h^{n+m+q_1+\ldots+q_k} = 1$.
Case 3: Prove is similarly as in Case 2 of (1) and (2) we get $fg = l$ with $l^{n+m+q_1+\ldots+q_k} = 1$.
Theorem 1.6 is proved.

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